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Beatrice Pataracchia

Design Limits in Regime-Switching Cases

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Abstract

This paper characterizes the derivation and the assessment of *design limits* of monetary policies in the case of a regime-switching economy. The object of the analysis of design limits is to derive the restrictions on how feedback rules, the Taylor-type rules typically used in monetary economics, affect the frequency fluctuations underlying the state variable of interest (Brock and Durlauf (2004)). The presence of switching modifies the characteristics of design limits of linear frameworks in two main aspects. First, model specification, in terms of both coefficients and transition probabilities, affects the measure of design limits. In general, in presence of high unconditional probability of switching, the variance minimizing rule is associated to important exacerbations of design limitations. On the contrary, when the probability of switching is quite low and the Markov Switching framework is suitable to represent model uncertainty, the robust policy rule smoothes important frequency peaks alleviating frequency specific tradeoffs. Second, design limits are affected by the particular policy rule chosen. It follows that, while in linear cases they are regarded as a constraint of the stabilizing control problem, in regime switching cases they can be considered an externality generated by the policy rule.

JEL Classification Codes: C52; E6

Keywords: Design Limits; Stabilization policy; Regime-switching; Model Uncertainty.

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1 Non technical summary

The paper explores the restrictions on the effects of stabilizing policies on fluctuations from the perspective of the frequency domain. Brock and Durlauf (2004) and Brock, Durlauf and Rondina (2008a) - from now on, respectively, BD and BDR - BDR have defined such restrictions *design limits*. The object of the analysis is the study of some fundamental limitations regarding frequency-specific effects that alternative monetary policy rules may imply.[1] All the existing studies have focused on the linear contexts. In this paper we extend the notion of design limits to a non linear framework which can account for a quite structured form of model uncertainty using Markov Switching ARMA models. Zang and Iglesias (2003) have shown that design limitations can be interpreted by means of information entropy (or Shannon entropy), a notion which has recently received a renewed interest in macroeconomics due to the literature on robustness as a way to deal with model misspecification, rigorously developed in Hansen and Sargent (2008): in contexts of model uncertainty the policymaker, seeking to robustify against model misspecification, minimizes the entropy associated to the economic model. It seems natural then to extend the theory of design limits to contexts which account for an additional source of uncertainty regarding the true economic model.

Design limits are relevant in the decision process of a monetary authority as long as there may be reasons to consider frequency-specific effects produced by the policy rule, as an alternative or in addition to the conventional object of monetary policy, namely, the minimization of the overall variability of the macroeconomic aggregates under control. For instance, it seems plausible to suppose that the central bank is more interested in the business cycle performance of its decisions (medium frequencies) rather than the long run (or low-frequency) effects.

After describing the general procedure, the paper focuses on the simple case of a Markov Switching model composed by two simple AR(1) models and considers the problem of a stabilizing policymaker who cannot observe the true state of the economy. While this is an extremely simplifying case, it is significant because it allows to grasp the main novelties due to the hidden switching between potential models. This non-linearity modifies the characteristics of design limits of linear frameworks in two main aspects. First, contrary to linear cases, design limits depend on the policy rule. It follows that they may be thought of an externality generated by the policymaker's action, rather than a constraint. Second, the probability of switching plays an important role in the determination of the frequency specific restrictions. In general, we observe that the more frequent the switching, the higher are the frequency specific tradeoffs associated with a variance minimizing rule.

The paper is structured as follows. Section 2 introduces the notion of design limits. Section 3 presents the derivation of the analysis in the regime-switching case.

[1] Related contributions and monetary policy applications are Brock et al.(2007), Brock et al. (2008b).

In Section 4 we present simple examples of a Markov Switching AR(1) models in order to shed light on the main novelties of the analysis in regime switching contexts. Section 5 presents a monetary policy application and Section 6 concludes.

2 The theory of Design Limits

Conventional monetary policy's objectives consist of the minimization of the overall unconditional variance of a vector of state variables of interest. Suppose that in a backward looking context, the policymaker wants to stabilize the model

$$x_t = A(L)\, x_{t-1} + B(L) u_t + \varepsilon_t \tag{1}$$

where x_t is a vector of state variables, u_t is the control variables vector, ε_t is the disturbance vector which we suppose, for simplicity, to be a vector of independent and identically normally distributed zero mean shocks with known and constant covariance matrix.[2] The matrices $A(L)$ and $B(L)$ are lag polynomials matrices, where L, the lag operator, is such that $L^a x_t = x_{t-a}$. When $u_t = 0$ we are in the *free dynamics* case: the model evolves independently from the control of the policymaker and the autoregressive part of the system depends only on $A(L)$. For convenience, we label this case *NC*, standing for "no-control", C otherwise.

Suppose for simplicity that x_t is a scalar. There is an important relation between the unconditional variance of a stationary stochastic process, $var(x_t \mid C)$, and its spectral density function $f_{x|C}(\omega)$:

$$var(x_t \mid C) = \int_{-\pi}^{\pi} f_{x|C}(\omega)\, d\omega$$

The area under the spectral density function, $f_{x|C}(\omega)$, defined in the real interval $[-\pi, \pi]$, corresponds to the overall variance of the process. The spectral density and the variance convey the same information about the second order moments of x_t. The frequency domain, however, adds additional information: the area under the spectrum between two frequencies represents the contributions of that frequency interval to the overall variance. The total variability can indeed be considered as the weighted average of the spectral density across the frequencies. Therefore, the frequency domain is relevant as long as the policymaker associates different losses to different frequency ranges. There may be different situations in economics where this may be the case. A central banker can focus primarily on the business cycle frequencies, rather than on the very low frequencies corresponding to cycles longer than ten years. The recent economic turmoil offers a second example, leading financial institutions and policy makers to focus mainly on short term fluctuations, while many wonder about the lower frequency effects of such policies. Further, nonseparable preferences

[2] The disturbance vector, ε_t, can have, in general, its own moving average representation.

for policymakers can lead to different losses for different frequency-specific fluctuations. Examples of this property are found in Otrok (2001) and Otrok, Ravikumar and Whiteman (2002).

Let's start the discussion with a simple example. Let's consider a univariate AR(1) model with $A(L) = 0.5$ and ε_t being a simple white noise with unit variance. The correspondent spectral representation is plotted against the frequency in figure 1 and represented by the solid line: the spectral density function presents a peak at the low frequencies and decreases at the high frequencies.[3] Let's suppose now that $B(L) = 1$ and $u_t = -f x_{t-1}$, where f is set by the policymaker. In order to stabilize the economy, the policy will be set so to kill off all the temporal dependences ($f = 0.5$) and obtain a white noise or a rectangular spectrum (the flat dashed line in figure 1): we are reducing, by construction, the overall variance (the variance of the controlled process is unity, the part of the variation deriving from the shock process that the policymaker cannot control). However the high frequency fluctuations are exacerbated. As in BD, we refer to those effects as *frequency trade-offs.* If we were able to control the process in the way represented by the dashed-dotted line, C^*, in figure 1, then uncertainty about preferences would not be a great deal: no matter frequency specific trade-offs, C^* would constitute an improvement. Unfortunately, in backward looking contexts, C^* is not feasible. This idea has constituted the main content of BD who first introduced the notion of the *Bode's integral* in macroeconomic contexts. In what follows, we briefly explain its technical background[4].

Suppose we consider the scalar version of (1):

$$x_t = A(L)\, x_{t-1} - B(L)\, u_t + \varepsilon_t \tag{2}$$

where x_t is a zero mean, second order stationary process and ε_t is a zero-mean white noise with variance σ_ε^2. The control rule is a typical Taylor type feedback rule

$$u_t = F(L)\, x_{t-1}$$

The optimal policy requires the control removes all the temporal dependences so that x_t is shaped into a white noise, namely, $A(L)x_{t-1} - B(L)F(L)\, x_{t-1} = 0$. In general, however, every non destabilizing control rule, even if not optimal, allows to shape the autoregressive representation of the model into a moving average one. In other words, every solution may be expressed in the form:

$$x_t = D^C(L)\, \varepsilon_t$$

[3] Notice that the domain of the spectral representation is the close interval $[-\pi, \pi]$. The spectral representation is always symmetric with the respect to the frequency 0, so that, alternatively it can be completely defined just in the close interval $[0, \pi]$.

[4] We invite the interested reader to refer to Brock and Durlauf (2004) and Brock, Durlauf and Rondina (2006) for an introduction to the control literature on the Bode's integral applied to economic contexts and for the extension to forward looking environments.

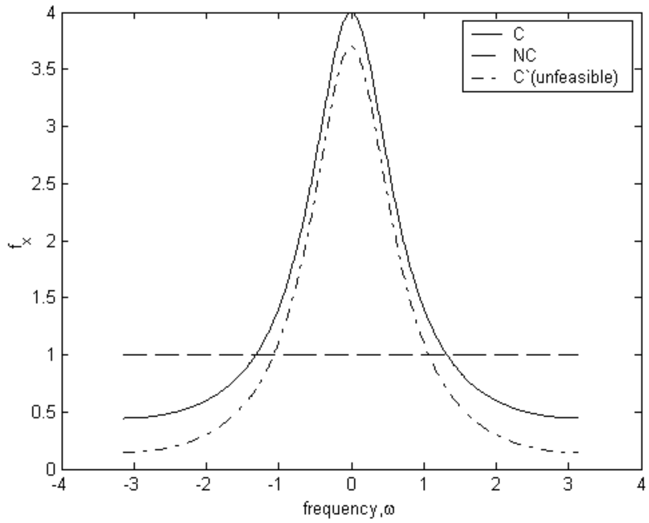

Figure 1: The spectral representation for the model $x_t = 0.5x_{t-1} + \varepsilon_t$ (solid line) with $var(\varepsilon_t) = 1$. The dashed line shows the spectral representation of the optimally controlled process which reduces to a white noise. The dashed-dot line reproduces an unfeasible result.

where $D^C(L)$ is defined the *transfer function.*[5] Equivalently, in the frequency domain, we can state that every control rule shapes the spectral representation of the unconstrained model

$$f_{x|NC} = \frac{\sigma_\varepsilon^2}{2\pi} D^{NC}\left(e^{-i\omega}\right) D^{NC}\left(e^{i\omega}\right)^6 \tag{3}$$

into

$$f_{x|C} = \frac{\sigma_\varepsilon^2}{2\pi} D^{C}\left(e^{-i\omega}\right) D^{C}\left(e^{i\omega}\right) \tag{4}$$

where the frequency ω belongs to the closed interval $[-\pi, \pi]$, $e^{-i\omega}$ is a complex number, $D^{NC}(e^{-i\omega})$ and $D^C(e^{-i\omega})$ represent the *Fourier transforms,* the analogues of the transfer functions in frequency domain.[7] Formulas (3) and (4) can be equivalently expressed

$$f_{x|NC} = \frac{\sigma_\varepsilon^2}{2\pi} \left|D^{NC}\left(e^{-i\omega}\right)\right|^2$$

[5] Given the model $x_t = D^c(L)\varepsilon_t$, the *transfer function* is the mapping from the shock (input), ε_t, to the target vector (output), x_t.

[6] This expression represents the covariance generating function of the process in terms of the coefficients of $D^{NC}(L)$ and the variance of the white noise ε_t (Sargent (1987), page 244).

[7] The interested reader is invited to refer to chapter 13 of Sargent (1987) for a comprehensive introduction to the theory of Fourier transform.

and

$$f_{x|C} = \frac{\sigma_{\varepsilon}^{2}}{2\pi} \left| D^{C} \left(e^{-i\omega} \right) \right|^{2}$$

where $|\cdot|^2$ denotes the complex and conjugate product operator.

We are now ready to define the object of interest of the Bode's integral, called *sensitivity function*:

$$S\left(e^{-i\omega}\right) \triangleq \frac{D^{C}\left(e^{-i\omega}\right)}{D^{NC}\left(e^{-i\omega}\right)}. \tag{5}$$

From (3) and (4) it follows that

$$\left|S\left(e^{-i\omega}\right)\right|^{2} \triangleq \frac{f_{x|C}}{f_{x|NC}} \tag{6}$$

which helps in understanding the role of $S\left(e^{-i\omega}\right)$: it describes how the spectrum of the unconstrained process is shaped into the controlled one. A stabilizer policymaker would always choose the policy rule such that $S\left(e^{-i\omega}\right) = 0$, $\forall\omega$. This is naturally not possible because the realizations of the driving process do not belong to the policymaker's information set. Furthermore, there exists a more stringent feasibility constraint described by the celebrated Bode's integral Theorem[8] that we state after defining the Bode's integral.

Definition 1 *Given a stochastic model, as described by (2), and a feasible, stabilizing rule F, with associated sensitivity function $S\left(e^{-i\omega}\right)$, as defined in (6), the Bode's integral (KB) is defined as follows*

$$KB = \int_{-\pi}^{\pi} \log\left|S\left(e^{-i\omega}\right)\right|^{2} d\omega.$$

Theorem 2 *Let us consider the process (2). If the roots of $A(L)$ – the eigenvalues of the free dynamics of (2) – are stable, then*

$$\int_{-\pi}^{\pi} \log\left|S\left(e^{-i\omega}\right)\right|^{2} d\omega = 0$$

Proof. *See Wu and Jonckheere (1992).* ■

This theorem states that, in backward looking environments, even if the policymaker is able to stabilize the state variable of interest, the variance contributions at some frequencies will necessarily exacerbate since the sensitivity function, $S(\omega)$,[9] cannot be less than one at all the frequencies: in other words, it must be that

$$f_{x|C}(\omega) > f_{x|NC}(\omega) \text{ for some } \omega.$$

[8] The original result is stated in Bode (1945)'s classical monograph.

[9] Trough the paper, we frequently switch from the equivalent notations $S\left(e^{-i\omega}\right)$ and $S(\omega)$. The former is more formal, the latter is more intuitive and synthetic: in covariance stationary cases, the spectral representation and, consequently, the sensitivity function, are always real objects and functions of the frequency, ω.

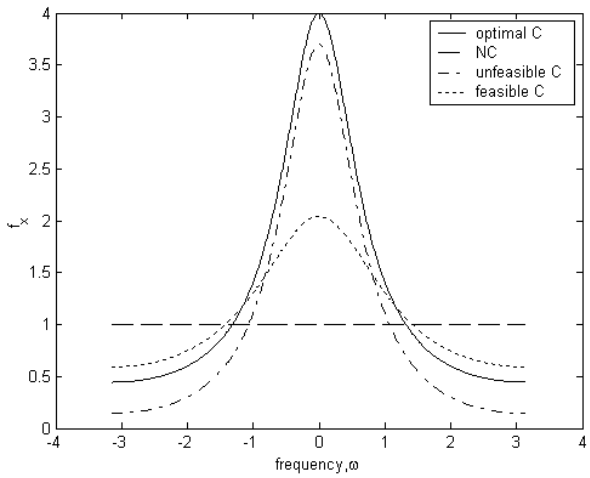

Figure 2: The spectral representation of the process $x_t = 0.5x_{t-1} + \varepsilon_t$ (solid line) and for the optimally controlled process which reduces to a white noise (dashed line). The dotted line represents the process resulting from a feasible control. The dashed-dot line reproduces an unfeasible result.

Notice that, for all the stable free dynamics, the value of the Bode's integral is always zero and independent of the particular model under investigation. Further, it is independent of the policy rule. Figure 2 proposes again the spectral representation of a stable autoregressive process (the solid line) and the spectrum of the process when a variance minimized rule is adopted (the dashed line). As anticipated before, Theorem 2 ensures that the dashed-dotted line constitutes an unfeasible control. Further, it shows that any randomly chosen feasible control (represented, for instance, by the dotted line), even if able to lower the overall variance, produces some frequency trade-offs.

There exists also a second formulation related to the Bode's integral for unstable cases.

Theorem 3 *Let us consider the process (1). If at least one root of $A(L)$ is greater than 1 in absolute value (unstable), then*

$$\int_{-\pi}^{\pi} \log |S(\omega)|^2 \, d\omega = 4\pi \sum_i |\log p_i|$$

where p_i are the unstable roots of $A(L)$.

Proof. *See Wu and Jonckheere (1992).* ■

The economic intuition is straightforward. Stabilization has some costs in terms of performance. In backward looking contexts Theorem 3 implies that the Bode's integral may take either zero or positive values, meaning that frequency-specific trade-offs are unavoidable. This is the reason why the Bode's integral has been termed "the Bode's integral *constraint*" in BD and in Brock and Dulrauf (2005) who provide an example of a formal control problem including the Bode's integral as a constraint. Notice that a positive Bode's integral depends only on the unstable roots of the free dynamics and not on the policy rule.

The time domain computation of the Bode's integral may sound more familiar to economists. It has been proved that the Bode's integral corresponds to the exacerbation of the information-theoretic entropy of the model due to the policy rule (Zang and Iglesias (2003)). In other words, the higher the value of the Bode's integral, the higher the entropy associated to a particular policy intervention. BDR show that the Bode's integral may also take negative values in forward looking models. This is due to agents' expectations which enrich the information set of the policymaker. However, the feasible rules which could, in principle, minimize the variance at all frequencies are, not always, the optimal response.

In this paper, we provide an extension of the Bode's integral in presence of model uncertainty: the policymaker knows that the economy may be represented by different potential models, whose probabilities of occurrence and switching are described by an ergodic and irreducible Markov Chain. We consider a static problem where Markov Chain's realizations are not observed. For a simple interpretation, we analyze a model independent policy rule, but the methodology presented is sufficiently flexible to handle model dependent reactions. The computation procedure relies on the fact that the sensitivity function is derived from the knowledge of the spectral densities as described in (6), rather than from the transfer function, as it is usually done in linear frameworks.

3 Design Limits for MSARMA models

3.1 The Spectral Representation of MSARMA models

Following Pataracchia (2008), we consider a MSARMA(p,q) model of the following type:

$$x_t = \sum_{i=1}^{p} a_i(\xi_t)\, x_{t-i} + \varepsilon_t + \sum_{j=1}^{q} b_j(\xi_t)\, \varepsilon_{t-j} \tag{7}$$

where x_t is a zero mean purely indeterministic process in $\mathbb{R}^K$, $\varepsilon_t \sim WN\ (0,\ \Omega)$, $\xi_t = 1, 2$ is an irreducible, aperiodic and ergodic two states Markov Chain with finite space $\Xi = \{1, 2, ..., d\}$ with stationary transition probabilities denoted by $p_{ij} = pr\left(\xi_t = j \mid \xi_{t-1} = i\right)$ and unconditional (or steady state) probabilities $\pi_i = pr\left(\xi_t = i\right)$,

$1 \leq i \leq d$, where $\sum_{i=1}^{d} \pi_i = 1$.[10] Neither the noise (ε_t) nor the Markov Chain (ξ_t) are observed (the latter is said to be hidden).

We briefly summarize the computation of the spectral representation for (7)[11]. We define the $K(p+q) \times K(p+q)$ matrix

$$\Phi(\xi_t) = \begin{bmatrix} a_1(\xi_t) & & \cdots & & a_p(\xi_t) & b_1(\xi_t) & & \cdots & & b_q(\xi_t) \\ I_K & 0 & \cdots & & 0 & 0 & & \cdots & & 0 \\ 0 & I_K & \cdots & & 0 & 0 & & & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & & \cdots & I_K & 0 & 0 & & \cdots & 0 & 0 \\ 0 & & \cdots & & 0 & 0 & & \cdots & & 0 \\ 0 & & \cdots & & 0 & I_K & 0 & \cdots & & 0 \\ 0 & & \cdots & 0 & 0 & 0 & I_K & \cdots & & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & & \cdots & 0 & 0 & 0 & & \cdots & I_K & 0 \end{bmatrix}$$

The spectral representation of the second order stationary MSARMA model (7) is given by

$$F_x(\omega) = \sum_{\tau=-\infty}^{\infty} (e' \otimes f') P^{*|\tau|} W(0) f e^{-i\omega\tau}$$

where

$$P^* = \begin{bmatrix} p_{11}\Phi(1) & p_{21}\Phi(1) & \cdots & p_{d1}\Phi(d) \\ p_{12}\Phi(2) & p_{22}\Phi(2) & \cdots & p_{d2}\Phi(d) \\ \vdots & \vdots & & \vdots \\ p_{1d}\Phi(d) & p_{2d}\Phi(d) & \cdots & p_{dd}\Phi(d) \end{bmatrix}$$

is a $dK(p+q) \times dK(p+q)$ square matrix, $e = (1, ..., 1)' \in \mathbb{R}^d$ and $f' = (I_K, 0, ..., 0)$ is a $K \times K(p+q)$ matrix. The matrix $W(0)$ is the $dK(p+q) \times K(p+q)$ matrix whose i^{th} block, for $i = 1, .., d$, is given by $\pi_i E(z_t z_t' | \xi_t = i)$, associated with the second order moments of (7). The spectral densities of each element of the state variable vector x_t correspond to the diagonal elements of $F_x(\omega)$.

For example, in the simple univariate case of a MSAR(1) with two potential states $(d = 2)$

$$x_t = a(\xi_t) x_{t-1} + \varepsilon_t, \ var(\varepsilon_t) = \sigma_\varepsilon^2 \tag{8}$$

[10] On the calculation of the ergodic probabilities π_i see Hamilton (2004), page 684. In the simple two-states case, given the Markov Chain $M = \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p22 \end{bmatrix}$, the steady state probabilities are such that $\pi_1 = \frac{(1-p_{22})}{(2-p_{11}-p_{22})}$ and $\pi_1 + \pi_2 = 1$.

[11] See Pataracchia(2008) for details on the computation.

the spectral density function can be written as

$$f_x\left(e^{-i\omega}\right)=\frac{\sigma_\varepsilon^2}{2\pi}\begin{pmatrix} H_1\frac{1}{\left(1+\lambda_1^2-2\lambda_1\cos\omega\right)}+ \\ H_2\frac{1}{\left(1+\lambda_2^2-2\lambda_2\cos\omega\right)} \end{pmatrix} \tag{9}$$

where H_1 and H_2 are functions of $a(i)$ and $p_{ij,}$ $\forall i,j=1,2$.

In order to appreciate the well behaving properties of (9), let's recall the linear framework. Any second order stationary ARMA model with the Wold representation

$$y_t=G(L)\,\eta_t$$

where η_t is a zero mean white noise with known and constant variance σ_η^2, possesses the following spectral representation

$$f_y\left(e^{-i\omega}\right)=\frac{\sigma_\eta^2}{2\pi}\left|G\left(e^{-i\omega}\right)\right|^2 \tag{10}$$

Comparing (9) and (10) we notice that the structure of the spectral density of Markov Switching model (with a model independent shock process) is similar to any other linear stationary ARMA models. Indeed, rewriting (9) as follows

$$f_x\left(e^{-i\omega}\right)=\frac{\sigma_\varepsilon^2}{2\pi}A(\omega) \tag{11}$$

we can state that $A(\omega)$ "plays the role" of the complex and conjugate product in (10).

In what follows we exploit (11) to characterize the design limits of regime switching models. While we recognize that the MSAR(1) is a very simple example which hardly allows a relevant economic application, we think it is important to consider it as a starting point because, while calculations remain tractable, it allows to derive quite general considerations peculiar of regime switching cases.

3.2 Design Limits for MSAR(1)

Let us consider again the following MSAR(1) model:

$$x_t=a(\xi_t)\,x_{t-1}+u_t+\varepsilon_t \tag{12}$$

where $u_t=Fx_{t-1}$ is the control variable and where, for simplicity, $F(L)=F$ so that we remain in the convenient MSAR(1) framework

$$\begin{aligned} x_t &= (a(\xi_t)-F)x_{t-1}+\varepsilon_t= \\ &= a^C(\xi_t)\,x_{t-1}+\varepsilon_t \end{aligned} \tag{13}$$

where $a^C(\xi_t)=a(\xi_t)-F$. In the simple case of $d=2$,we compare:

$$f_{x|NC}\left(e^{-i\omega}\right)=\frac{\sigma_{\varepsilon}^{2}}{2\pi}\left(\begin{array}{c} H_{1}^{NC}\frac{1}{\left(1-\lambda_{1}^{NC}e^{-i\omega}\right)\left(1-\lambda_{1}^{NC}e^{i\omega}\right)}+\\ H_{2}^{NC}\frac{1}{\left(1-\lambda_{2}^{NC}e^{-i\omega}\right)\left(1-\lambda_{2}^{NC}e^{i\omega}\right)}\end{array}\right)$$

with

$$f_{x|C}\left(e^{-i\omega}\right)=\frac{\sigma_{\varepsilon}^{2}}{2\pi}\left(\begin{array}{c} H_{1}^{C}\frac{1}{\left(1-\lambda_{1}^{C}e^{-i\omega}\right)\left(1-\lambda_{1}^{C}e^{i\omega}\right)}+\\ H_{2}^{C}\frac{1}{\left(1-\lambda_{2}^{C}e^{-i\omega}\right)\left(1-\lambda_{2}^{C}e^{i\omega}\right)}\end{array}\right)$$

where C and NC correspond, respectively, to the case in which we suppose the policymaker intervenes and the case in which $u_t = 0$. We are interested in these two cases because, as we showed before, the Bode's integral represents an aggregate measure of design limitations the policymaker must face with respect to the case without control.

In linear time invariant (LTI) frameworks (for instance, in the analysis of model (11)), we are used to consider $f_{y|NC}\left(e^{-i\omega}\right)$ and $f_{y|C}\left(e^{-i\omega}\right)$ in the following forms:

$$f_{y|NC}(\omega)=f_{\eta}(\omega)\mid G^{NC}(\omega)\mid^{2} \tag{14}$$

and

$$f_{y|C}(\omega)=f_{\eta}(\omega)\mid G^{C}(\omega)\mid^{2} \tag{15}$$

where $f_{\eta}(\omega)=\sigma_{\eta}^{2}$ and $\mid G^{NC}(\omega)\mid^{2}$and $\mid G^{C}(\omega)\mid^{2}$ represent, respectively, the complex and conjugate products of the transfer functions of the unconstrained and constrained systems. As described in (5), in linear frameworks, the sensitivity function is usually derived by the knowledge of the transfer functions of the controlled and uncontrolled models. In our case, even if we deal with nonlinear objects, we can still define the complex and conjugate product of the sensitivity function via the knowledge of the spectra, as described in (6), where

$$\begin{aligned}\frac{f_{x|C}(\omega)}{f_{x|NC}(\omega)} &= \left(\begin{array}{c} H_{1}^{C}\frac{1}{\left(1-\lambda_{1}^{C}e^{-i\omega}\right)\left(1-\lambda_{1}^{C}e^{i\omega}\right)}+\\ H_{2}^{C}\frac{1}{\left(1-\lambda_{2}^{C}e^{-i\omega}\right)\left(1-\lambda_{2}^{C}e^{i\omega}\right)}\end{array}\right)\times\\ &\times\left(\begin{array}{c} H_{1}^{NC}\frac{1}{\left(1-\lambda_{1}^{NC}e^{-i\omega}\right)\left(1-\lambda_{1}^{NC}e^{i\omega}\right)}+\\ H_{2}^{NC}\frac{1}{\left(1-\lambda_{2}^{NC}e^{-i\omega}\right)\left(1-\lambda_{2}^{NC}e^{i\omega}\right)}\end{array}\right)^{-1}\end{aligned}$$

The Bode's integral can now be computed

$$KB=\int_{-\pi}^{\pi}\log\mid S(\omega)\mid^{2}d\omega.$$

Taking the log of $\mid S(\omega)\mid^{2}$, we end up with an expression on the form

$$\log\mid S(\omega)\mid^{2}=\log X^{C}(\omega)+\log Y^{NC}(\omega)-\log Y^{C}(\omega)-\log X^{NC}(\omega)$$

where

$$\begin{aligned}
X^{C}(\omega) &= H_1^C\left(1-\lambda_2^C e^{-i\omega}\right)\left(1-\lambda_2^C e^{i\omega}\right)+ \\
&\quad H_2^C\left(1-\lambda_1^C e^{-i\omega}\right)\left(1-\lambda_1^C e^{i\omega}\right) \\
Y^{NC}(\omega) &= \left(1-\lambda_1^{NC} e^{-i\omega}\right)\left(1-\lambda_1^{NC} e^{i\omega}\right)\times \\
&\quad \left(1-\lambda_2^{NC} e^{-i\omega}\right)\left(1-\lambda_2^{NC} e^{i\omega}\right) \\
Y^{C}(\omega) &= \left(1-\lambda_1^C e^{-i\omega}\right)\left(1-\lambda_1^C e^{i\omega}\right)\times \\
&\quad \left(1-\lambda_2^C e^{-i\omega}\right)\left(1-\lambda_2^C e^{i\omega}\right) \\
X^{NC}(\omega) &= H_1^{NC}\left(1-\lambda_2^{NC} e^{-i\omega}\right)\left(1-\lambda_2^{NC} e^{i\omega}\right)+ \\
&\quad H_2^{NC}\left(1-\lambda_1^{NC} e^{-i\omega}\right)\left(1-\lambda_1^{NC} e^{i\omega}\right).
\end{aligned}$$

The Bode's constraint can therefore be rewritten as

$$KB=\int_{-\pi}^{\pi}\log X^{C}(\omega)\,d\omega+\int_{-\pi}^{\pi}\log Y^{NC}(\omega)\,d\omega-\int_{-\pi}^{\pi}\log Y^{C}(\omega)\,d\omega-\int_{-\pi}^{\pi}\log X^{NC}(\omega)\,d\omega.$$

The terms $Y^{NC}(\omega)$ and $Y^{C}(\omega)$ appear familiar because they characterize the transfer functions of LTI models. By simple algebra we can write

$$\log Y^{NC}(\omega)=\log\left(\mid 1-\lambda_1^{NC}e^{-i\omega}\mid^2\mid 1-\lambda_2^{NC}e^{-i\omega}\mid^2\right)=\log\left(\mid e^{i\omega}-\lambda_1^{NC}\mid^2\mid e^{i\omega}-\lambda_2^{NC}\mid^2\right)$$

which allows to immediately apply the Wu and Jonckheere lemma (see Brock et al. (2006) for details).

Lemma 4 $\int_{-\pi}^{\pi}\log\mid e^{i\omega}-r\mid^2 d\omega=0$ *if* $\mid r\mid<1$, $=2\pi\log\mid r\mid^2$ *otherwise.*

Now we can write

$$\int_{-\pi}^{\pi}\log Y^{NC}(\omega)\,d\omega=4\pi\sum_{v_i}\log\mid\lambda_{v_i}^{NC}\mid,\ i\in v_i\ \text{ if }\ \mid\lambda_i^{NC}\mid>1,\forall i=1,2.$$

Similar observations apply to $Y^{C}(\omega)$, so that

$$\begin{aligned}
&\int_{-\pi}^{\pi}\log Y^{NC}(\omega)\,d\omega-\int_{-\pi}^{\pi}\log Y^{C}(\omega)\,d\omega= \\
&4\pi\left(\sum_{v_i}\log\mid\lambda_{v_i}^{NC}\mid-\sum_{r_i}\log\mid\lambda_{r_i}^{C}\mid\right), \qquad (16)\\
i &\in v_i \text{ if }\ \mid\lambda_i^{NC}\mid>1, i\in r_i \text{ if }\ \mid\lambda_i^{C}\mid>1,\forall i=1,2.
\end{aligned}$$

From (16) one is tempted to conclude that one part of the design limits formula resembles the linear framework's result: the Bode's integral is zero in case of a (global) stable free dynamics. Notice, however, that, contrary to BDR, we used the notion of the spectral representation to get the complex and conjugate product of the sensitivity function and the condition of stationarity is necessary to derive the spectral representation.[12] Therefore, the condition of global stationarity must hold in both the uncontrolled and controlled model. This observation does not necessarily rule out interesting comparisons with positive Bode values in the underlying models: instabilities in each AR(1) model are neither a necessary nor a sufficient condition for global instability (Costa et al. (2005)).

We are now left with the terms $X^C(\omega)$ and $X^{NC}(\omega)$. Let's start from the former.

$$\begin{aligned} X^C(\omega) &= K^C\left(1-\lambda_2^C e^{-i\omega}\right)\left(1-\lambda_2^C e^{i\omega}\right)+ \\ & \quad H^C\left(1-\lambda_1^C e^{-i\omega}\right)\left(1-\lambda_1^C e^{i\omega}\right) \end{aligned}$$

which is on the form

$$X^C(\omega) = A^C + B^C\cos\omega$$

so that

$$\int_{-\pi}^{\pi}\log X^C(\omega)\,d\omega = \int_{-\pi}^{\pi}\log\left(A^C + B^C\cos\omega\right)d\omega \tag{17}$$

where

$$\begin{aligned} A^C &= K^C(1+\lambda_2^{C2}) + H^C\left(1+\lambda_1^{C2}\right) \\ B^C &= -2\left(K^C\lambda_2^C + H^C\lambda_1^C\right). \end{aligned} \tag{18}$$

Similarly,

$$\int_{-\pi}^{\pi}\log X^{NC}(\omega)\,d\omega = \int_{-\pi}^{\pi}\log\left(A^{NC} + B^{NC}\cos\omega\right)d\omega \tag{19}$$

From Gradshteyn and Ryzhik (1980)[13]

$$\int_{-\pi}^{\pi}\log\left(a+b\cos\omega\right)d\omega = 2\pi\log\frac{a+\sqrt{a^2-b^2}}{2} \tag{20}$$

so that

$$\int_{-\pi}^{\pi}\log X^C(\omega)\,d\omega - \int_{-\pi}^{\pi}\log X^{NC}(\omega)\,d\omega =$$

$$\int_{-\pi}^{\pi}\log\left(A^C + B^C\cos\omega\right) - \int_{-\pi}^{\pi}\log\left(A^{NC} + B^{NC}\cos\omega\right)d\omega =$$

[12] We invite the reader interested in the discussion on the technical issue of existence of the mathematical objects under exam to BDR, page 6.

[13] See Gradshteyn and Ryshink (1980), page 527.

$$= 2\pi \log \frac{A^C + \sqrt{A^{C2} - B^{C2}}}{A^{NC} + \sqrt{A^{NC2} - B^{NC2}}}$$[14]

The above results can be collected and summarized in the following Bode's integral theorem for regime-switching cases.

Theorem 5 *Given model (15), the value of the Bode' integral corresponds to*

$$\begin{aligned} KB &= 2\pi \log \frac{A^C + \sqrt{A^{C2} - B^{C2}}}{A^{NC} + \sqrt{A^{NC2} - B^{C2}}} + 4\pi \left(\sum_{v_i} \log \mid \lambda_{v_i}^{NC} \mid - \sum_{r_i} \log \mid \lambda_{r_i}^{C} \mid \right) \quad (21) \\ \textit{with } i &\in v_i \textit{ if } \mid \lambda_i^{NC} \mid > 1, i \in r_i \textit{ if } \mid \lambda_i^{C} \mid > 1, \forall i = 1, 2 \end{aligned}$$

where A^{NC}, A^C, B^{NC} and B^{NC} are defined as in (18).

Theorem 5 defines design limits in case of a MSAR(1) model with $d = 2$. Its generic analytic formula depends on the policy rule, on the $a(i)\,s$ and the $p_{ij}s$ with $i, j = 1, 2$. This leads us to notice two important differences with respect to the LTI cases. First, the Bode's integral is *model specific*, and therefore subject to model misspecifications. Second, the presence of the terms A^C and B^C suggests that the value of the Bode's integral can be affected by the policy rule, consequently, one can associate to it the role of an *endogenous* constraint, similar to an externality effect, rather than the exogenous constraint, typical of linear frameworks. This certainly constitutes an additional motivation to consider the analysis of design limits important in any policy evaluation exercise.

4 The dynamics of the Bode's integral

4.1 The Bode's integral across the models

4.1.1 The case of *symmetric* transition probabilities

In this section, we compute the value of the integral in the simple case of MSAR(1). Unless otherwise stated, we consider the variance minimizing rule and a disturbance term with constant and unit variance. We start considering the case of both p_{11} and p_{22} equal to 0.5 (the case of *symmetric* transition probabilities). In Section 4.1.2 we extend the analysis for asymmetric values of the transition probabilities to investigate the effects of the frequency of the switching on the Bode's integral. Finally, we consider different simple policy rules and compare their performances in terms of the variance minimizing rule.

[14]Notice that the explicit solutions of the integral exists only for $A \geq |B|$ and $A^c \geq |B^c|$. This restriction, however, is not binding in any of the simulations presented next.

Figures 3, 4 and 5 present the spectral densities of the global MSAR(1) model together with the one of the two underlying models. The solid line refers to the free dynamics case while the dashed line illustrates the spectral density of the constrained processes when the optimal rule which minimizes the overall variability of the regime-switching model is applied to uncover the effects of the policy on the potential models which may realize. Figure 3 is related to a MSAR(1) with $a(1) = 0.8$ and $a(2) = 0.2$ and figure 4 shows the case of a MSAR(1) model with coefficients $a(1) = -0.8$ and $a(2) = 0.2$. The Bode's integral relative to these exercises is -0.21. The Bode's integral is negative, even if the controlled spectrum presents some frequency trade-offs (stabilization is improved at the low frequencies but exacerbated at the high ones). A negative Bode's integral implies that, even though some frequency trade-offs result due to policy intervention, overall, the frequency-specific variability contributions are reduced in comparison with the free dynamics case, in the sense that, while we are able to reduce the overall variability, the frequency-specific trade-offs overall are diminished. In this particular case we can notice that, even if the low frequency fluctuations appear exacerbated there is an important stabilization at the higher frequencies, where the model presents a quite pronounced peak. The second and third panels show that the high frequency peak comes from the model with autoregressive coefficient of 0.8, which is the more unstable. This example shows that, when the spectral representation of one underlying model presents pronounced peaks, this is going to drive the control rule in such a way that this peak is smoothed at the expense of the frequency performance at other frequencies.

Figure 4 shows similar conclusions. In this case the more pronounced peaks appear at the high frequencies because the negative autoregressive coefficient of one underlying model plays a much more important role than the positive coefficient of the more stable underlying model. The value of the Bode's integral is, again, -0.21. Again, the more unstable model drives the policy rule, which tends to smooth the most important frequency contribution, together with the minimization of the overall variance.

Finally, we present a particular case in which the policymaker is indifferent between intervening or not. Figure 5 shows that, in the symmetric probabilities case, when the two underlying models have opposite autoregressive coefficients, the perceived Markov Switching model has a rectangular spectrum, so that the optimal reaction is no intervention. The value of the Bode's integral is, by definition, zero.

In order to have a much more complete picture of the analysis, next we plot the value of the Bode's integral against the autoregressive coefficient of one underlying model, keeping the other model's autoregressive coefficient fixed. Figures 6 and 7 present the case of the autoregressive coefficient of underlying model fixed at, respectively, 0.8 and 0.5.

Several considerations can be advanced. First, the Bode's integral is always negative or equal to zero. This is, at first glance, counterintuitive: the assumption of the switching regimes introduces an additional source of uncertainty and it is reasonable

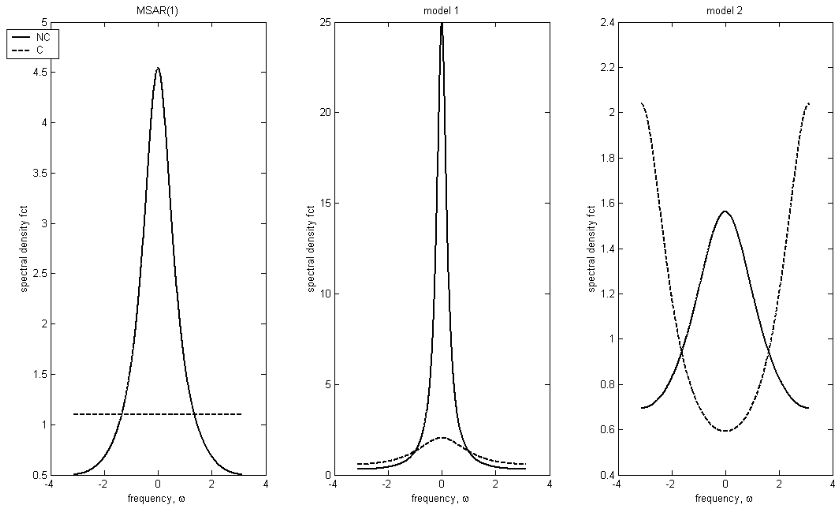

Figure 3: The spectral representation of the MSAR(1) model with coefficients $a\,(1) = 0.8$ and $a\,(2) = 0.2$ (first panel). The second and third panel represent the spectral representation of the underlying models. The solid line denotes the non control case, the dashed line denotes the control case.

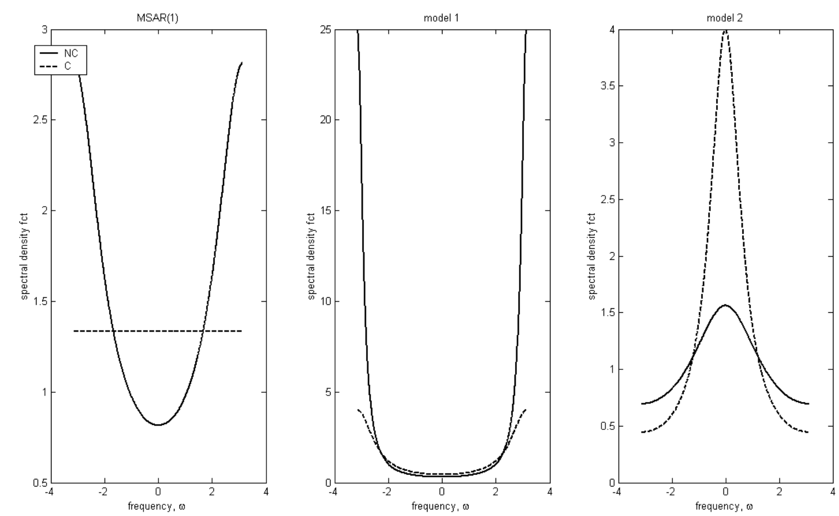

Figure 4: MSAR(1) model with coefficients $a\,(1) = -0.8$ and $a\,(2) = 0.2$ (first panel). The second and third panel represent the spectral representation of the underlying models. The solid line denotes the non control case, the dashed line denotes the control case.

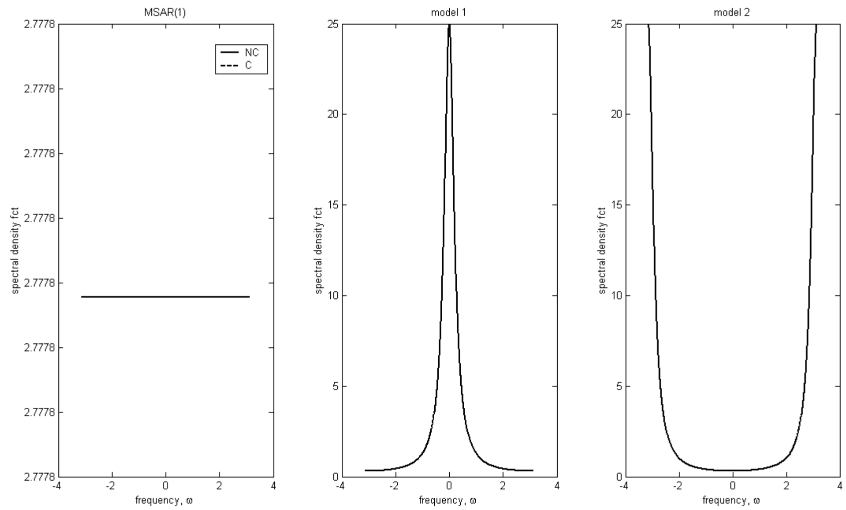

Figure 5: MSAR(1) model with coefficients $a\,(1) = 0.8$ and $a\,(2) = -0.8$ (first panel). The second and third panel represent the spectral representation of the underlying models. The solid line denotes the non control case, the dashed line denotes the control case.

to expect that this may cause an exacerbation to design limits. Second, the values of the Bode's integral are symmetric around $a\,(2) = 0$.

The Bode's integral is null in two cases: one is the case in which no intervention is the optimal reaction so that the Bode's integral is zero by definition. The second case is the linear case, where the two models coincide. We saw above that for stable linear models the value of the Bode's integral is always zero. These figures make clear that the analysis of the regime switching cases can be considered as a generalization of the linear case, where the two underlying models coincide. As we already discussed above, when one model presents important frequency peaks, like the model with autoregressive coefficient equal to 0.8, the policy rule is very biased towards this model such that the peak is smoothed. This property allows negative values of the Bode's integral. If we interpret the presence of different potential models as a way to model "model uncertainty", we can consider the policy rule a robust policy rule: it will not be optimal with respect to any model, but it will bring satisfactory performance in all possible realizations. This usually implies that the policy rule will be biased towards the worst performing model so to ensure that no big losses will be produced in the case in which this model realizes. Hansen and Sargent (2008) describe the frequency specific effects of the robust feedback rules in frequency domain. They emphasize that, while an optimal rule targets the minimization of the overall variance, that is the area beneath the spectrum, a robust rule will perform such that the most relevant frequency density peaks will be lowered, as if concerns about robustness in frequency

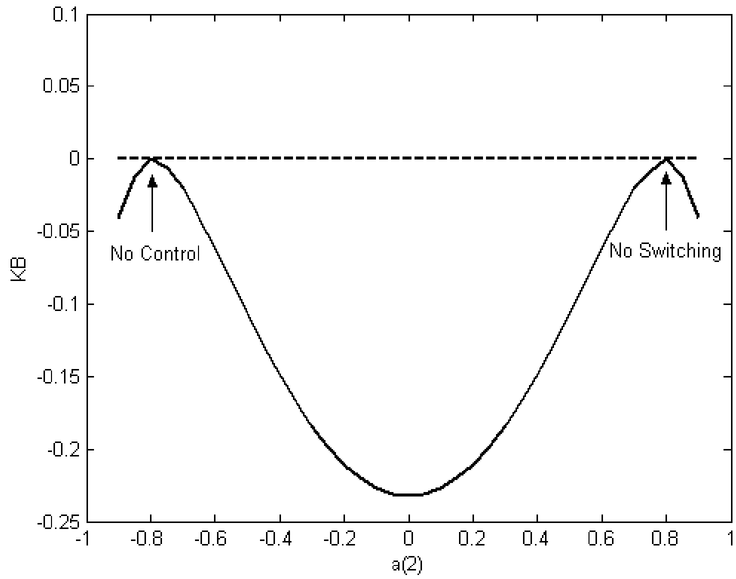

Figure 6: The dynamics of the Bode's integral constraint with $a(1) = 0.8$, $p_{11} = 0.5$ and $p_{22} = 0.5$.

domain can be interpreted as risk aversion across the frequencies. This property implies less severe frequency tradeoffs, lowering, therefore, the value of the Bode's integral.

4.1.2 The Bode's integral across the probabilities of switching

Next, we abandon the symmetry of the transition probabilities to explore the effect of the frequency of the switching on the Bode's integral value. We consider two special cases, setting the transition probabilities $p_{11} = p_{22}$ equal to 0.8 and 0.3.

Figures 8 and 9 represent, as above, the dynamics of the Bode's integral for different models. The probabilities of switching place an important qualitative role in the analysis of the frequency specific effects. When the frequency of the switching is sustained the value of the Bode's integral is always negative and its dynamics appears completely reverted: the non control case and the linear case constitute now a lower bound for the measure of the frequency specific tradeoffs. Figure **??** helps grasping the intuition of the result for this simple first order regime switching case. It reproduces the spectral densities of the regime switching model for different transition probabilities.

When one model is very unstable compared to the other, then the variance minimizing rule is very effective in reducing the pronounced peak and this constitutes

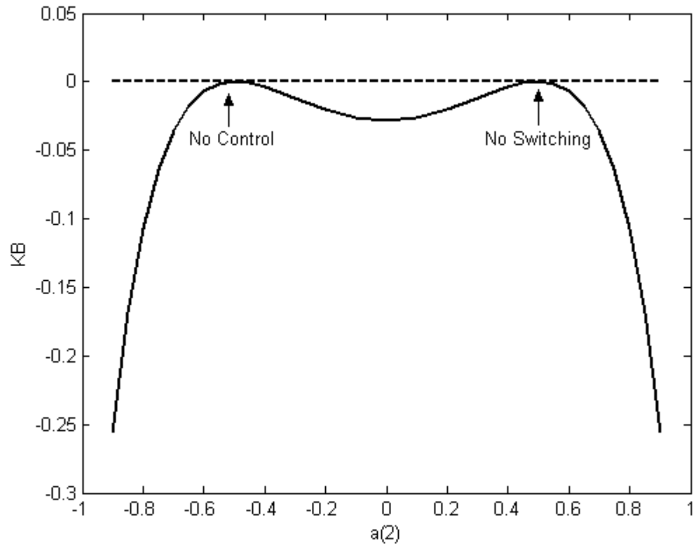

Figure 7: The dynamics of the Bode's integral constraint with $a\,(1) = 0.5$, $p_{11} = 0.5$ and $p_{22} = 0.5$.

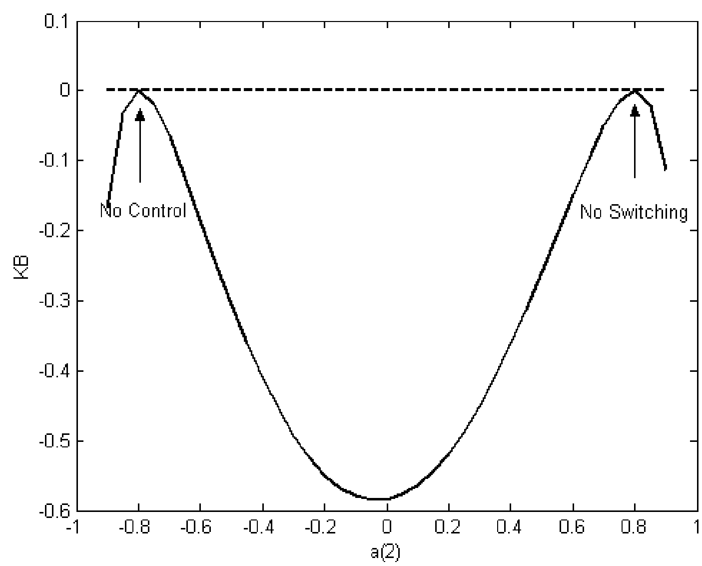

Figure 8: The dynamics of the Bode's integral constraint with $a\,(1) = 0.8$, $p_{11} = 0.8$ and $p_{22} = 0.8$.

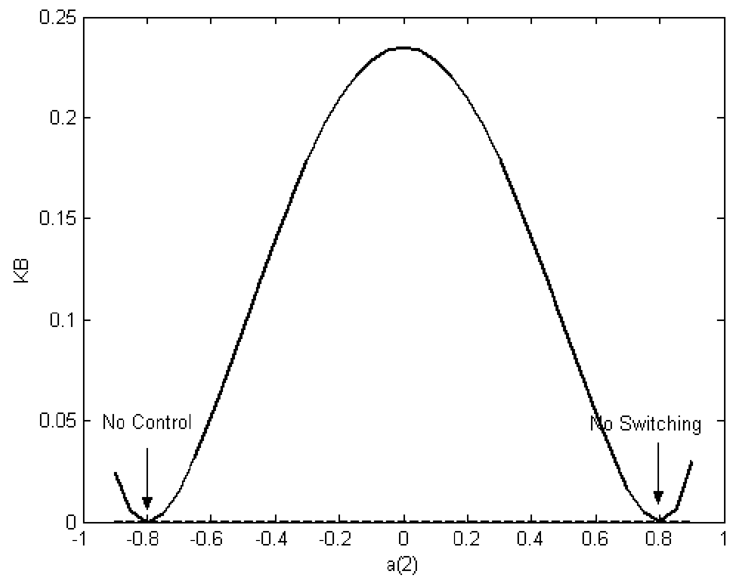

Figure 9: The dynamics of the Bode's integral constraint with $a\,(1) = 0.8$, $p_{11} = 0.3$ and $p_{22} = 0.3$.

an improvement in the frequency specific performance. This improvement is much more evident when the probability of switching is low. In this case the peak of the unstable model is very pronounced since the MSAR(1) model is closer, in terms of frequency specific characteristics, to the more unstable model. A higher probability of switching makes the MSAR(1) less close to both underlying models and therefore the peak at the low frequencies is also less pronounced. In other words, a very high probability of switching modifies completely the distribution of the volatility at the different frequencies and this impedes the state independent feedback rule of the type assumed in this framework to be effective in reducing frequency specific peaks. Here, however, we are mostly conceiving the regime switching case as a convenient way to represent model uncertainty. Typically macroeconomic studies which account for model uncertainty with Markov Switching models, consider a very low probability of switching since phases of growth or recessions have typically medium or low frequencies characteristics.

4.2 The Bode's integral across the policy rules

We have already noticed that the policy rule not only shapes the spectral characteristics of the model, but it does so in a way which affects also the measure of frequency tradeoffs. Figure 10 shows the dynamics of the Bode's integral across several models with $a\,(1) = 0.8$, $p_{11} = 0.2$, and $p_{22} = 0.8$, and compares different policies along

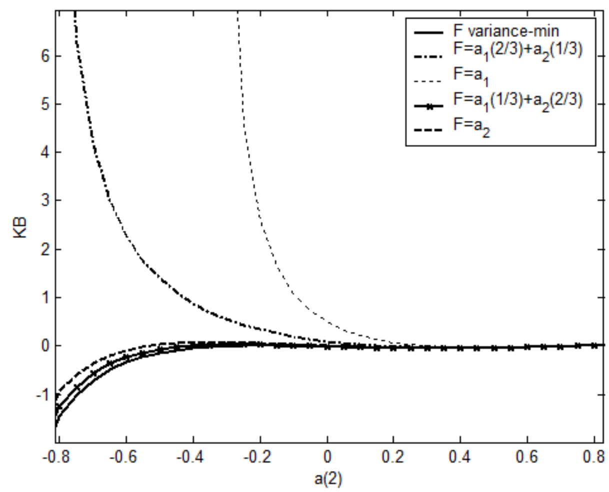

Figure 10: The dynamics of the Bode's integral constraint for different policies. $[a(1) = 0.8, p_{11} = 0.2, p_{22} = 0.8]$.

with the variance minimizing rule. In absence of switching ($a(1) = a(2) = 0.8$), the dynamics of the Bode' s integral converges to zero (we are back to the linear framework). As we move leftward the dynamics of design limits varies substantially according to the policy rule. Notice that the variance minimizing rule (the solid line case) always dominates all the other proposed rules while the farther is the policy rule from the optimal one, the worse the performance in terms of design limits. In general cases, however, the statement that the variance minimizing rule always produces the smallest amount of design limits does not always hold, as figure **??** shows. It depicts the measure of the Bode's constraint with the same set of rules considered before but in a context of a high probability of switching.

When the probability of switching is high, the variance minimizing rule is associated with the highest level of fundamental limitations of the design.

To conclude, we state that proposing the minimization of design limits through the minimum Bode's integral as an object is certainly an argument too strong. However, we want to emphasize that conventional monetary policy calculations, based on the overall variance minimization, may hide important exacerbations of frequency specific performance of the policy rule.

5 Application: monetary policy rules

In order to understand frequency-specific tradeoffs, we follow BDR and compute the frequency-specific losses that are implicit in the tradeoffs associated with the variance-based Phillips curve in the backward looking regime switching model estimated by

Svensson and Williams (2005). We compute the parameters for the simple interest rate rule

$$i_t = g_\pi \pi_{t-1} + g_y y_{t-1}$$

to minimize the loss function

$$L = \lambda var(\pi_t) + (1 - \lambda) var(y_t)$$

where λ is a real number. Varying λ between 0 and 1, one traces out the efficient frontier of inflation/output variance pairs from which a policymaker may choose. For each point on the frontier we decompose the variance values between low frequencies (cycles of 8 years or more), business cycle frequencies (cycles of 2 to 8 years), and high frequencies (cycles of less than 2 years) following the NBER classification of minor and major business cycle. The decomposition is represented in figure 11.[15] The squares and the diamond in the figure correspond to the case of a policymaker who distastes output variance over inflation variance (*dove*), so that $\lambda = 0.05$, and a policymaker which possesses a relative higher distaste for inflation variance (*hawk*), so that $\lambda = 0.95$, respectively.

Similarly to the related analysis in linear frameworks presented in BDR, the negative sloping frontier of the overall variance features are also reproduced for the low frequencies decomposition. Tradeoffs are very different for business cycle frequencies where the frontier has a positive slope and for the high frequencies in which case the frontier has a negative slope but with a reversed direction. Contrary to linear cases, we notice that the inflation volatility contribution due to the high frequencies is much more important than the one due to the business cycle frequencies. The almost vertical shape of the high frequencies frontier is also peculiar of the switching case. In this case, a strong distaste for inflation volatility brings an exacerbation of both inflation and output variance contributions and an even more important exacerbation of inflation volatility at the higher frequencies.

This simple exercise shows how unpacking the variance tradeoffs in different frequencies intervals may reveal unexpected frequency tradeoffs that a policymaker with frequency specific losses may want to take into account.

6 Conclusion

In this paper we extend the theory of design limits in regime-switching contexts, deriving the analogue of the Bode's integral value, which has been recently introduced in macroeconomic studies by Brock and Durlauf (2004). The Bode's integral value quantifies the amount of the frequency trade-offs (design limits) the controller (typically, a central bank) faces in the process of stabilization. Positive values of the

[15] The analysis is the regime-switching analogue of the trade-off frontier analysis of linear backward looking contexts with costless control proposed in BDR.

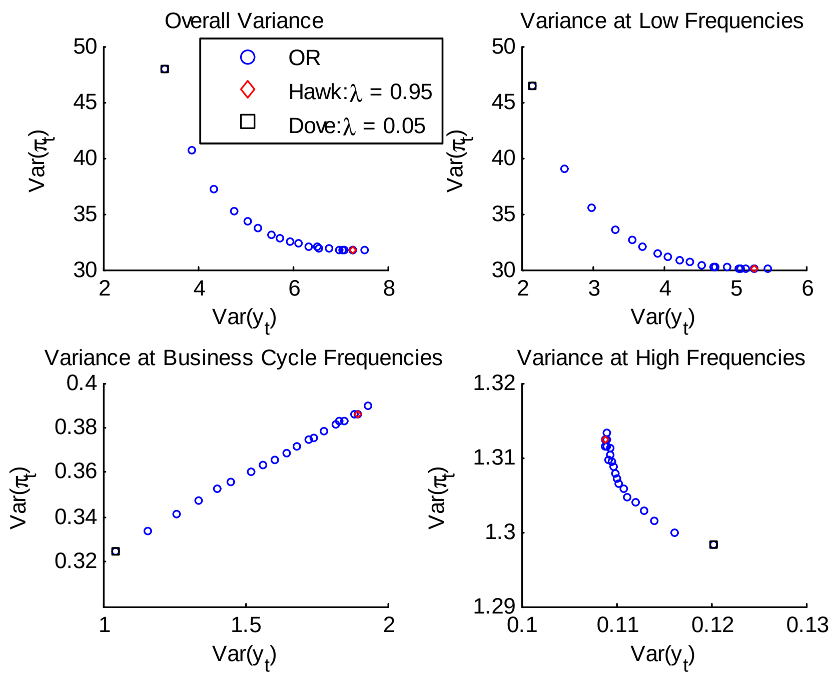

Figure 11: Tradeoff Frontiers: Aspects of inflation and output processes that correspond to the minimization of the loss function as λ is varied between 0 and 1. The coefficients are derived using a grid search over the space $g_{\pi,y} \in [0.0, 10.0]$.

Bode's integral imply necessary exacerbations of the contribution of some frequency ranges to the total fluctuations of the model. Negative values do not imply the absence of frequency trade-offs, but denote less stringent limits. While we describe the general procedure, we show explicitly the computation for the simple case of a Markov Switching model composed by two simple AR(1) models and consider the problem of a stabilizing policymaker who cannot observe the realizations of the Markov Chain process so to grasp the main novelties of the analysis due to the switching between models.

We show that the analysis of the design limits in regime switching cases can be viewed as a generalization of the linear case where the potential models coincide. Some main features, peculiar to regime-switching cases, are revealed. First, the behavior of the Bode's integral strongly depends on the particular model considered. Second, different policy rules shape the spectral density of the process under examination in different ways, affecting the measure of design limits. Therefore, while in linear cases the Bode's integral is included as constraint in control problems (Brock and Dulrauf (2005)), we propose to consider it as an externality which a policymaker, who associates different losses to different frequency ranges, may want to include in his analysis.

The transition probabilities play an important role in the analysis. When the unconditional probability of switching is low, such that the mean duration of the potential states is high, the variance-minimizing policy is associated to less stringent frequency trade-offs and the value of the Bode's integral is typically negative. The conclusion is different in the presence of a high unconditional probability of switching between the models. In these cases, the stabilization has a price in terms of frequency specific performance. The values of the Bode's integral are typically positive. In the former case, the Markov Switching framework is suitable to represent model uncertainty and, similarly to the analysis of robust feedback rules in frequency domain (see Hansen and Sargent (2008) and Tetlow and von zur Muehlen(2001)), the policy rules which account for model uncertainty targets the pronounced peaks of the spectral density of the model rather than the area beneath it, which corresponds to the overall variance. This effect alleviates the frequency specific tradeoffs.

The second peculiar characteristic of the Bode's integral value in regime switching models is represented by its dependence on the policy rule. In this sense, the frequency-specific performance plays the role of an externality of the policymaker's action. This observation can open the way to a reconsideration of the Bode's integral and a possible further extension in which the minimization of the frequency tradeoffs may be associated a relative weight in the target vector of a monetary economic analysis.

The examples shown make clear that general conclusions cannot be derived and that each particular case should be evaluated separately in order to derive policy relevant considerations. This study does not have the conceit to propose the minimization of the Bode's integral values as a pure object. However, we regard the communication of the frequency specific effects of any policy rule, including the knowledge of design limits, as an important practice which should contribute to a more complete monetary policy evaluation analysis.

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