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The Spectral Representation of Markov-Switching Arma Models

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Abstract

In this paper we propose a method to derive the spectral density function of Markov Switching ARMA models. We apply the Riesz-Fisher Theorem which defines the spectral representation as the Fourier Transform of the autocovariance functions.

JEL Classification: C32

Keywords: Multivariate ARMA models; Regime-switching models, Markov-switching models, Frequency domain

1 Introduction

This paper proposes a tractable method to derive the spectral representation of a general class of Markov Switching (MS) ARMA models. The procedure simply relies on the Riesz-Fisher theorem, which defines the *spectral density function* of a covariance-stationary stochastic process as the Fourier Transform of the autocovariance functions. Markov Switching models are widely used for modelling dynamics in different fields, for instance in economic studies where applications have found a great development from the seminal work of Hamilton (1989). However, to the best of our knowledge, this is the first attempt to derive the spectral representation for regime-switching cases[1].

We consider a MSARMA *(p,q)* model of the following type:

$$x_t = \sum_{i=1}^{p} a_i\left(\xi_t\right) x_{t-i} + \varepsilon_t + \sum_{j=1}^{q} b_j\left(\xi_t\right) \varepsilon_{t-j} \tag{1}$$

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[1]Traditional studies of non linear models in frequency domain rest on the complex concepts of *Volterra Series Expansion* or *higher order cumulants* and the corresponding Fourier transforms, the *polyspectra*. See Priestley (1981), ch.11.

where x_t is a zero mean purely indeterministic process in $\mathbb{R}^K$, ξ_t is an irreducible, aperiodic and ergodic Markov Chain with finite space $\Xi = \{1, 2, ..., d\}$, stationary transition probabilities denoted by $p_{ij} = pr\left(\xi_t = j \mid \xi_{t-1} = i\right)$ and unconditional (or steady state) probabilities, $\pi_i = pr\left(\xi_t = i\right), 1 \leq i \leq d$, where $\sum_{i=1}^{d} \pi_i = 1$[2]. The $a_i\left(\xi_t\right)$ and $b_j\left(\xi_t\right)$ are $K \times K$ real random matrices. To allow for the possibility of change in variance, it is assumed that $\varepsilon_t = \sigma\left(\xi_t\right)\eta_t$, where $\sigma\left(\xi_t\right)$ is a $K \times K$ random matrix and η_t is supposed to be a white noise vector with $E\left(\eta_t' \eta_t\right) = \Omega$.

The paper is structured as follows. Section 2 reviews the main results of Francq and Zakoïan (2001), who define the second order moments for covariance stationary Markov Switching models. We complete the characterization of autocovariance functions including also the case of negative time lags. The derivation of the spectral matrix follows. In Section 3 we propose an economic application as a simple example of a MSVAR(4) model. Section 4 concludes.

2 Markovian Representation: Stationarity, Second Order Moments and Spectral Density

Francq and Zakoïan (2001), FZ hereinafter, propose the following Markovian representation of (1): $z_t = \Phi_t z_{t-1} + \Sigma_t \eta_t$ where $z_t = \begin{bmatrix} x_t & x_{t-1} & \cdots & x_{t-p+1} & \varepsilon_t & \varepsilon_{t-1} & \cdots & \varepsilon_{t-q+1} \end{bmatrix}' \in \mathbb{R}^{K(p+q)}$, $\Sigma\left(\xi_t\right) = \begin{bmatrix} \sigma\left(\xi_t\right) & 0 & \cdots & 0 & \sigma\left(\xi_t\right) & 0 & \cdots & 0 \end{bmatrix}' \in \mathbb{R}^{(p+q)}$ and

$$\Phi_t = \begin{bmatrix}
a_1\left(\xi_t\right) & & \cdots & & a_p\left(\xi_t\right) & b_1\left(\xi_t\right) & & \cdots & & b_q\left(\xi_t\right) \\
I_K & 0 & \cdots & & 0 & 0 & & \cdots & & 0 \\
0 & I_K & \cdots & & 0 & 0 & & & & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & & \vdots \\
0 & & \cdots & I_K & 0 & 0 & & \cdots & 0 & 0 \\
0 & & \cdots & & 0 & 0 & & \cdots & & 0 \\
0 & & \cdots & & 0 & I_K & 0 & \cdots & & 0 \\
0 & & \cdots & 0 & 0 & 0 & I_K & \cdots & & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & & \vdots \\
0 & & \cdots & 0 & 0 & 0 & & \cdots & I_K & 0
\end{bmatrix}$$

is a $K\left(p+q\right) \times K\left(p+q\right)$ matrix. Letting $\Phi\left(k\right)$ be the matrix obtained by replacing ξ_t by k in Φ_t, the following matrices are defined:

$$P = \begin{bmatrix}
p_{11}\left\{\Phi\left(1\right) \otimes \Phi\left(1\right)\right\} & p_{21}\left\{\Phi\left(1\right) \otimes \Phi\left(1\right)\right\} & \cdots & p_{d1}\left\{\Phi\left(1\right) \otimes \Phi\left(1\right)\right\} \\
p_{12}\left\{\Phi\left(2\right) \otimes \Phi\left(2\right)\right\} & p_{22}\left\{\Phi\left(2\right) \otimes \Phi\left(2\right)\right\} & \cdots & p_{d2}\left\{\Phi\left(2\right) \otimes \Phi\left(2\right)\right\} \\
\vdots & \vdots & \ddots & \vdots \\
p_{1d}\left\{\Phi\left(d\right) \otimes \Phi\left(d\right)\right\} & p_{2d}\left\{\Phi\left(d\right) \otimes \Phi\left(d\right)\right\} & \cdots & p_{dd}\left\{\Phi\left(d\right) \otimes \Phi\left(d\right)\right\}
\end{bmatrix}$$

[2]Hamilton (2004), page 684, shows how to compute the ergodic probabilities π_i.

and

$$S=\begin{bmatrix}\pi_1\left\{\Sigma\left(1\right)\otimes\Sigma\left(1\right)\right\}\\ \pi_2\left\{\Sigma\left(2\right)\otimes\Sigma\left(2\right)\right\}\\ \vdots\\ \pi_d\left\{\Sigma\left(d\right)\otimes\Sigma\left(d\right)\right\}\end{bmatrix},$$

which are, respectively, a $dK^2\left(p+q\right)^2\times dK^2\left(p+q\right)^2$ and a $dK^2\left(p+q\right)^2\times K^2$ matrix.

The following theorem states the necessary and sufficient condition for second-order stationarity of MSARMA models which is assumed to hold in the rest of the paper.

Theorem 1 *Suppose that*

$$\varrho(P)<1 \tag{2}$$

where $\varrho\left(\cdot\right)$ denotes the spectral radius, then, for all $t\in\mathbb{Z}$, the series $x_t=\varepsilon_t+\sum\limits_{k=1}^{\infty}A_tA_{t-1}...A_{t-k+1}\varepsilon_{t-k}$ converges in L^2 and the process x_t is the unique second-order stationary solution of (1). Suppose that (1) admits a second order stationary solution, then we have $\sum\limits_{k=0}^{\infty}\left\|\mathcal{I}'P^kS\sigma_\varepsilon^2\right\|<\infty$ where $\mathcal{I}'=\left(I_{K^2(p+q)},...,I_{K^2(p+q)}\right)$ which holds as long as (2) is true.

Proof. *See FZ, page 347.* ■

2.1 Second order moments

FZ define the conditional variance of z_t as follows:

$$\begin{aligned}\pi_iE\left\{vec(z_tz_t')\mid\xi_t=i\right\} &= \pi_i\left\{\Sigma\left(i\right)\otimes\Sigma\left(i\right)\right\}vec\left(\Omega\right)+ \\ &+\left\{\Phi\left(i\right)\otimes\Phi\left(i\right)\right\}\sum_{j=1}^{d}p_{ji}\pi_jE\left(vec\left(z_{t-1}z_{t-1}'\right)\left|\xi_{t-1}=j\right.\right).\end{aligned} \tag{3}$$

Let $V=\left(\left(E\left(vec\left(z_tz_t'\right)\right)\mid\xi_t=1\right)\pi_1,...,\left(E(vec\left(z_tz_t'\right))\mid\xi_t=d\right)\pi_d\right)'$; we then have

$$V=\left(I-P\right)^{-1}Svec\left(\Omega\right). \tag{4}$$

The construction of the conditional expectations in (3) is quite intuitive: they are made up by the sum of the conditional objects relative to the previous period weighted by the respective probabilities. Notice that, by (2), $I-P$ is an invertible matrix. We can therefore compute the variance-covariance matrix of the vector $x_t:vec\left(Ex_tx_t'\right)=\left(g'\otimes f'\otimes f'\right)V$ where $g=\left(1,...,1\right)'\in\mathbb{R}^d$ and $f'=\left(I_K,0,...,0\right)$ is a $K\times K\left(p+q\right)$ matrix.

Similar calculations can be used to define the autocovariance functions of x_t, $\Gamma_x\left(\tau\right)$, for all $\tau>0$. Let $W\left(\tau\right)$ be the matrix of size $dK\left(p+q\right)\times K\left(p+q\right)$ whose

ith block $(i = 1, ..., d)$ is the $K(p+q) \times K(p+q)$ matrix $\pi_i E\left\{z_t z'_{t-\tau} \mid \xi_t = i\right\}$. For $\tau > 0$,

$$\begin{aligned} W(\tau) &= \begin{bmatrix} \pi_1 E\left\{z_t z'_{t-\tau} \mid \xi_t = 1\right\} \\ \pi_2 E\left\{z_t z_{t-\tau} \mid \xi_t = 2\right\} \\ \vdots \\ \pi_d E\left\{z_t z_{t-\tau} \mid \xi_t = d\right\} \end{bmatrix} \\ &= \begin{bmatrix} \pi_1 \left\{\Gamma_z(\tau) \mid \xi_t = 1\right\} \\ \pi_2 \left\{\Gamma_z(\tau) \mid \xi_t = 2\right\} \\ \vdots \\ \pi_d \left\{\Gamma_z(\tau) \mid \xi_t = d\right\} \end{bmatrix} \end{aligned}$$

where $\Gamma_z(\tau) = E\left(z_t z'_{t-\tau}\right)$ is the autocovariance of z_t[3]. Then,

$$\begin{aligned} \pi_i \left\{\Gamma_z(\tau) \left| \xi_t = i\right.\right\} &= \sum_{j=1}^{d} E\left\{\Phi(i) z_{t-1} z'_{t-\tau} \left| \xi_{t-1} = j\right.\right\} p_{ji} \pi_j \\ &= \sum_{j=1}^{d} \Phi(i) \left\{\Gamma_z(\tau - 1) \left| \xi_{t-1} = j\right.\right\} p_{ji} \pi_j \end{aligned}$$

from which we have

$$\begin{aligned} W(\tau) &= P^* W(\tau - 1) \\ &= P^{*\tau} W(0), \quad \forall \tau > 0 \end{aligned} \tag{5}$$

where

$$P^* = \begin{bmatrix} p_{11}\Phi(1) & p_{21}\Phi(1) & \cdots & p_{d1}\Phi(1) \\ p_{12}\Phi(2) & p_{22}\Phi(2) & \cdots & p_{d2}\Phi(2) \\ \vdots & \vdots & \ddots & \vdots \\ p_{1d}\Phi(d) & p_{2d}\Phi(d) & \cdots & p_{dd}\Phi(d) \end{bmatrix}$$

is a $dK(p+q) \times dK(p+q)$ matrix. Finally, we can compute the autocovariance of the vector process x_t: $\Gamma_x(\tau) = (g' \otimes f') W(\tau) f$.

For $\tau < 0$, let's define $\widetilde{W}(\tau)$ be the matrix of size $dK(p+q) \times K(p+q)$ whose ith block $(i = 1, ..., d)$ is the $K(p+q) \times K(p+q)$ matrix $\pi_i E\left\{z_t z'_{t-\tau} \mid \xi_{t-\tau} = i\right\}$. It

[3]Notice that the matrix $W(0)$ has the same elements as the matrix V. The latter is a vector composed by $K(p+q)$ blocks corresponding to the rows of $W(0)$.

is defined as follows

$$\begin{aligned}
\widetilde{W}(\tau) &= \begin{bmatrix} \pi_1 E\left\{z_t z'_{t-\tau} \mid \xi_{t-\tau}=1\right\} \\ \pi_2 E\left\{z_t z'_{t-\tau} \mid \xi_{t-\tau}=2\right\} \\ \vdots \\ \pi_d E\left\{z_t z'_{t-\tau} \mid \xi_{t-\tau}=d\right\} \end{bmatrix} \\
&= \begin{bmatrix} \pi_1 \left\{\Gamma_z(\tau) \mid \xi_{t-\tau}=1\right\} \\ \pi_2 \left\{\Gamma_z(\tau) \mid \xi_{t-\tau}=2\right\} \\ \vdots \\ \pi_d \left\{\Gamma_z(\tau) \mid \xi_{t-\tau}=d\right\} \end{bmatrix}
\end{aligned}$$

where $\Gamma_z(\tau) = E\left(z_t z'_{t-\tau}\right)$ is the autocovariance of z_t. Then for $\tau < 0$,

$$\begin{aligned}
\widetilde{W}^i(\tau) &= \pi_i \left\{\Gamma_z(\tau) \left|\xi_{t-\tau}=i\right.\right\} = \pi_i \left\{E z_t z'_{t-\tau} |\xi_{t-\tau}=i\right\} \\
&= \left[\pi_i \left\{E z_{t-\tau} z'_t |\xi_{t-\tau}=i\right\}\right]' \\
&= \left[W^i(-\tau)\right]'
\end{aligned}$$

from which we have

$$\begin{aligned}
\widetilde{W}(\tau) &= \left[P^* W(-\tau-1)\right]^{b'} \\
&= \left[P^{*^{-\tau}} W(0)\right]^{b'} \\
&= \left[P^{*|\tau|} W(0)\right]^{b'}
\end{aligned} \tag{6}$$

where $W^i(\cdot)$ represents the i-th block of matrix $W(\cdot)$. Finally, for negative τ, we can compute the autocovariance of the vector process x_t: $\Gamma_x(\tau) = (g' \otimes f')\,\widetilde{W}(\tau)\, f$ from which it can be verified that $\Gamma_x(\tau) = \Gamma'_x(|\tau|), \forall \tau < 0$.

2.2 Spectral Representation

In this section we apply the Riesz-Fisher theorem which defines the spectral matrix as the Fourier Transform of the autocovariance function:

$$\begin{aligned}
F_x(\omega) &= \sum_{\tau=-\infty}^{\infty} \Gamma_x(\tau)\, e^{-i\omega\tau} \\
&= \sum_{\tau=0}^{\infty} \Gamma_x(\tau)\, e^{-i\omega\tau} + \sum_{\tau=-\infty}^{-1} \Gamma'_x(\tau)\, e^{-i\omega\tau}
\end{aligned} \tag{7}$$

1. The multivariate spectral matrix described the spectral density functions of each element of the state vector in the diagonal terms. The off-diagonal terms are

defined cross spectral density functions and they are typically complex numbers. In this paper we are only interested to the diagonal terms. Therefore, we can compute them, without loss of generality, considering the summation

$$\begin{aligned}
F_x(\omega) &= \sum_{\tau=-\infty}^{\infty} \Gamma_x(\tau)\, e^{-i\omega\tau} \\
&= \sum_{\tau=-\infty}^{\infty} \Gamma_x(|\tau|)\, e^{-i\omega|\tau|} \\
&= \sum_{\tau=-\infty}^{\infty} (g' \otimes f')\, W(\tau)\, f e^{-i\omega|\tau|} \\
&= \sum_{\tau=-\infty}^{\infty} (g' \otimes f')\, P^{*|\tau|} W(0)\, f e^{-i\omega|\tau|}
\end{aligned}$$

where $F_x(\omega)$ is the spectral density matrix of x_t and ω, the frequency, belongs to $[-\pi, \pi]$. If P^* is diagonalizable[4], it holds that $P^* = TDT^{-1}$where T is a matrix made up by the linear independent eigenvectors of P^* and D is a diagonal matrix whose elements are the distinct eigenvalues of P^*. By the properties of the series of diagonalizable matrices we can write:

$$\begin{aligned}
F_x(\omega) &= \sum_{\tau=-\infty}^{\infty} \Gamma_x(\tau)\, e^{-i\omega\tau} \\
&= \sum_{\tau=-\infty}^{\infty} (g' \otimes f')\, TD^{|\tau|}T^{-1}W(0)\, f e^{-i\omega\tau} \\
&= (g' \otimes f')\, T \sum_{\tau=-\infty}^{\infty} D^{|\tau|} e^{-i\omega\tau} T^{-1} W(0)\, f \\
&= (g' \otimes f')\, T diag \left[\sum_{\tau=-\infty}^{\infty} \lambda_1^{|\tau|} e^{-i\omega\tau}, \sum_{\tau=-\infty}^{\infty} \lambda_2^{|\tau|} e^{-i\omega\tau}, \ldots, \sum_{\tau=-\infty}^{\infty} \lambda_{dK(p+q)}^{|\tau|} e^{-i\omega\tau} \right] \times \\
&\quad \times T^{-1} W(0)\, f
\end{aligned}$$

that we rewrite more compactly as

$$F_x(\omega) = (g' \otimes f')\, T \left(\oplus_{k=1}^{dK(p+q)} \sum_{\tau=-\infty}^{\infty} \lambda_k^{|\tau|} e^{-i\omega\tau} \right) T^{-1} W(0)\, f \tag{8}$$

[4]A necessary and sufficient condition for a $n \times n$ matrix to be diagonalizable is that it has n linearly independent eigenvectors. A natural relaxion of this requirement is the use of the Jordan Canonical Form, whose properties still allow to compute the power series of a matrix as power series of its elements. See Appendix A.

where λ_k, with $k = 1, .., dK(p+q)$, are the eigenvalues of the matrix P^*. It is known that each sum into the bracket converges to $\sum_{\tau=-\infty}^{\infty} \lambda_i^{|\tau|} e^{-i\tau\omega} = \frac{(1-\lambda_i^2)}{(1+\lambda_i^2 - 2\lambda_i \cos\omega)}$ if and only if

$$| \lambda_i | < 1. \tag{9}$$

Condition (9) is always satisfied in our context. Indeed, Costa et al. (2005)[5] show that $\varrho(P) < 1 \Rightarrow \varrho(P^*) < 1$.

Substituting in (8), we get:

$$F_x(\omega) = (g' \otimes f') T \left(\oplus_{k=1}^{dK(p+q)} \frac{(1-\lambda_k^2)}{(1+\lambda_k^2 - 2\lambda_k \cos\omega)} \right) T^{-1} W(0) f. \tag{10}$$

which defines the spectral density matrix of model (11)[6]. In the next Section, we present an example to investigate its characteristics.

3 Example: the case of a MSVAR(4)

The example is based on an estimated model of the US economy: a regime-switching version of the quarterly backward looking model of Rudebush and Svensson (1999), presented in Svensson and Williams (2005). The key variables are quarterly annualized inflation v_t, the output gap y_t and the instrument rate (the federal fund rate), r_t. The model is composed by a Phillips curve and an aggregate demand of the following forms:

$$\begin{aligned} v_t &= \sum_{i=1}^{3} \alpha_i(\xi_t) v_{t-i} + \left(1 - \sum_{i=1}^{3} \alpha_i(\xi_t)\right) v_{t-4} + \alpha_4(\xi_t) y_{t-1} + \sigma_\pi(\xi_t) \eta_{v,t}, \quad (11) \\ y_t &= \beta_1(\xi_t) y_{t-1} + \beta_2(\xi_t) y_{t-2} + \beta_3(\xi_t)(\overline{r}_{t-1} - \overline{v}_{t-1}) + \sigma_y(\xi_t) \eta_{y,t} \end{aligned}$$

where $\xi_t \in \{1,2,3\}$ indexes the regime, $\overline{r}_{t-1} \equiv \sum_{i=1}^{4} r_{t-i}/4$ and $\overline{v}_{t-1} \equiv \sum_{i=1}^{4} v_{t-i}/4$ are 4-quarter averages and the shocks $\eta_{v,t}$ and $\eta_{y,t}$ are each independent standard normal variables. The estimated coefficients are reported in Table 1, together with the estimates for the linear case.

The estimated transition matrix $\mathcal{P}$ with elements $p(j,i)$ [7] and its implied stationary distribution $\pi = [\pi_1\ \pi_2\ \pi_3]'$ are

$$\mathcal{P} = \begin{bmatrix} 0.83 & 0.03 & 0.04 \\ 0.09 & 0.92 & 0.05 \\ 0.08 & 0.05 & 0.91 \end{bmatrix}, \quad \pi = \begin{bmatrix} 0.17 \\ 0.45 \\ 0.38 \end{bmatrix}.$$

[5] See Costa et al. (2005), page 35, Proposition 3.6.

[6] It is possible to check that for $d = 1$ formula (10) reduces to the known expression of the spectral density of a VAR(1) of dimensions $K(p+q)$ .We thank the referee for suggesting this observation.

[7] Each element p_{ji} represents the probability of moving from state j to i, so that columns elements sum up to 1.

Parameter	Model 1	Model 2	Model 3	Constant
α_1	0.2402	0.4236	1.2387	0.5697
α_2	0.1654	-0.2219	-0.6911	0.0752
α_3	1.0388	0.0714	0.5491	0.1276
α_4	0.1514	0.2755	-0.0304	0.1451
β_1	1.0015	1.0302	1.8943	1.1834
β_2	-0.0853	-0.1069	-1.0312	-0.2651
β_3	-0.3244	0.0315	-0.1011	-0.0510
σ_π	1.5504	0.1798	0.1562	1.0070
σ_y	1.2696	0.1447	0.2365	0.7540

Table 1: Estimated coefficients of model (10). Source: Svensson and Williams (2005).

For both models we consider the simple model-independent Taylor rule

$$i_t = \gamma_\pi \pi_t + \gamma_y y_t \tag{12}$$

which minimizes the loss function $L_t = (1/2)\,Var(\pi_t) + (1/2)\,Var(y_t)$ [8] so that, substituting in (11), we get a MSVAR(4) bivariate model. In the terminology of Section 1 we have $d = 3$, $K = 2$, $p = 4$ and $q = 0$. The idea is that the policymaker set the policy facing uncertainty about the regime in which the economy is.

Figure 1 shows the comparison of the spectral representation of the estimated Markov Switching model and the constant coefficients version. Given quarterly data, business cycle frequencies range from 0.2 to 1.05, approximately.

[Picture 1 about here]

The distributions of the volatility of the inflation processes are quite similar. However, the case of regime-switching presents slightly higher volatility components at all the frequencies. Further, it is better able to capture the high frequency component (corresponding to a period cycle of one-two years) usually detected in the postwar US inflation time series (see Balakrishman and Ouliaris (2006) for instance). The output gap spectral dynamics shows important differences on the frequency decomposition of the volatility. This is plausibly due to the fact that, being the regimes quite different in their natures, in particular regarding β_3, the coefficient which determines the effect of the policy on the real activity, the policy intervention is quite moderate compared to the optimal response in the case of a linear model. Further, the volatility component of the business cycle is quite exacerbated. This is the range at which the switching occurs[9] suggesting that the switching characteristics can play an important

[8] The coefficients are chosen to minimize the loss function using a grid search algorithm over the space $\gamma_\pi \in [0.00, 10.00]$ and $\gamma_y \in [-0.50, 5.00]$. The two policies are $\gamma_\pi = 1.27$ and $\gamma_y = -0.07$ for the regime switching model and $\gamma_\pi = 3.89$ and $\gamma_y = 2.93$ for the linear version.

[9] The mean duration (md) of regime i can be computed knowing the transition probabilities: $md_i = 1/(1 - p_{ii})$.

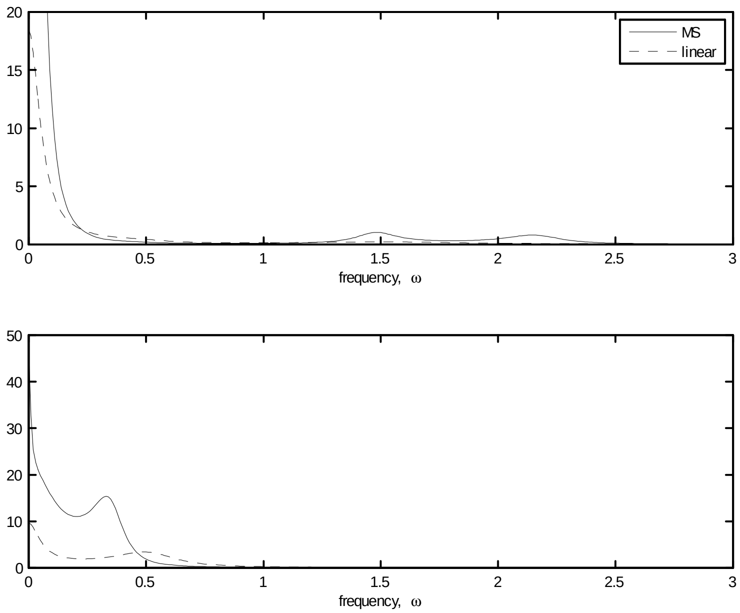

Figure 1: The spectral representations of the inflation (above panel) and the output gap processes (below panel) of the regime switching (solid line) and the linear version (dashed line) of model (10).

role in the determination of the frequency distribution of the volatility. We should notice, however, that the comparison must be discussed with caution since different policies are considered in the two models.

Further, Francq and Zakoïan(2002) claim that a VAR(p) switching model of dimension K with d number of states has an ARMA$\left(d\left(Kp\right)^2, d\left(Kp\right)^2 - 1\right)$ representation. In our example, it would imply that appropriate comparison between the spectral densities should be made comparing the estimated MSVAR(4) with a linear ARMA(192,192), which seems hardly correctly specified. In general, if important differences between the spectral densities exist, this can be considered as an indicator of misspecification of the MSVAR or inaccurate estimation[10].

Nonetheless, spectral analyses are recently receiving a renew interest in macroeconomic studies where they are used to investigate policy evaluations based on frequency-specific effects (Brock et al. (2007)). We regard these contributions as a promising area for future research and we consider our work as a first step of the extension of such analyses in non linear contexts.

[10] We thank the referee for suggesting this observation.

4 Conclusion

In this paper we propose a simple procedure to compute the spectral representation of covariance-stationary Markov Switching ARMA models. We complete the characterization of the autocovariance function of such models, showing their correspondence with the second order moments of linear stationary ARMA models. The spectral representation is then obtained as the Fourier Transform of the autocovariance function using the Riesz-Fisher Theorem as in linear frameworks. The example provided suggests that the switching recurrence plays an important role in the frequency decomposition of regime-switching models.

5 Appendix A

If matrix P^* does not have $dK(p+q)$ linearly independent eigenvectors, it is not a diagonalizable matrix. However, there still exists an invertible matrix T such that $P^* = TJT^{-1}$. This is called the Jordan Decomposition where J is a block diagonal matrix:

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_n \end{bmatrix}$$

where each J_i is a square matrix

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ \vdots & & \ddots & \lambda_i & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_i \end{bmatrix}$$

of dimension m_i : it can be indicated as J_{λ_i,m_i}. Using the notation introduced in Section 2, matrix J can therefore also be compactly defined as $J_{\lambda_1,m_1} \oplus J_{\lambda_2,m_2} \oplus \ldots \oplus J_{\lambda_n,m_n}$ or $diag\left(J_{\lambda_1,m_1}, J_{\lambda_2,m_2}, \ldots, J_{\lambda_n,m_n}\right)$. The Jordan Decomposition is very useful because it still allows the computation of infinite series. Indeed, the following properties hold:

$$f(P^*) = T\left(\oplus_{k=1}^{n} f\left(J_{\lambda_k,m_k}\right)\right)T^{-1}$$

and

$$P^{*k} = TJ^kT^{-1}$$

Therefore, we can compute the series directly via power series of every Jordan blocks. Further,

$$
f\left(J_{\lambda_i,m_i}\right)=\begin{bmatrix} f\left(\lambda_i\right) & f'\left(\lambda_i\right) & \frac{f''\left(\lambda_i\right)}{2!} & \cdots & \frac{f^{\left(m_i-1\right)}\left(\lambda_i\right)}{\left(m_i-1\right)!} \\ 0 & f\left(\lambda_i\right) & f'\left(\lambda_i\right) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{f''\left(\lambda_i\right)}{2!} \\ \vdots & & \ddots & \ddots & f'\left(\lambda_i\right) \\ 0 & \cdots & \cdots & 0 & f\left(\lambda_i\right) \end{bmatrix}. \tag{13}
$$

Back to our computation, when P^* is not diagonalizable, we can rewrite (**??**) as

$$
\begin{aligned}
F_x\left(\omega\right) &= \sum_{\tau=-\infty}^{\infty}\left(g'\otimes f'\right)TJ^{\tau}T^{-1}W\left(0\right)fe^{-i\omega\tau} \\
&= \left(g'\otimes f'\right)T\sum_{\tau=-\infty}^{\infty}J^{\tau}e^{-i\tau\omega}T^{-1}W\left(0\right)f \\
&= \left(g'\otimes f'\right)T\begin{bmatrix} \sum_{\tau=-\infty}^{\infty}J_1^{\tau}e^{-i\tau\omega} & 0 & \cdots & \cdots & 0 \\ 0 & \sum_{\tau=-\infty}^{\infty}J_2^{\tau}e^{-i\tau\omega} & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \sum_{\tau=-\infty}^{\infty}J_n^{\tau}e^{-i\tau\omega} \end{bmatrix}T^{-1}W\left(0\right)f
\end{aligned}
$$

Considering (13) and $f\left(\lambda_i\right)=\sum\limits_{\tau=-\infty}^{\infty}\lambda_i^{\tau}e^{-i\tau\omega}=\frac{\left(1-\lambda_i^2\right)}{\left(1+\lambda_i^2-2\lambda_i\cos\omega\right)}$, we can write:

$$
F_x\left(\omega\right)=\left(g'\otimes f'\right)T\oplus_{k=1}^{n}\begin{bmatrix} f\left(\lambda_k\right) & f'\left(\lambda_k\right) & \frac{f''\left(\lambda_k\right)}{2!} & \cdots & \frac{f^{\left(m_k-1\right)}\left(\lambda\right)}{\left(m_k-1\right)!} \\ 0 & f\left(\lambda_k\right) & f'\left(\lambda_k\right) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{f''\left(\lambda_k\right)}{2!} \\ \vdots & & \ddots & \ddots & f'\left(\lambda_k\right) \\ 0 & \cdots & \cdots & 0 & f\left(\lambda_k\right) \end{bmatrix}T^{-1}W\left(0\right)f
$$

which is easily computable.

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