UNIVERSITÀ DEGLI STUDI DI SIENA

QUADERNI DEL DIPARTIMENTO
DI ECONOMIA POLITICA

**Stefano Vannucci**

# Choosing VNM-stable sets of the revealed dominance digraph

**n. 576 – Settembre 2009**
Revised April 2010

**Abstract -** The choice functions that are consistent with selections of VNM-stable sets of an underlying revealed dominance digraph are characterized both under VNM-perfection of the latter and in the general case.

**JEL classification**: C70, C71, D01, D03
**Keywords**: VNM-stable sets, kernel-perfect digraphs, choice functions

**Stefano Vannucci**, Dipartimento di Economia Politica, Università di Siena – vannucci@unisi.it

# 1 Introduction

A Von Neumann-Morgenstern stable set is meant to model a *standard of behaviour* within an ongoing social interaction: it consists of a set of alternative outcomes that do not occasion serious objections against each other, while providing strong objections against any other available alternative. The use of Von Neumann-Morgenstern stable sets (henceforth VNM-stable sets) as a solution concept for coalitional games has a long tradition dating back to *Theory of Games and Economic Behavior* (Von Neumann and Morgenstern (1953)). Since then, VNM-stable sets and kernels of irreflexive simple digraphs, their dual graph-theoretic counterpart, have been the main focus of a substantial flow of literature (see e.g. Harary, Norman and Cartwright (1965) for an early, classic graph-theoretic treatment, Schmidt and Ströhlein (1985) and Ghoshal, Laskar and Pillone(1998) for comprehensive surveys covering the game-theoretic and the graph-theoretic literature, and Greenberg (1990) for a highly stimulating if controversial attempt at using VNM-stable sets as an unifying solution scheme for both cooperative and noncooperative game theory).

Now, consider the interaction underlying the ongoing operation of an organization, a team or indeed any suitably complex decision-making system. Clearly enough, if the set of available options happens to change at a faster pace than the behavioural attitudes of the relevant players *and* the latter interact as predicted by VNM-stable sets, then the corresponding choice behaviour of the system should be related in certain specific ways to the *dominance digraph* (to be defined below) representing the underlying interaction process, and the resulting choice function should somehow *reveal* that fact. But then, what are the characteristic features of such a choice function, hence the *testable* behavioural predictions of VNM-stable sets as a solution concept? Or to put it more succinctly, *which choice functions may be regarded as revealed VNM-solutions*?

To the best of the author's knowledge, such an issue has never been addressed *in its full generality.* To be sure, there exist at least *two* remarkable contributions to that topic, namely Wilson (1970) and Suzumura (1983), that consider a class of revealed VNM-solutions in the wider setting of *partial* choice functions, i.e. choice functions as possibly defined on just *some* subsets of the 'universal' outcome set. In particular, the seminal Wilson(1970) -which is by the way mainly focused on rationalizability by total preorders and strict partial orders- is to be credited for pointing out the pivotal role of the *revealed conjugation relation* (to be defined below) when it comes to identifying VNM-solutions. However, Wilson (1970) is only concerned with *proper* i.e. *nonempty-valued* partial choice functions. Thus, it only provides a characterization of *nonempty-valued* revealed VNM-solutions. Moreover, its characterization of such *subclass* of revealed VNM-solutions does not refer to VNM-stable sets of a dominance digraph, but rather to the cryptomorphic notion of (a suitably defined type of) fixed point solutions of an *undominance* or reflexive digraph, and relies on a *single* condition which makes explicit reference to revealed conjugation relations. Thus, such a condition is in fact rather cumbersome and scarcely amenable to comparisons with other, simpler properties of choice functions as currently used in related characterization problems and results. For instance, Wilson's condition and related characterization makes it scarcely possible to single out the peculiar combination of contraction-consistency and expansion-consistency properties of choice sets which is characteristic of VNM-solutions as opposed to, say, core-solutions or total-preorder-maximizing choice functions.

Relying on Wilson's work, Suzumura (1983) offers a similar characterization of an even *smaller* subclass of (nonempty-valued) revealed VNM-solutions by means of a single condition. Indeed, Suzumura *does* proceed to unpack such a condition via a set of simpler and more familiar properties, to the effect of producing a further, and much more transparent characterization. But of course, as just mentioned, Suzumura's result only concerns a *very specialized* class of *nonempty-valued* VNM-solutions, namely those *proper* choice functions that may be regarded as both VNM-solutions *and* core-solutions of a *suitably defined, asymmetric* revealed dominance digraph.

Now, it is of course well-known that there exist nontrivial dominance digraphs with induced subdigraphs having *no VNM-stable sets,* hence generating *locally empty-valued* VNM-solutions. Thus, the Wilson-Suzumura theorems mentioned above concern anyway just some *proper subclasses* of the entire collection of revealed VNM-solutions of an *arbitrary (* namely, *irreflexive) dominance digraph.*

Of course, that paucity and incompleteness of contributions on revealed VNM-solutions is to be contrasted with the impressive body of literature on 'revealed preference' i.e. essentially on revealed maximization in different domains (see e.g. among many others Sen (1971,1977), Suzumura (1983), Moulin (1985), Aizerman and Aleskerov (1995), Danilov and Koshevoy (2009)). Moreover, there exists some work on several revealed (*nonempty-valued*) tournament solutions and their *general* characterizations (see e.g.

the seminal Moulin (1986), and Laslier (1997), Ehlers and Sprumont (2008), Lombardi (2008, 2009)), and on characterizations of revealed noncooperative equilibrium behaviour for some wide classes of games admitting equilibria (e.g. Yanovskaya (1980), Ray and Zhou (2001), Galambos (2005)). Clearly, this state of affairs may be partly explained by the traditional emphasis on domains that ensure *nonempty* sets of solutions of the required type. Nevertheless, it is striking that a general characterization of revealed VNM-solutions is still missing, while the only known characterization of nonempty-valued VNM-solutions due to Wilson(1970) as discussed above is arguably somewhat opaque.

The present paper is aimed at filling this gap in the literature by addressing the *general* VNM-solution revelation problem as formulated above both for proper and for possibly empty-valued choice functions. The first task to accomplish is to devise a simple, transparent characterization of the class of all nonempty-valued VNM-solutions, based upon pure choice-theoretic properties with no direct reference to 'revealed' binary relations (along the lines of Suzumura's results about the specialized subclass of nonempty-valued VNM–and-core-solutions as mentioned above). The second objective is to characterize at last *the entire collection of VNM-solutions*. Accordingly, two characterization results are provided: the first one concerns revealed VNM-solutions under VNM-perfection of the revealed dominance digraph (i.e. existence of VNM-stable sets at any subset) hence refers to *proper* choice functions, offering an useful, more transparent alternative to Wilson's own characterization, while the second one covers the case of *arbitrary*, possibly empty-valued, choice functions. In particular, a characterization of nonempty-valued VNM-solutions based upon two new, simple pure choice-theoretic properties is offered. Moreover, it turns out that two very mild-looking local nonemptiness conditions for choice sets and a somewhat more ad hoc 'boundary' property forbidding the existence of certain VNM-stable sets are all that is needed in order to lift that characterization to the more general case of arbitrary VNM-solutions. From a more substantive perspective, and apart from the latter 'boundary' property, both of those characterizations rely on a combination of nonemptiness requirements for choice sets and of two weakened versions of the Chernoff contraction-consistency property (also known in the relevant literature as 'heritage', 'heredity', or '$\alpha$ contraction-consistency') as applied to choice sets *and* rejection sets, respectively.

Finally, the foregoing characterizations enable a global study of the corresponding posets of revealed VNM-solutions: the basic properties of such posets are also briefly analysed.

The paper is organized as follows: section 2 is devoted to a presentation of the model, of the two characterization results; section 3 presents the basic properties of the posets of revealed VNM-solutions and revealed nonempty-valued VNM-solutions, respectively, as mentioned above; section 4 comprises the main characterization results; section 5 is devoted to a study of the basic order-theoretic structure of the set of revealed VNM-solutions; section 6 includes a discussion of the related literature; section 7 consists of a few concluding remarks. An Appendix details a coalitional (strategic) representation for irreflexive (asymmetric) simple digraphs in order to illustrate the full game-theoretic import of the results presented in this paper.

# 2 Choice functions and revealed VNM-solutions: basic definitions and examples

Let $X$ be a set denoting the 'universal' outcome set, with cardinality $\#X \geq 3$, and $\mathcal{P}(X)$ its power set. It is also assumed for the sake of convenience that $X$ is *finite* (but it should be remarked that the bulk of the ensuing analysis is easily lifted to the case of an infinite outcome set). A *choice function on* $X$ is a *deflationary* operator on $\mathcal{P}(X)$ i.e. a function $c : \mathcal{P}(X) \to \mathcal{P}(X)$ such that $c(A) \subseteq A$ for any $A \subseteq X$ (empty choice sets are allowed, but we only focus on choice functions that are defined on the 'full' domain $\mathcal{P}(X)$; a set $A \subseteq X$ is also referred to as an agenda or menu, and is said to be rejected by $c$ if $c(A) = \varnothing$, and accepted otherwise). A choice function $c$ is *proper* (or nonempty-valued) if $c(A) \neq \varnothing$ whenever $\varnothing \neq A \subseteq X$, and *single-valued* if $\#c(A) = 1$ for any nonempty $A \subseteq X$. We denote by $C_X$ the set of all choice functions on $X$, by $C_X^{\circ}$ the set of all *proper* choice functions on $X$, and by $C_X^1$ the set of all *single-valued* choice functions on $X$: clearly, by definition, $C_X^1 \subset C_X^{\circ} \subset C_X$. Notice that any $c \in C_X$ fully determines its *dual choice* or *rejection function* $c^c$ by the rule: for any $A \subseteq X$, $c^c(A) = A \setminus c(A)$; clearly, $c^c \in C_X$ as well, and $c = (c^c)^c$. The latter fact licenses a dual interpretation of choice functions as representations of a discriminating behaviour resulting either in *selection* or in *rejection* of the outcomes included in a choice set: in the ensuing analysis, we shall allow and exploit such a dual view of choice functions. Nevertheless, when it comes to informal description, I shall indulge sometimes for the sake of convenience in the common interpretation of chosen (unchosen) outcomes as selected (rejected) outcomes.

For any binary relation $R \subseteq X \times X$, $R^a$ and $R^s$ denote the asymmetric and symmetric components of $R$, respectively, $R^{-1}$denotes the inverse of $R$ (namely $(x, y) \in R^{-1}$ iff $(y, x) \in R$), while $\Delta^R \subseteq X \times X$ denotes the binary relation defined by the following rule: $x\Delta^R y$ iff *not* $yRx$ (or equivalently *not* $xR^{-1}y$). Moreover, an $\forall$-*type fixed point* of $R$ in $\mathcal{P}(X)$ is a set $Y = \{x \in X : xRy \text{ for all } y \in Y\}$, and $\mathcal{F}^{\forall}(X, R)$ denotes the set of all $\forall$-*type fixed points* of $R$ in $\mathcal{P}(X)$.

Let $\Delta \subseteq X \times X$ be an *irreflexive* binary relation on $X$ i.e. such that *not* $x\Delta x$ for all $x \in X$, denoting a suitably defined *dominance* relation: $(X, \Delta)$ is the corresponding *dominance digraph* (in graph-theoretic parlance, $(X, \Delta)$ is in particular a *simple, loopless digraph* i.e. a directed graph with at most one arc between any ordered pair of vertices, and with no arc from a vertex to itself). Observe that, by definition, $(X, \Delta)$ *is a dominance digraph iff both its inverse* $(X, \Delta^{-1})$ *and its symmetric closure* $(X, \Delta \cup \Delta^{-1})$ *are also dominance digraphs.* In particular, $(X, \Delta)$ is an *asymmetric dominance digraph* if $\Delta = \Delta^a$ i.e. $x\Delta y$ entails *not* $y\Delta x$, for any $x, y \in X$. Similarly, a digraph $(X, \Delta)$ is *reflexive* (respectively, *total*) if $\Delta$ is reflexive i.e. $x\Delta x$ for all $x \in X$ (respectively, total i.e. $x\Delta y$ or $y\Delta x$ for any $x, y \in X$).

For any $Y \subseteq X$, $\Delta_Y = \Delta \cap (Y \times Y)$ denotes the dominance relation induced by $\Delta$ on $Y$ (of course $\Delta_X = \Delta$), and $(Y, \Delta_Y)$ is the induced *dominance subdigraph* on $Y$. A *VNM-stable set* of $(Y, \Delta_Y)$ is a set $S \subseteq Y$ that satisfies both *internal stability* (i.e. *not* $x\Delta_Y y$ for any $x, y \in S$) and *external stability* (i.e. for any $y \in Y \setminus S$ there exists $x \in S$ such that $x\Delta_Y y$). The set of all VNM-stable sets of $(Y, \Delta_Y)$ is denoted $\mathcal{S}(Y, \Delta_Y)$. A dominance digraph $(X, \Delta)$ is said to be *VNM-perfect* if $\mathcal{S}(Y, \Delta_Y) \neq \varnothing$ for any $Y \subseteq X$. The *core* of $(Y, \Delta_Y)$, denoted $\mathbb{C}(Y, \Delta_Y)$, is the set of all $y \in Y$ such that *not* $z\Delta y$ for each $z \in Y$.

**Remark 1** *It should be emphasized here that only irreflexive digraphs admit a natural representation as dominance relations of an underlying game in coalitional or strategic form, hence the suggested definition of domination digraphs. Moreover,* any *dominance digraph as defined here may arise in a natural way from an underlying game in coalitional form* and *from a related game in strategic form (see the Appendix for the relevant details).*

The *(asymmetric) basic revealed dominance digraph* $(X, \Delta(c))$ of a choice function $c \in C_X$ is defined by the following rule: for any $x, y \in X$, $x\Delta(c)y$

if and only if $x \neq y$ and $c(\{x,y\}) = \{x\}$. Similarly, the *revealed dominance digraph* $(X, \Delta^*(c))$ of a choice function $c \in C_X$ is defined as follows: for any $x, y \in X$, $x\Delta^*(c)y$ if and only if there exists $A \subseteq X$ such that $x \in c(A)$ and $y \notin c(A)$. Clearly enough, $\Delta(c)$ is *asymmetric* hence in particular *irreflexive* by definition, $\Delta^*(c)$ is *irreflexive*, and $\Delta(c) \subseteq \Delta^*(c)$.

We shall also make some use of three related digraphs attached to a choice function $c \in C_X$, namely the *basic revealed digraph* $(X, R_c))$ of $c$, the *extended revealed digraph* $(X, R(c))$ of $c$, and the *revealed choice-conjugation digraph* $(X, \widehat{R}(c))$ of $c$, defined as follows: for any $x, y \in X$, $xR_c y$ if and only if $x \in c(\{x,y\})$, $xR(c)y$ if and only if there exists $S \subseteq X$ such that $x \in c(S)$ and $y \in S$, $x\widehat{R}(c)y$ if and only if there exists $S \subseteq X$ such that $\{x,y\} \subseteq c(S)$.

A choice function $c \in C_X$ is a *revealed VNM-solution (or VNMS-rationalizable)* if there exists an *irreflexive* relation $\Delta \subseteq X \times X$ such that for any *nonempty* $Y \subseteq X$, $c(Y) \in \mathcal{S}(Y, \Delta_Y)$ if $\mathcal{S}(Y, \Delta_Y) \neq \varnothing$ and $c(Y) = \varnothing$ if $\mathcal{S}(Y, \Delta_Y) = \varnothing$ (observe that in particular $c(\varnothing) = \varnothing$ and $\mathcal{S}(\varnothing, \Delta_\varnothing) = \{\varnothing\}$ whence $c(\varnothing) \in \mathcal{S}(\varnothing, \Delta_\varnothing)$ anyway, and that -by external stability- for any *nonempty* $A \subseteq X$ if $B \in \mathcal{S}(A, \Delta_A)$ then $B \neq \varnothing$). Then, we also say that $c$ is *VNMS-rationalizable by dominance digraph* $(X, \Delta)$. Moreover, $c \in C_X$ is said to be a *fixed point solution of* $\forall$*-type* for $(X, R)$ -where $R \subseteq X \times X$- if $c(A) \in \mathcal{F}^{\forall}(Y, R_Y)$ for any $Y \subseteq X$ (and a *fixed point solution of* $\forall$*-type* if there exists an $R$ such that $c$ is a *fixed point solution of* $\forall$*-type* for $(X, R)$). Let us take notice from the outset of an elementary equivalence between revealed VNM-solutions and fixed point solutions of $\forall$-type, namely

**Claim 2** *A choice function $c \in C_X$ is a revealed VNM-solution iff there exists a* reflexive *$R \subseteq X \times X$ such that $c$ is a fixed point solution of $\forall$-type for $(X, R)$.*

**Proof.** Just consider $R = R^\Delta$, and $\Delta = \Delta^R$, respectively. ■

Now, consider the following list of examples:

**Example 3** *Notice that the digraph $(X, \varnothing)$ is also a dominance digraph and $\mathcal{S}(A, \varnothing_A) = \{A\}$ for any $A \subseteq X$. Therefore, the* identity operator *$c^{id_X}$ : $\mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is a revealed VNM-solution.*

**Example 4** *Similarly, the complete digraph $(X, (X \times X)^{-diag}))$ where $X \times X^{-diag} = (X \times X) \smallsetminus \{(x,x) : x \in X\}$ is of course a dominance digraph and*

$\mathcal{S}(A,(X\times X)_A^{-diag}) = \{\{x\}: x\in A\}$ *for any* $A\subseteq X$. *Thus, any* single-valued $c\in C_X$ *is a revealed VNM-solution.*

**Example 5** *Take* $\varnothing\subset G\subset X$ *and consider the dichotomic choice function* $c^G:\mathcal{P}(X)\rightarrow\mathcal{P}(X)$ *as defined by the* satisficing *rule* $c^G(A)=A\cap G$ *for any* $A\subseteq X$. *It is easily checked that* $c^G$ *is* not *a revealed VNM-solution: to see why, take any* $x\in X\setminus G$. *Then,* $c^G(\{x\})=\varnothing$ *while for any dominance digraph* $(X,\Delta)$ *and any* $x\in X$, *it cannot be the case that* $x\Delta x$ *hence* $\mathcal{S}(\{x\},\Delta_{\{x\}})=\{\{x\}\}$.

**Example 6** *Take* $X=\{x,y,z\}$, $\Delta=\{(x,y),(y,x),(x,z),(z,x)\}$ *and* $c\in C_X$ *such that* $c(\{x\})=\{x\}$, $c(\{y\})=\{y\}$, $c(\{z\})=\{z\}$, $c(\{x,y\})=c(\{x,z\})=\{x\}$, *and* $c(\{y,z\})=c(\{x,y,z\})=\{y,z\}$. *It is easily checked that* $\mathcal{S}(X,\Delta)=\{\{x\},\{y,z\}\}$, $\mathcal{S}(\{x,y\},\Delta_{\{x,y\}})=\{\{x\},\{y\}\}$, $\mathcal{S}(\{x,z\},\Delta_{\{x,z\}})=\{\{x\},\{z\}\},\mathcal{S}(\{y,z\},\Delta_{\{y,z\}})=\{\{y,z\}\}$, *and that* $c$ *is proper by construction, and VNMS-rationalizable by* $(X,\Delta)$, *an* irreflexive but not asymmetric *dominance digraph: in fact,* $\Delta^a=\varnothing$. *Moreover,* $\Delta(c)=\{(x,y),(x,z)\}$, $\Delta^*(c)=\Delta$. *Of course,* $\Delta(c)$ *is asymmetric and in particular* $\Delta(c)=(\Delta^*(c))^a$, *but* $\mathcal{S}(X,\Delta(c))=\{\{x\}\}$ *hence* $c$ *is not VNMS-rationalizable by* $(X,\Delta(c))$. *Indeed,* $c$ *is a VNMS-rationalizable choice function that is covered by Wilson's result but* not *captured by Suzumura's characterization as mentioned in the Introduction. Notice, however, that by positing* $xR^{\Delta^*(c)}y$ *iff* $(y,x)\notin(\Delta^*(c))^a$ *for any* $x,y\in X$, *and denoting* $\max R_Y^{\Delta^*(c)}$ *the set of* $R^{\Delta^*(c)}$*-maxima of* $Y$ *for any* $Y\subseteq X$, *it follows that* $c(Y)=\max R_Y^{\Delta^*(c)}$ *for all* $Y\subseteq X$.

**Example 7** *Consider* $X=\{x,y,z\},x\neq y\neq z\neq x$, *and define* $c_1\in C_X$ *as follows:* $c_1(\{x\})=c_1(\{x,y\})=\{x\}$, $c_1(\{y\})=c_1(\{y,z\})=\{y\}$, $c_1(\{z\})=c_1(\{x,z\})=\{z\}$, *and* $c_1(\{x,y,z\})=\varnothing$. *It is easily checked that* $c_1$ *is VNMS-rationalizable by the asymmetric dominance digraph* $(X,\Delta)$, *where* $\Delta=\{(x,y),(y,z),(z,x)\}$. *Clearly, however,* $c_1$ *is not proper hence is not covered by Wilson's more general result (see the Introduction).*

The main objective of this article is precisely to provide a *characterization of all revealed VNM-solutions* both in $C_X^{\circ}$ and in $C_X$.

# 3 Properties of choice functions

The following two properties of a choice function $c \in C_X$ play a prominent role under various labels in the extant literature:

*Properness (PR)*: $c(A) \neq \varnothing$ for any *nonempty* $A \subseteq X$.

*Chernoff contraction-consistency (C):* for any $A, B \subseteq X$ such that $A \subseteq B$, $c(B) \cap A \subseteq c(A)$.

Of course, PR amounts to the nonempty-valuedness requirement mentioned previously, and is a minimal condition of universal resoluteness on choice behaviour.

Condition C dictates that any outcome chosen out of a certain set should also be chosen out of any subset of the former: essentially, it says that *any good reason to choose a certain option out of a given menu should retain its strenght in every submenu of the former containing that option.*

Since the co-choice (or rejection) function attached to a choice function is also a choice function each property admits its counterpart as obtained by interchanging the roles of 'choice sets' and 'rejection sets' (it goes without saying that a contraction-consistency property concerning rejection sets amounts to an expansion-consistency requirement on choice sets, and conversely). Hence, in particular, we have:

*Co- Chernoff contraction-consistency (CC):* for any $A, B \subseteq X$ such that $A \subseteq B$, $c^c(B) \cap A \subseteq c^c(A)$.

Thus, condition CC is a context-independency property requiring in turn that *any good reason to reject a certain option out of a given menu retain its strenght in any submenu of the former containing that option.* Notice that this amounts to *set-inclusion monotonicity of* $c$ (while conversely C amounts to *set-inclusion monotonicity* of $\overline{c}$). This basic fact and the resulting bijection between choice functions satisfying C and *set-inclusion monotonic* choice functions is indeed duly exploited by Echenique (2007) in order to evaluate the number of choice functions satisfying C.

**Remark 8** *It is easily checked that a VNMS-rationalizable choice function may well violate PR, C and CC. To see this, consider* $X = \{x, y, z, w\}$ *with* $\#X = 4$, $\Delta = \{(x, y), (y, z), (z, x), (w, x)\}$, $A = \{x, z\}, A' = \{x, y, z\}$ ,

*$B = X = \{x, y, z, w\}$, and $c' \in C_X$ such that for any nonempty $Y \subseteq X$, $c'(Y) \in \mathcal{S}(Y, \Delta_Y)$ if $\mathcal{S}(Y, \Delta_Y) \neq \varnothing$ and $c'(Y) = \varnothing$ if $\mathcal{S}(Y, \Delta_Y) = \varnothing$ (thus, by definition, $c'$ is VNMS-rationalizable). Now, $c'(A') = \varnothing$ because $\mathcal{S}(A', \Delta_{A'}) = \varnothing$ (hence PR fails). Moreover, $c'(\{x, y, z, w\}) = \{x, w\}$ (since $\mathcal{S}(X, \Delta) = \{x, w\}$), hence $c'(\{x, y, z, w\}) \cap \{x, z\} = \{x\}$ while $c'(\{x, z\}) = \{z\}$, and C is violated. Furthermore, take $A = \{x, z\}$: then $c'^c(\{x, y, z, w\}) = \{y, z\}$, hence $z \in c'^c(\{x, y, z, w\}) \cap \{x, z\}$: however, $c(\{x, z\}) = \{z\}$ i.e. $z \notin c'^c(\{x, z\})$ and CC is also violated.*

As explained in the previous Remark, VNMS-rationalizable choice functions need not satisfy PR, C, or CC. Let us then consider some considerably weakened versions of the foregoing properties, namely

*No-dummy property (ND)*: $c(\{x\}) = \{x\}$ for any $x \in X$.

*2- Properness (2-PR):* $c(A) \neq \varnothing$ for any $A \subseteq X$ such that $\#A = 2$.

*Idempotence (IDP):* $c(c(A)) = c(A)$ for any $A \subseteq X$.

*Binary non-discrimination of jointly chosen outcomes (BNC):* for any $A \subseteq X$ and $x, y \in A$, if $\{x, y\} \subseteq c(A)$ *then either* $c(\{x, y\}) = \{x, y\}$ or $c(\{x, y\}) = \varnothing$.

*Weak binary co- Chernoff contraction-consistency (WBCC):* for any $B \subseteq X$ such that $c(B) \neq \varnothing$, and any $y \in c^c(B)$ there exists $z \in c(B)$ such that $y \in c^c(\{y, z\})$ whenever $z \in c(\{y, z\})$.

Clearly, ND and 2-PR are simply restricted versions of PR.

On the other hand, IDP, BNC and WBCC are context-independence requirements for choice sets.

Indeed, IDP -sometimes also referred to as 'Stability'- is a basic self-consistency requirement of the contraction-consistency family that is easily shown to be implied by C.

BNC requires that any two chosen outcomes *be treated in a symmetric way when compared to each other in a binary contest.* Thus, it is a kind of contraction-consistency property for selected outcomes, and it can be shown that it is indeed also implied by C.

WBCC is by construction a restricted version of CC, dictating that *any good reason to reject a certain option out of a given not totally rejected menu* *should retain its strenght in certain binary submenus of the former containing that option and sharing a chosen option with it.*

In other terms, WBCC is a contraction-consistency requirements for *rejection* sets, hence, as observed above, it may also be construed as an *expansion-consistency property for choice sets*. Thus, in particular, it is worth asking what is the relationship of WBCC to the most widely used choice expansion-consistency properties such as Superset consistency, Concordance (also known as Generalized Condorcet expansion-consistency, or $\gamma$), Regular Expansion-consistency (or $\beta$), and Strong Expansion-consistency (or $\beta+$) (see e.g. Sen (1971, 1977), Suzumura (1983)), namely

*Superset consistency (SS):* for any $A, B \subseteq X$ , if $A \subseteq B$ and $\varnothing \neq c(B) \subseteq c(A)$ then $c(A) \subseteq c(B)$.

*Concordance (CO):* for any $A, B \subseteq X$ , $c(A) \cap c(B) \subseteq c(A \cup B)$.

*Regular expansion-consistency (RE):* for any $A, B \subseteq X$ , if $A \subseteq B$ and $c(B) \cap c(A) \neq \varnothing$ then $c(A) \subseteq c(B)$.

*Strong expansion-consistency (SE):* for any $A, B \subseteq X$ , if $A \subseteq B$ and $c(B) \cap A \neq \varnothing$ then $c(A) \subseteq c(B)$.

It turns out that WBCC is implied by RE (but not conversely), and is independent of CO and SS (observe that CC implies SE which implies RE which in turn implies SS, by definition). Some of the salient relationships among PR, C, CC, SS, CO, RE, SE, ND, 2-PR, IDP, BNC, and WBCC as alluded to above (and not necessarily mutually independent) are made precise by the following

**Proposition 9** *Let $c \in C_X$ . Then,*

*(i) if c satisfies PR then it also satisfies ND, 2-PR (while the converse need not hold);*

*(ii) if c satisfies C then it also satisfies IDP (while the converse need not hold);*

*(iii) if c satisfies C then it also satisfies BNC (while the converse need not hold );*

*(iv) if c satisfies RE then it also satisfies WBCC;*

*(v) even under PR, properties BNC and IDP are mutually independent;*

*(vi) WBCCA is independent of properties CO and SS (even under PR);*

*(viii) if $c$ satisfies ND, 2-PR, BNC, and WBCC then it also satisfies SS, but need not satisfy C, CO, or RE;*

*(ix) if $c$ satisfies PR, C, CO, and SS then it also satisfies WBCC.*

**Proof.** (i) That PR implies ND and 2-PR but not conversely is trivial since by hypothesis $\#X \geqslant 3$.

(ii) Just take $A = c(B)$. To check that the converse does not hold, see the example provided below under point (iii) of this proof;

(iii) Let $c$ satisfy C, $A \subseteq X$ , $x, y \in A$, and $\{x, y\} \subseteq c(A)$. Then, by C, $c(A) \cap \{x, y\} \subseteq c(\{x, y\})$ hence $\{x, y\} \subseteq c(\{x, y\})$ i.e. $c(\{x, y\}) = \{x, y\}$ and BNC holds. However, take the choice function introduced in the previous Remark i.e. consider $X = \{x, y, z, w\}$ , $\#X = 4$, and $c' \in C_X$ such that $c'(\{x, y\}) = c'(\{x\}) = \{x\}$, $c'(\{y, z\}) = c'(\{y\}) = \{y\}$, $c'(\{x, z\}) = c'(\{z\}) = \{z\}$, $c'(\{z, w\}) = c'(\{w\}) = \{w\}$, $c'(\{x, w\}) = c'(\{x, y, w\}) = c'(\{x, z, w\}) = \{x, w\}$, $c'(\{y, w\}) = c'(\{y, z, w\}) = \{y, w\}$, $c'(\{x, y, z\}) = \varnothing$, and $c'(\{x, y, z, w\}) = \{x, w\}$. Clearly, BNC is satisfied by $c'$, by construction. However, $c'(\{x, y, z, w\}) \cap \{x, z\} = \{x\}$ while $c'(\{x, z\}) = \{z\}$ hence C is violated;

(iv) Let us assume that $c$ satisfies RE but not WBCC. Then, by definition of WBCC, there exist $B \subseteq X$ and $x \in B$ s.t. $x \notin c(B) \neq \varnothing$ and $c(\{x, y\}) = \{x, y\}$ for any $y \in c(B)$. It follows that, by RE, $c(\{x, y\}) \subseteq c(B)$, whence $x \in c(B)$, a contradiction.

(v) To check that the conjunction of PR and BNC does not imply IDP, consider $X = \{x, w, y, z\}$, $\#X = 4$, and take $c \in C_X$ defined as follows: $c(A) = A$ for any $A \subseteq X$, $A \notin \{X, \{x, y, z\}\}$, $c(X) = \{x, y, z\}$, $c(\{x, y, z\}) = \{x, y\}$. It is immediately checked that $c$ satisfies both PR and BNC, however $c(c(X)) = \{x, y\} \neq c(X)$ i.e. IDP is violated.To check that the conjunction of PR and IDP does not imply BNC, consider $X = \{x, y, z\}$, $\#X = 3$, and take $c \in C_X$ defined as follows: $c(\{h\}) = \{h\}$ for any $h \in X$, $c(X) = \{x, y, z\}$, $c(\{x, y\}) = \{x\}$, $c(\{x, z\}) = \{z\}$, $c(\{y, z\}) = \{y\}$. Clearly, $c$ satisfies PR and IDP, but violates BNC.

(vii) To check that WBCC does not imply CO, consider $X = \{x, y, z, u, v\}$, $\#X = 5$, $\Delta = \{(x, y), (y, z), (z, u), (u, v), (v, x)\}$, and take a choice function $c^* \in C_X$ which is VNMS-rationalizable by dominance digraph $(X, \Delta)$ on any $A \subset X$, while $c^*(X) = \{y, z, u, v\}$. It is easily checked that $c^*$does indeed satisfy WBCC since $c^*(\{x, y\}) = \{x\}$ (and PR as well). On the other hand, take $A = \{x, y, z\}$ , $B = \{x, u, v\}$: it is immediately checked that

$c^*(A) = \{x, z\}, c^*(B) = \{x, u\}$, while $x \in c^*(A \cup B)$ hence CO fails. Next, take $X = \{x, y, z, u, v\}$, $\#X = 5$, and consider $c^\circ \in C_X$ defined as follows: $c^\circ(\{h\}) = \{h\}$ for any $h \in X$,

$c^\circ(\{x, y\}) = c^\circ(\{x, y, u\}) = c^\circ(\{x, y, u, v\})) = c^\circ(\{x, u\}) = c^\circ(\{x, u, v\}) = \{x\}$,

$c^\circ(\{x, v\}) = c^\circ(\{x, y, v\}) = \{x, v\}$,

$c^\circ(\{x, z\}) = c^\circ(\{x, z, u\}) = c^\circ(\{x, z, u, v\})) = \{z\}$,

$c^\circ(\{x, y, z\}) = c^\circ(\{x, y, z, u\}) = c^\circ(\{x, y, z, u, v\}) = \{x, y, z\}$,

$c^\circ(\{y, z\}) = \{y\}$,

$c^\circ(\{y, z, u\}) = \{y, u\}$,

$c^\circ(\{y, v\}) = c^\circ(\{y, z, v\}) = \{y, v\}$,

$c^\circ(\{y, u\}) = c^\circ(\{y, u, v\}) = c^\circ(\{y, z, u, v\}) = \{y, u\}$,

$c^\circ(\{z, u\}) = c^\circ(\{z, u, v\}) = \{z, u\}$,

$c^\circ(\{z, v\}) = c^\circ(\{x, z, v\}) = \{z, v\}$,

$c^\circ(\{u, v\}) = \{u\}$,

$c^\circ(\{x, y, z, v\}) = \{x, y, z, v\}$.

It can be checked that $c^\circ$ satisfies CO (and PR), but fails to satisfy WBCC: indeed, $v \notin c^\circ(\{x, y, z, u, v\}) = \{x, y, z\}$. However, $c^\circ(\{x, v\}) = \{x, v\}$, $c^\circ(\{y, v\}) = \{y, v)\}$, and $c^\circ(\{z, v\}) = \{z, v\}$.

To see that WBCC does not entail SS, take $X = \{x, y, z, w\}$ and consider $c'' \in C_X$ defined as follows: $c''(\{x, y)\} = \{x\}$, $c''(\{z, w\}) = \{w\}$, $c''(X) = \{x, z\}$, and $c''(A) = A$ for any other $A \subseteq X$. It is easily checked, that by construction WBCC is satisfied (and PR as well). However, e.g. $\{x, y\} = c''(X) \subset c''(\{x, y, z\})$. To check that SS does not entail WBCC, take $X = \{x, y, z\}$ and $c''' \in C_X$ defined as follows: $c''(A) = A$ for any $A \subset X$, and $c'''(X) = \{x, y\}$. Clearly, by construction SS is satisfied (and PR as well), but WBCC is violated.

(viii) Let $c$ satisfy 2-PR, BNC and WBCC. Then, suppose SS is violated. Thus, there exist $A, B \subseteq X$ and $x \in A$ such that $A \subseteq B$, $\varnothing \neq c(B) \subseteq c(A)$ and $x \in c(A) \setminus c(B)$. It follows, by WBCC and 2-PR, that there exists $y \in c(B)$ such that $\varnothing \neq c(\{x, y\}) \neq \{x, y\}$. Assume that $y \notin c(\{x, y)\}$. Then, $c(\{x, y)\} = \{x\}$. Otherwise, $c(\{x, y)\} = \{y\}$. In any case, by BNC, it follows that there exists no $A \subseteq X$ such that $\{x, y\} \subseteq c(A)$, a contradiction since $y \in c(B) \subseteq c(A)$.

On the other hand, consider again $X = \{x, y, z, u, v\}$, $\#X = 5$, $\Delta = \{(x, y), (y, z), (z, u), (u, v), (v, x)\}$, and take any choice function $c^{**} \in C_X$ which is VNMS-rationalizable by dominance digraph $(X, \Delta)$. Finally, consider again $c'$ as defined above under point (iii) of this proof. It can be readily checked that both of them satisfy WBCC, BNC, ND and 2-CPR. However, $c^{**}$ violates CO (for the very same reason its variant $c^{*}$ as defined under point (iii) does), while $c'$ fails to satisfy C (as observed above) and RE (because, recall, $c'(\{y,w\}) = \{y,w\}$, while $c'(\{x,y,z,w\}) = \{x,w\}$, hence $c'(\{y,w\}) \cap c'(\{x,y,z,w\}) \neq \varnothing$, but $c'(\{y,w\}) \nsubseteq c'(\{x,y,z,w\})$).

(ix) Let $c$ satisfy PR,C, CO and SS. Then, by a well-known result (see e.g. Theorem 2.6 in Suzumura (1983)) there exists a binary relation $R$ with a *transitive asymmetric component such that* $c(A) = \max R_A$ *for any* $A \subseteq X$; moreover, $R = R_c$ which is defined as follows: for any $y, z \in X$, $yR_cz$ iff there exists $Y \subseteq X$ such that $y \in c(Y)$ and $z \in Y$. Let us now assume that WBCC is *not* satisfied by $c$. Then, there exists $B \subseteq X$ and $x \in B$ such that $c(B) \neq \varnothing$, $x \notin c(B)$, while $c(\{x,z\}) = \{x,z\}$ for each $z \in c(B)$. It follows that $xRz$ for all $z \in c(B)$ hence, by definition of $R$, $x \in c(c(B))$. However, by C, $c$ also satisfies IDP (see point (ii) of this claim) whence $c(c(B)) = c(B)$. But then, $x \in c(B)$, a contradiction. ■

**Remark 10** *It is easily checked that any VNMS-rationalizable choice function does indeed satisfy IDP. To see this, take any* $A \subseteq X$*, and* $x \in c(A) \in \mathcal{S}(A, \Delta_A)$ *(for some dominance digraph* $(X, \Delta)$*). Clearly,* not $x\Delta_A y$ *for any* $y \in c(A)$*, hence by definition* $\mathcal{S}(A, \Delta_A) = \{c(A)\}$*. It follows that* $\mathcal{S}(c(A), \Delta_{c(A)}) = \{c(A)\}$ *as well, whence* $c(c(A)) = c(A)$*. The proof that VNMS-rationalizable choice functions satisfy ND, 2-PR, BNC, and WBCC (hence SS as well) will be provided below (as a part of the proof of Theorem 13 below).*

*It is also worth noticing here that any choice function c that is VNMS-rationalizable by an* asymmetric *dominance digraph* $(X, \Delta)$ *does also satisfy a weaker asymmetric version of CO, namely*

Asymmetric Generalized Condorcet expansion-consistency (AGC)*: for any* $A \subseteq X$ *, and any* $x \in A$*, if* $c(\{x,y\}) = \{x\}$ *for all* $y \in A$*, then* $c(A) = \{x\}$ *.*

*Indeed, if c is VNMS-rationalizable by an asymmetric dominance digraph* $(X, \Delta)$*, then for any* $A \subseteq X$*,* $x \in A$*, if for all* $y \in A$*,* $c(\{x,y\}) = \{x\}$ *then* $\mathcal{S}(\{x,y\}, \Delta_{\{x,y\}}) = \{\{x\}\}$ *for all* $y \in A$*. Therefore, by* external *stability of VNM-stable sets,* $x\Delta y$ *(whence* not $y\Delta x$*, by asymmetry) for any* $y \in A$*. Now, let* $(X, \Delta')$ *be any dominance digraph such that c is VNMS-rationalizable by* $(X, \Delta')$*. By construction,* $\Delta'_A = \Delta_A$*, hence* $\mathcal{S}(A, \Delta'_A) = \mathcal{S}(A, \Delta_A) = \{\{x\}\}$*(because it is easily checked that by construction* $\{x\} \in \mathcal{S}(A, \Delta_A)$*,* *and that* any *VNM-stable set of* $(A, \Delta_A)$ *must contain* $x$ *by external stability whence by internal stability must also exclude every* $y \in A \setminus \{x\}$*). It follows that* $c(A) = \{x\}$ *as required, and AGC is satisfied.*

*To see that the asymmetry requirement on* $(X, \Delta)$ *cannot be dispensed with, just take* $X = \{x, y, z\}$, $\Delta = \{(x, y), (y, x), (x, z), (z, x)\}$ *and* $c \in C_X$ *such that* $c(\{x\}) = \{x\}$, $c(\{y\}) = \{y\}$, $c(\{z\}) = \{z\}$, $c(\{x, y\}) = \{x\}$, $c(\{x, z\}) = \{x\}$*, and* $c(\{y, z\}) = c(\{x, y, z\}) = \{y, z\}$*. In fact, it is easily checked that* $\mathcal{S}(X, \Delta) = \{\{x\}, \{y, z\}\}$ *and that* $c$ *is VNMS-rationalizable by* $(X, \Delta)$*, while clearly violating AGC.*

# 4 Characterizing revealed VNM-solutions

Let us now proceed to characterizations of VNMS-rationalizable choice functions by means of suitable combinations of some the properties discussed in the previous section.

To begin with, it is worth emphasizing that a certain dominance digraph may provide a VNMS-rationalization of *several* distinct choice functions and, perhaps less obviously, a choice function may be VNM-rationalizable by *several* distinct dominance digraphs. Those simple facts about revealed VNM-solutions are illustrated by the following

**Example 11** *Consider* $X = \{x, y, z\}$ *and dominance digraph* $(X, \Delta)$ *with* $\Delta = \{(x, y), (y, z), (z, x), (x, z)\}$*, and choice functions* $c, c' \in X$ *defined as follows:* $c(\{u\}) = c'(\{u\}) = \{u\}$ *for any* $u \in X$, $c(\{x, y\}) = c'(\{x, y\}) = \{x\}$, $c(\{y, z\}) = c'(\{y, z\}) = \{y\}$, $c(X) = c'(X) = \{x\}$*, but* $c(\{x, z\}) = \{x\}$ *while* $c'(\{x, z\}) = \{z\}$*. Clearly, both* $c$ *and* $c'$ *are VNMS-rationalizable by* $(X, \Delta)$*. Conversely, take again* $X = \{x, y, z\}$ *and* $c'' \in C_X$ *defined as follows:* $c(\{u\}) = c'(\{u\}) = \{u\}$ *for any* $u \in X$, $c(\{x, y\}) = c'(\{x, y\}) = \{x\}$, $c(\{y, z\}) = \{y\}$, $c(\{x, z\}) = \{x\}$ *, and* $c(X) = \{x\}$*. Now consider* $\Delta = \{(x, y), (x, z), (y, z)\}$ *and, say,* $\Delta' = \{(x, y), (x, z), (y, z), (z, y)\}$*. Clearly* $c$ *is VNMS-rationalizable by both* $(X, \Delta)$ *and* $(X, \Delta')$*.*

In order to proceed towards our characterizations, let us now introduce a new piece of notation, and prove a pair of simple and useful lemmas, namely

**Notation 12** *Let $R \subseteq R' \subseteq X \times X$. Then $R'$ is a* local symmetric closure *of $R$ - written $R' \in cl^s(R)$ - iff [for any $x, y \in X$, $(x, y) \in R' \smallsetminus R$ entails $(y, x) \in R$].*

**Lemma 13** *Let $c \in C_X$ . Then, (i) $c$ satisfies 2-PR and is VNMS-rationalizable by dominance digraph $(X, \Delta)$ only if $\Delta^{R_c} \subseteq \Delta \subset \Delta^{R_c} \cup (\Delta^{R_c})^{-1}$ and $\Delta \in cl^s(\Delta^{R_c})$; (ii) if $c$ satisfies 2-PR and BNC, then $\Delta^{R_c} \cup (\Delta^{R_c})^{-1} = \Delta^{\widehat{R}(c)}$, and for any $Z \subseteq Y \subseteq X$, $Z \in \mathcal{S}(Y, \Delta_Y^{\widehat{R}(c)})$ whenever [$Z \in \mathcal{S}(Y, \Delta_Y^{R_c})$ or $Z \in \mathcal{S}(Y, (\Delta^{R_c})_Y^{-1})$]; (iii) if $c$ satisfies 2-PR, is VNMS-rationalizable by $(X, \Delta)$ and $(X, \Delta'')$, and $\Delta \subseteq \Delta' \subseteq \Delta''$ then $c$ is also VNMS-rationalizable by $(X, \Delta')$.*

**Proof.** (i) Let $c$ be VNMS-rationalizable by dominance digraph $(X, \Delta)$. Then, for any $x \in X$, $c(\{x\}) \in \mathcal{S}(\{x\}, \Delta_{\{x\}}) = \{\{x\}\}$, i.e. $c(\{x\}) = \{x\}$ hence $c$ satisfies ND and by definition both $(x, x) \notin \Delta^{R_c}$ and $(x, x) \notin (\Delta^{R_c})^{-1}$. Now, let $x, y \in X$: of course there are four possible cases to consider, namely a) $x\Delta y$ and $y\Delta x$, whence $\mathcal{S}(\{x, y\}, \Delta_{\{x,y\}}) = \{\{x\}, \{y\}\}$; b) $x\Delta y$ and *not* $y\Delta x$, whence $\mathcal{S}(\{x, y\}, \Delta_{\{x,y\}}) = \{\{x\}\}$, c) $y\Delta x$ and *not* $x\Delta y$, whence $\mathcal{S}(\{x, y\}, \Delta_{\{x,y\}}) = \{\{y\}\}$; d) *not* $x\Delta y$ and *not* $y\Delta x$ whence $\mathcal{S}(\{x, y\}, \Delta_{\{x,y\}}) = \{\{x, y\}\}$. Under a), either $c(\{x, y\}) = \{x\}$ or $c(\{x, y\}) = \{y\}$ (but not both of them) hence by definition either ($x\Delta^{R_c}y$ and *not* $y\Delta^{R_c}x$) or ($y\Delta^{R_c}x$ and *not* $x\Delta^{R_c}y$); under b), $c(\{x, y\}) = \{x\}$ hence $x\Delta^{R_c}y$ and *not* $y\Delta^{R_c}x$; under c), $c(\{x, y\}) = \{y\}$ hence $y\Delta^{R_c}x$ and *not* $x\Delta^{R_c}y$; under d), $c(\{x, y\}) = \{x, y\}$ hence *not* $x\Delta^{R_c}y$ and *not* $y\Delta^{R_c}x$. Thus, in any case $x\Delta y$ entails either $x\Delta^{R_c}y$ or $x(\Delta^{R_c})^{-1}y$ i.e. $\Delta \subseteq \Delta^{R_c} \cup (\Delta^{R_c})^{-1}$. Next, suppose that $\Delta^{R_c} \nsubseteq \Delta$ i.e. there exist $x, y \in X$ such that $x\Delta^{R_c}y$ and *not* $x\Delta y$. Thus, *not* $yR_cx$ i.e. by 2-PR $c(\{x, y\}) = \{x\} \in \mathcal{S}(\{x, y\}, \Delta_{\{x,y\}})$, while as it is easily checked *not* $x\Delta y$ entails $\mathcal{S}(\{x, y\}, \Delta_{\{x,y\}}) \subseteq \{\{y\}, \{x, y\}\}$, a contradiction: it follows that $\Delta^{R_c} \subseteq \Delta$.

Moreover, let $x\Delta y$ and *not* $y\Delta^{R_c}x$. Then $\{x\} \in \mathcal{S}(\{x, y\}, \Delta_{\{x,y\}})$, and $(x, y) \notin \Delta^{R_c}$ i.e. by definition $y \in c(\{x, y\})$. Clearly, $\{x, y\}$ is not internally stable in $(\{x, y\}, \Delta_{\{x,y\}})$ since $\{x\} \in \mathcal{S}(\{x, y\}, \Delta_{\{x,y\}})$ and is therefore an externally stable set. Thus, $\{x, y\} \notin \mathcal{S}(\{x, y\}, \Delta_{\{x,y\}})$: as a consequence, $y \in c(\{x, y\}) \in \mathcal{S}(\{x, y\}, \Delta_{\{x,y\}})$ entails that $\{y\} \in \mathcal{S}(\{x, y\}, \Delta_{\{x,y\}})$ as well, whence $c(\{x, y\}) = \{y\}$ and $y\Delta x$. It follows that, by definition, $\Delta \in cl^s(\Delta^{R_c})$.

(ii) Let $c$ satisfy ND, 2-PR and BNC, and suppose $(x, y) \in \Delta^{R_c} \cup (\Delta^{R_c})^{-1}$ i.e. *not* $yR_cx$ or *not* $xR_cy$.

If *not* $yR_cx$ then $c(\{x,y\}) \neq \{x,y\}$ hence by BNC there is no $Y \subseteq X$ such that $\{x,y\} \subseteq c(Y)$ and therefore *not* $x\widehat{R}(c)y$ and *not* $y\widehat{R}(c)x$. Thus, by definition, $x\Delta^{\widehat{R}(c)}y$ and $y\Delta^{\widehat{R}(c)}x$. By the very same argument, if *not* $xR_cy$ then $x\Delta^{\widehat{R}(c)}y$ and $y\Delta^{\widehat{R}(c)}x$.

Conversely, suppose that $x\Delta^{\widehat{R}(c)}y$ i.e. by ND $x \neq y$ and there is no $Y \subseteq X$ such that $\{x,y\} \subseteq c(Y)$. Then in particular $c(\{x,y\}) \neq \{x,y\}$ hence by 2-PR $c(\{x,y\}) = \{x\}$ or $c(\{x,y\}) = \{y\}$. If $c(\{x,y\}) = \{x\}$ then *not* $yR_cx$ i.e. $x\Delta^{R_c}y$, and similarly if $c(\{x,y\}) = \{y\}$ then *not* $xR_cy$ i.e. $y\Delta^{R_c}c$, namely $x(\Delta^{R_c} \cup (\Delta^{R_c})^{-1})y$.

Moreover, let $Z \in \mathcal{S}(Y, \Delta_Y^{R_c}) \cup \mathcal{S}(Y, (\Delta^{R_c})_Y^{-1})$. Then, in any case, by internal stability *not* $x\Delta^{R_c}y$ and *not* $y\Delta^{R_c}x$ whence *not* $x(\Delta_Y^{R_c} \cup (\Delta^{R_c})_Y^{-1})y$ for any $x, y \in Z$. Also, by external stability, for any $y \in Y \smallsetminus Z$ there exists $z \in Z$ such that $z\Delta^{R_c}y$ or $z(\Delta^{R_c})^{-1}y$ i.e. $z(\Delta^{R_c} \cup (\Delta^{R_c})^{-1})y$. It follows that indeed $Z \in \mathcal{S}(Y, \Delta_Y^{R_c} \cup \Delta^{R_c})_Y^{-1}$ i.e. by the previous argument $Z \in \mathcal{S}(Y, \Delta_Y^{\widehat{R}_c})$;

(iii) Let us now suppose that $c$ is VNMS-rationalizable by $(X, \Delta)$ and $(X, \Delta'')$, and $\Delta \subseteq \Delta' \subseteq \Delta''$: notice that, by point (i) of this Lemma, $\Delta^{R_c} \subseteq \Delta$ and $\Delta'' \subseteq \Delta^{R_c} \cup (\Delta^{R_c})^{-1}$. Then, for any $Y \subseteq X$ such that $c(Y) \neq \varnothing$, $c(Y) \in \mathcal{S}(Y, \Delta_Y) \cap \mathcal{S}(Y, \Delta''_Y)$. Hence, in particular *not* $x\Delta''y$ and therefore by hypothesis *not* $x\Delta'y$ for any $x, y \in c(Y)$. On the other hand, for any $y \in Y \smallsetminus Z$ there exists in particular $z \in c(Y)$ such that $z\Delta y$ hence by hypothesis $z\Delta'y$. It follows that $c(Y) \in \mathcal{S}(Y, \Delta'_Y)$ as well. Moreover, let $Y \subseteq X$ be nonempty and such that $c(Y) = \varnothing$. Then, by hypothesis, $\mathcal{S}(Y, \Delta_Y) = \mathcal{S}(Y, \Delta''_Y) = \varnothing$. Suppose however that $\mathcal{S}(Y, \Delta'_Y) \neq \varnothing$ and let $S \in \mathcal{S}(Y, \Delta'_Y)$. By internal stability, *not* $x\Delta'y$ for any $x, y \in S$ hence by definition *not* $x\Delta^{R_c}y$ -and *not* $y\Delta^{R_c}x$- and therefore *not* $x\Delta''y$. Moreover, for any $y \in Y \smallsetminus S$ there exists $z \in S$ such that $z\Delta'y$, hence $z\Delta''y$ as well, i.e. $S \in \mathcal{S}(Y, \Delta''_Y)$, a contradiction. It follows that $\mathcal{S}(Y, \Delta'_Y) = \varnothing$ and $c$ is also VNMS-rationalizable by $(X, \Delta')$. ■

**Lemma 14** *Let $c \in C_X$ satisfy ND, 2-PR, BNC and WBCC. Then, for any $Y \subseteq X$, if $c(Y) \neq \varnothing$ then $\mathcal{S}(Y, \Delta_Y^{\widehat{R}(c)}) \neq \varnothing$ and $c(Y) \in \mathcal{S}(Y, \Delta_Y^{\widehat{R}(c)})$.*

**Proof.** Let $Y \subseteq X$ and $c(Y) \neq \varnothing$: then, by BNC, for any $x, y \in c(Y)$ either $c(\{x,y\}) = \{x,y\}$ or $c(\{x,y\}) = \varnothing$. Hence $c(\{x,y\}) = \{x,y\}$ by ND and 2-PR. Therefore, $x\widehat{R}(c)y$ and $y\widehat{R}(c)x$ i.e. by definition of $\Delta^{\widehat{R}(c)}$, *not* $y\Delta^{\widehat{R}(c)}x$ and *not* $x\Delta^{\widehat{R}(c)}y$ hence $c(Y)$ is an *internally stable* set of $(Y, \Delta_Y^{\widehat{R}(c)})$. Moreover, let $z \in Y \setminus c(Y)$: then, by WBCC, there exists $x \in c(Y)$ such that

$z \in c^c(Y) \cap \{x,z\} \subseteq c^c(\{x,z\})$ unless $x \notin c(\{x,z\})$. Now, if $x \notin c(\{x,z\})$ then by 2-PR $c(\{x,z\}) = \{z\}$, otherwise $c(\{x,z\}) = \{x\}$ whence in any case by BNC there is no $A \subseteq X$ such that $\{x,z\} \subseteq c(A)$ i.e. *not* $x\widehat{R}(c)z$ and *not* $z\widehat{R}(c)x$. It follows that, by definition, $z\Delta^{\widehat{R}(c)}y$ (and $y\Delta^{\widehat{R}(c)}z$), hence in particular $c(Y)$ is also an *externally stable* set, and therefore a *VNM-stable set* of $(Y, \Delta_Y^{\widehat{R}(c)})$. ■

The foregoing lemmas motivate the introduction of a new notion, namely

**Definition 15** *The* Von Neumann-Morgenstern basis $\mathcal{B}^{VNM}(c)$ of the proper subdomain *of* $c \in C_X$ *is the set of* set-inclusion minimal *dominance digraphs* $(X, \Delta^i)_{i \in I}$ *such that* $c(B) \in \mathcal{S}(B, \Delta^i_B)$ *for any* $B \subseteq X$ *with* $c(B) \neq \varnothing$.

Such a notion of Von Neumann-Morgenstern basis of a choice function in turn enables the introduction of the following 'boundary' property concerning the behaviour of a choice function $c \in C_X$ on its (possibly empty) improper subdomain.

*Revealed VNM-Failure (RVNMF): For any nonempty* $A \subseteq X$, if $c(A) = \varnothing$ then there exists a $\Delta \in \mathcal{B}^{VNM}(c)$ such that $\mathcal{S}(A, \Delta_A) = \varnothing$.

Clearly enough, RVNMF rules out the existence of VNM-stable sets of *some* revealed dominance digraphs whenever the choice set of a nonempty set is empty, as aptly illustrated by the following example.

**Example 16** *Take* $X = \{x,y,z,w\}$, *and consider* $c, c', c'' \in C_X$ *defined as follows: for any* $u \in X$, $c(\{u\}) = \{u\}$, $c(\{x,y\}) = \{x\}$, $c(\{y,z\}) = \{y\}$, $c(\{x,z\}) = \{z\}$, $c(Y) = \{w\}$ *for any* $Y \subseteq X$ *such that* $w \in Y$ *and* $\#Y \leq 3$ , $c(\{x,y,z\}) = c(X) = \varnothing$, *while* $c'(Y) = c(Y)$ *for any* $Y \subset X$ *and* $c'(X) = \{w\}$, *and* $c''(Y) = c'(Y)$ *for any* $Y \subseteq X$ *such that* $Y \neq \{x,y,z\}$, *and* $c''(\{x,y,z\}) = \{x\}$. *It can be easily checked that* $c'$ *is a VNM-solution of dominance digraph* $(X, \Delta^{R_{c'}})$: *notice that* $\Delta^{R_{c'}} = \Delta^{R_c} = \{(x,y),(y,z),(z,x),(w,x),(w,y),(w,z)\}$. *Since* $\mathcal{S}(X, \Delta^{R_{c'}}) = \mathcal{S}(X, \Delta^{R_c}) = \{\{w\}\}$ *and* $\mathcal{S}(\{x,y,z\}, \Delta^{R_{c'}}_{\{x,y,z\}}) = \mathcal{S}(\{x,y,z\}, (\Delta^{R_{c'}})^{-1}_{\{x,y,z\}}) = \varnothing$ *it follows that* $c$ *violates RVNMF while* $c'$ *satisfies it. On the other hand,* $c''$ *is nonempty-valued and therefore trivially satisfies RVNMF.*

Admittedly, RVNMF smacks of ad-hoc-ness in that it dictates an *explicit* connection between choice sets and VNM-stable sets of certain digraphs. However, unpacking that condition in a more transparent manner seems to be hardly a trivial task. Arguably, that apparent difficulty might reflect the fact that the characterization of digraphs admitting a VNM-stable set (or, dually, a kernel) is still an open problem, and a resilient one at that.

Be it as it may, we are now ready to prove the main results of this paper. Let us then start from the class of *nonempty-valued* revealed VNM-solutions.

**Theorem 17** *Let $c \in C_X$. Then, c satisfies PR, BNC and WBCC iff there exists a VNM-perfect dominance digraph $(X, \Delta)$ such that $c(Y) \in \mathcal{S}(Y, \Delta_Y)$ for any $Y \subseteq X$. Moreover, $\Delta^a = \Delta(c)$ hence in particular $\Delta = \Delta(c)$ if $(X, \Delta)$ is an* asymmetric *dominance digraph.*

**Proof.** Let $c \in C_X$ satisfy PR, BNC and WBCC. Since PR obviously entails ND and 2-PR, the previous Lemma applies. It follows that, for any $Y \subseteq X$, $\mathcal{S}(Y, \Delta_Y^{\widehat{R}(c)}) \neq \varnothing$ and $c(Y) \in \mathcal{S}(Y, \Delta_Y^{\widehat{R}(c)})$ .

Conversely, suppose that there exists a VNM-perfect dominance digraph $(X, \Delta)$ such that $c(Y) \in \mathcal{S}(Y, \Delta_Y)$ for any $Y \subseteq X$. Since $(X, \Delta)$ is VNM-perfect, $\mathcal{S}(Y, \Delta_Y) \neq \varnothing$ for any $Y \subseteq X$. Since by external stability a VNM-stable set of a *nonempty* set is *nonempty*, $c(Y) \neq \varnothing$ for any *nonempty* $Y \subseteq X$, i.e. $c$ satisfies PR.

Also, for any $A \subseteq X$ and $x, y \in A$, if $\{x, y\} \subseteq c(A) \in \mathcal{S}(Y, \Delta_Y)$ then neither $x\Delta_Y y$ nor $y\Delta_Y x$ whence by definition *not* $x\Delta y$ and *not* $y\Delta x$. It follows that $\mathcal{S}(\{x, y\}, \Delta_{\{x,y\}}) = \{x, y\}$ hence $c(\{x, y\}) = \{x, y\}$ and therefore $c$ does satisfy BNC.

Furthermore, let $B \subseteq X$. By hypothesis, $\varnothing \neq c(B) \in \mathcal{S}(B, \Delta_B)$. If $c(B) = B$ i.e. $c^c(B) = \varnothing$, there is nothing to prove concerning WBCC. Otherwise, take any $x \in c^c(B)$: by external stability of $c(B)$ with respect to $(B, \Delta_B)$, there exists $z \in B$ such that $z\Delta y$. Therefore, it cannot be the case that $\{y, z\} \subseteq c\{y, z\}$ i.e. $y \in c^c(\{y, z\})$ whenever $z \in c(\{y, z\})$, and WBCC is satisfied.

Moreover, let $(X, \Delta)$ be a VNM-perfect asymmetric dominance digraph such that $c(Y) \in \mathcal{S}(Y, \Delta_Y)$ for any $Y \subseteq X$. For any $x, y \in X$, if $x\Delta(c)y$ then by definition $x \neq y$ and $c(\{x, y\}) = \{x\}$. Since by hypothesis $c(\{x, y\}) \in \mathcal{S}(\{x, y\}, \Delta_{\{x,y\}})$ it follows that $\{x\} \in \mathcal{S}(\{x, y\}, \Delta_{\{x,y\}})$ hence by external stability $x\Delta_{\{x,y\}}y$, and therefore $x\Delta y$. On the other hand, if $x\Delta y$ and *not*

$y\Delta x$ then $\mathcal{S}(\{x,y\}, \Delta_{\{x,y\}}) = \{\{x\}\}$ hence by hypothesis $c(\{x,y\}) = \{x\}$, and $x\Delta(c)y$. It follows that $\Delta^a = \Delta(c)$. ■

**Remark 18** *The characterization result provided above is tight as shown by the following examples. First, to check the independence of PR, let us consider without loss of generality $X = \{x,y,z\}, x \neq y \neq z \neq x$, and define $c_1 \in C_X$ as follows: $c_1(\{x\}) = c_1(\{x,y\}) = \{x\}$, $c_1(\{y\}) = c_1(\{y,z\}) = \{y\}$, $c_1(\{z\}) = c_1(\{x,z\}) = \{z\}$, and $c_1(\{x,y,z\}) = \varnothing$. It is easily checked that, by construction, $c_1$ satisfies both BNC and WBCC. However, $c_1(\{x,y,z\}) = \varnothing$ hence $c_1$ does not satisfy PR. Next, take $c_2 \in C_X$ defined as follows: $c_2(\{x\}) = c_2(\{x,y\}) = \{x\}$, $c_2(\{y\}) = c_2(\{y,z\}) = \{y\}$, $c_2(\{z\}) = c_2(\{x,z\}) = \{z\}$, and $c_2(\{x,y,z\}) = \{x,y,z\}$. It is immediately checked that $c_2$ satisfies PR and WBCC, but violates BNC. Furthermore, consider again $X = \{x,y,z\}$ and define $c_3 \in C_X$ as follows: $c_3(\{x\}) = c_3(\{x,y\}) = c_3(\{x,y,z\}) = \{x\}$, $c_3(\{y\}) = c_3(\{y,z\}) = \{y\}$, $c_3(\{z\}) = \{z\}$, and $c_3(\{x,z\}) = \{x,z\}$. It is easily checked that $c_3$ satisfies both PR and BNS, but violates WBCC. Hence, PR, BNC and WBCC are mutually independent properties.*

It should be remarked here that *any single-valued choice function* does satisfy PR, BNS and WBCC and is therefore a VNM-solution i.e. $C_X^1 \subset C_X^{s^\circ}$.

Let us now turn to the *general* case, i.e. to the characterization problem for choice functions in $C_X$ that are VNMS-rationalizable by some (possibly *not* VNM-perfect) dominance digraph $(X, \Delta)$. We shall provide here a characterization of *all* VNMS-rationalizable choice functions (including the very important subclass of VNMS-rationalizable *tournament* solutions).

**Theorem 19** *Let $c \in C_X$. Then $c$ is VNMS-rationalizable by a dominance digraph $(X, \Delta)$ iff it satisfies ND, 2-PR, BNC, WBCC and RVNMF.*

**Proof.** Let $c \in C_X$ be VNMS-rationalizable, namely, suppose there exists a dominance digraph $(X, \Delta)$ such that $c(Y) \in \mathcal{S}(Y, \Delta_Y)$ for any $Y \subseteq X$ if $\mathcal{S}(Y, \Delta_Y) \neq \varnothing$, and $c(Y) = \varnothing$ otherwise. Then, take any $A \subseteq X$ such that $1 \leq \#A \leq 2$. Clearly, if $x, y \in A$ then there are four cases to consider: (i) $x\Delta_{\{x,y\}}y$, (ii) $y\Delta_{\{x,y\}}x$, (iii) $\Delta_{\{x,y\}} = \varnothing$, (iv) $x\Delta_{\{x,y\}}y$ and $y\Delta_{\{x,y\}}x$ (in particular, case (iii) always applies if $x = y$).

Then, if $x = y$ we have $\mathcal{S}(\{x\}, \Delta_{\{x\}}) = \{\{x\}\}$ hence $c(\{x\}) = \{x\}$ i.e. $c$ satisfies ND.

Moreover, if $x \neq y$ then, respectively, $\mathcal{S}(\{x, y\}, \Delta_{\{x,y\}}) = \{\{x\}\}$, $\mathcal{S}(\{x, y\}, \Delta_{\{x,y\}}) = \{\{y\}\}$, $\mathcal{S}(\{x, y\}, \Delta_{\{x,y\}}) = \{\{x, y\}\}$, and $\mathcal{S}(\{x, y\}, \Delta_{\{x,y\}}) = \{\{x\}, \{y\}\}$.

In any case, it follows that $\mathcal{S}(\{x, y\}, \Delta_{\{x,y\}}) \neq \varnothing$ and therefore $c(\{x, y\}) \in \mathcal{S}(\{x, y\}, \Delta_{\{x,y\}})$. Now, $\varnothing \notin \mathcal{S}(\{x, y\}, \Delta_{\{x,y\}})$ by external stability of VNM-stable sets, hence $c(\{x, y\}) \neq \varnothing$, and 2-PR is also satisfied.

Next, consider any $A \subseteq X$ and $x, y \in A$, such that $\{x, y\} \subseteq c(A)$. Then, by our hypothesis, $\mathcal{S}(A, \Delta_A) \neq \varnothing$ and $c(A) \in \mathcal{S}(A, \Delta_A)$. It follows that, by internal stability of VNM-stable sets, *neither* $x\Delta_A y$ nor $y\Delta_A x$ i.e. in particular *not* $x\Delta_{\{x,y\}}y$ and *not* $y\Delta_{\{x,y\}}x$. Thus $\mathcal{S}(\{x, y\}, \Delta_{\{x,y\}}) = \{\{x, y\}\}$ whence $c(\{x, y\}) = \{x, y\}$, and BNC is also satisfied by $c$.

Moreover, take any $B \subseteq X$ such that $c(B) \neq \varnothing$. Then, by hypothesis, $c(B) \in \mathcal{S}(B, \Delta_B)$ hence, by external stability of VNM-stable sets, for any $y \in c^c(B)$ there exists $x_y \in c(B)$ such that $x_y\Delta_B y$ i.e. $x_y\Delta y$. Thus, $y \in c^c\{x_y, y\}$ whenever $x_y \in c(\{x_y, y\})$, and WBCC is satisfied by $c$.

Finally, let $(X, \Delta)$ be a dominance digraph such that $c$ is VNMS-rationalizable, and suppose $c$ fails to satisfy RVNMF. Then, there exists a nonempty $A \subseteq X$ such that such that $c(A) = \varnothing$ and $\mathcal{S}(A, \Delta_A) \neq \varnothing$ for any $\Delta \in \mathcal{B}^{VNM}(c)$, a contradiction since by hypothesis $c$ is VNMS-rationalizable.

Conversely, let $c \in C_X$ satisfy ND, 2-PR, BNS and WBCC, and RVNMF. Thus Lemma 15 above applies again, and for any $B \subseteq X$, if $c(B) \neq \varnothing$ then in particular there exists a dominance digraph such that $\mathcal{S}(B, \Delta_B) \neq \varnothing$ and $c(B) \in \mathcal{S}(B, \Delta_B)$, whence $c(B) \notin \mathcal{S}(B, \Delta_B)$ only if $c(B) = \varnothing$. Moreover, by RVNMF, for any nonempty $A \subseteq X$ such that $c(A) = \varnothing$ there exists a $\Delta' \in \mathcal{B}^{VNM}(c)$ such that $\mathcal{S}(A, \Delta'_A) = \varnothing$. Therefore, by definition of $\mathcal{B}^{VNM}(c)$, $c$ is VNMS-rationalizable by $(X, \Delta')$. ■

**Remark 20** *Notice that the foregoing characterization is also tight: indeed, consider again without loss of generality a set $X$ such that $\#X = 3$, and $c_4, c_5 \in C_X$ as defined below: $c_4(\{x\}) = c_4(\{x, y\}) = c_4(\{x, z\}) = \{x\}$, $c_4(\{y\}) = c_4(\{y, z\}) = \{y\}$, $c_4(\{z\}) = \{z\}$, and $c_4(\{x, y, z\}) = \varnothing$; $c_5(\{x\}) = c_5(\{x, y\}) = c_5(\{x, z\}) = \{x\}$, $c_5(\{y\}) = c_5(\{y, z\}) = \{y\}$, $c_5(\{z\}) = \{z\}$, and $c_5(\{x, y, z\}) = \{x\}$. By construction, both $c_4$ and $c_5$ do satisfy 2-CPR, BNS and WBCC. However, $c_4$ fails to satisfy RVNMF since for any $A \subseteq X$, $c_5(A) \neq \varnothing$ whenever $c_4(A) \neq \varnothing$, but $c_4(\{x, y, z\}) = \varnothing$ while $c_5(\{x, y, z\}) \neq \varnothing$.*

*Moreover, take any dominance digraph* $(X, \Delta)$ *such that there exist three distinct* $x, y, z \in X$ *with* $\Delta_{\{x,y,z\}} = \{(x, y), (y, z)\}$, *and consider* $c_6, c_7, c_8, c_9 \in C_X$ *as defined according to the following rules: for any nonempty* $A \subseteq X$,

$c_6(A) \in \mathcal{S}(A, \Delta_A)$ *if* $\mathcal{S}(A, \Delta_A) \neq \varnothing$ *and* $A \neq \{z\}$, *and* $c_6(\{z\}) = \varnothing$ *otherwise*

$c_7(A) \in \mathcal{S}(A, \Delta_A)$ *if* $\mathcal{S}(A, \Delta_A) \neq \varnothing$ *and* $A \neq \{y, z\}$, *and* $c_7(A) = \varnothing$ *otherwise*

$c_8(A) = TC(A, \Delta_A) \cup \{y \in A$*: for all* $z \in TC(A, \Delta_A)$, *not* $z\Delta_A y\}$, *where*

$$TC(A, \Delta_A) = \left\{ \begin{array}{c} x \in A\text{: for any } y \in A \text{ there exists a positive integer } k \\ \text{and } z_1, .., z_k \in A \\ \text{such that } x = z_1, .., z_k = y, \\ \text{and } z_i \Delta z_{i+1} \text{ for any } i,\ 1 \leq i \leq k-1 \end{array} \right\}$$

$c_9(A) = \{x \in A$*: for all* $y \in A$, *not* $y\Delta_A x\ \}$

*Finally, posit* $X = \{x, y, z\}$,

$c_{10}(\{u\}) = \{u\}$ *for any* $u \in X$, $c_{10}(\{x, y\}) = c_{10}(\{x, z\}) = \{x\}$, $c_{10}(\{y, z\}) = \{y\}$, $c_{10}(X) = \varnothing$,

*and*

$c_{11}(Y) = c_{10}(Y)$ *for any* $Y \subset X$, $c_{11}(X) = \{x\}$.

*Then, it is immediately checked that* $c_6$ *satisfies satisfies 2-PR, BNC and WBCC, but violates ND,* $c_7$*satisfies ND, BNC and WBCC, but violates 2-PR,* $c_8$ *satisfies ND, 2-PR and WBCC but violates BNC,* $c_9$ *satisfies ND, 2-PR and BNC but violates WBCC,* $c_{10}$ *satisfies ND, 2-PR, BNC and WBCC but violates RVNMF since* $c_{10}$ *and* $c_{11}$ *are VNMS-homologous with respect to dominance digraph* $(X, \{(x, y), (x, z), (y, z)\})$ *on* $\mathcal{P}(X) \smallsetminus \{X\}$, *and* $c_{10}(X) = \varnothing \neq c_{11}(X)$.

# 5 Posets of revealed VNM-solutions

Let us now turn to a global description of the order-theoretic structure of the class of all revealed VNM-solutions (of all *nonempty-valued* revealed VNM-solutions, respectively).

A partially ordered set or *poset* is a pair $\mathbf{C} = (C, \leqslant)$ where $C$ is a set and $\leqslant$ is a reflexive, transitive and antisymmetric binary relation on $C$ (i.e. for any $x \in C$, $x \leqslant x$ and for any $x, y, z \in C$, $x \leqslant z$ whenever $x \leqslant y$ and $y \leqslant z$, and $x = y$ whenever $x \leqslant y$ and $y \leqslant z$). A *coatom* of a poset $\mathbf{C} = (C, \leqslant)$ with a top element or maximum $1_C$ is any $j \in C$ which is covered by $1_c$- written

$j \lessdot 1_C$- i.e. $j < 1_C$ and $l = j$ for any $l \in C$ such that $j \leqslant l < 1_C$. The set of all coatoms of $\mathbf{C}$ is denoted $A^*_C$.

A poset $\mathbf{C} = (C, \leqslant)$ is a *meet semilattice* (*join semilattice*, respectively) if for any $x, y \in C$ the $\leqslant$-greatest lower bound $x \wedge y$ (the $\leqslant$-least upper bound $x \vee y$, respectively) of $\{x, y\}$ does exist. Moreover, $\mathbf{C}$ is a *lattice* if it is both a meet semilattice and a join semilattice. A lattice $\mathbf{C} = (C, \leqslant)$ is *bounded* if there exist both a bottom element $0_C$ and a top element $1_C$ (hence in particular a finite lattice is also bounded), *distributive* iff $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for any $x, y, z \in C$, *complemented* if it is bounded and for any $x \in C$ there exists $x' \in C$ such that $x \vee x' = 1_C$ and $x \wedge x' = 0_C$, and *Boolean* iff it is both distributive and complemented.

The set $C_X$ of all choice functions on $X$ can be endowed in a natural way with the point-wise set inclusion partial order $\leqslant$ by positing, for any $c, c' \in C_X$, $c \leqslant c'$ iff $c(A) \subseteq c'(A)$ for each $A \subseteq X$. Clearly, the identity operator $c^{id_X}$ is its top element, and the constant empty-valued choice function $c^{\emptyset_X}$ its bottom element. It is well-known, and easily checked, that $(C_X, \leqslant)$ is in fact a *Boolean lattice* with $\vee = \cup$ (i.e. set-union) and $\wedge = \cap$ (i.e. set-intersection): see e.g. Monjardet, Raderanirina (2004). The class of choice functions $C^- = \bigcup \{C^{-xy} : x, y \in X, x \neq y\}$, that will also be used in the ensuing analysis, is defined as follows: for any $x, y \in X$ such that $x \neq y$, and any $c \in C_X$, $c \in C^{-xy}$ iff for all $A \subseteq X$, $c(A) = A \smallsetminus \{y\}$ or $c(A) = A \smallsetminus \{x\}$ if $\{x, y\} \subseteq A$, and $c(A) = A$ otherwise.

Now, let $C^s_X \subseteq C_X$ denote the set of all revealed VNM-solutions on $X$, and $C^{s^\circ}_X = C^s_X \cap C^\circ_X$ the set of all nonempty–valued VNM-solutions on $X$. We also denote by $(C^s_X, \leqslant')$ and $(C^{s^\circ}_X, \leqslant'')$ the corresponding subposets of $(C_X, \leqslant)$ (i.e. $\leqslant' = \leqslant \cap (C^s_X \times C^s_X)$, and $\leqslant'' = \leqslant \cap (C^{s^\circ}_X \times C^{s^\circ}_X)$, respectively). It turns out that both $(C^s_X, \leqslant')$ and $(C^{s^\circ}_X, \leqslant'')$ share with $(C_X, \leqslant)$ their top element but neither of them is a sub-$\cap$-semilattice or a sub-$\cup$-semilattice of $(C_X, \leqslant)$, namely

**Theorem 21** *(i) The poset $(C^{s^\circ}_X, \leqslant'')$ has a maximum, $c^{id_X}$; $C^-$ as defined above is the set of its coatoms, and $C^1_X$ is the set of its minimal elements. However, $(C^{s^\circ}_X, \leqslant'')$ is neither a sub-$\cap$-semilattice nor a sub-$\cup$-semilattice of $(C_X, \leqslant)$; (ii) The poset $(C^s_X, \leqslant')$ has a maximum, $c^{id_X}$, and $C^-$ is the set of its coatoms. However, $(C^s_X, \leqslant')$ is neither a sub-$\cap$-semilattice nor a sub-$\cup$-semilattice of $(C_X, \leqslant)$.*

**Proof.** (i) First, recall that $c^{id_X}$ is VNMS-rationalizable by dominance digraph $(X, \varnothing)$ hence it is the maximum of $(C_X^{s^\circ}, \leqslant'')$.

Now, let $c \in C^-$ namely there exist $x, y \in X$ such that $c \in C^{-xy}$ as defined above, and suppose that $c$ is *not* a coatom of $(C_X^{s^\circ}, \leqslant'')$. Then, there exists a coatom $c'$of $(C_X^{s^\circ}, \leqslant'')$ such that $c \leqslant c'$ and $c(A) \subset c'(A)$ for some $A \subseteq X$. Since $c'$ is a coatom of $(C_X^{s^\circ}, \leqslant'')$ there also exists a $B \subseteq X$ such that $c(B) \subseteq c'(B) \subset B$. However, by definition, $\#(B \smallsetminus c(B)) \leq 1$ whence $c(B) = c'(B)$ for any such $B$. It follows that $c(A) \subset c'(A)$ only if $c'(A) = A$. By definition of $c$, it follows that there exists $A \subseteq X$ such that $\{x, y\} \subseteq A$, $c(A) \in \{A \smallsetminus \{x\}, A \smallsetminus \{y\}\}$and $c'(A) = A$, while for some $B \subseteq X$, $c'(B) = c(B) \in \{A \smallsetminus \{x\}, A \smallsetminus \{y\}\}$. Suppose, without any loss of generality, that $c'(B) = c(B) = A \smallsetminus \{y\}$. It follows, by WBCC, that there exists $z \in c'(B)$ such that $y \in c'^c(\{y, z\})$ whenever $z \in c'(\{y, z\})$. By the previous observation, $y \in c'^c(\{y, z\})$ entails $c'^c(\{y, z\}) = c^c(\{y, z\})$ i.e. $c'(\{y, z\}) = c(\{y, z\}) \neq \{y, z\}$: thus, since by hypothesis $c \in C^{-xy}$, it must be the case that $z = x$, and $c'(\{x, y\}) = c(\{x, y\}) = \{x\}$. Hence, $c'(\{x, y\}) = \{x\}$ while there exists $A \subseteq X$ such that $\{x, y\} \subseteq A$, and $c'(A) = A$, contradicting BNC. As a result, any $c \in C^-$ is a coatom of $(C_X^{s^\circ}, \leqslant'')$.

Conversely, let $c$ be a coatom of $(C_X^{s^\circ}, \leqslant'')$ i.e. there exists $A \subseteq X$ such that $\varnothing \neq c(A) \subset A$, and for any $c' \in C_X^{s^\circ}$, if $c(B) \subseteq c'(B)$ for all $B \subseteq X$ then $c = c'$. Notice that by ND $\#A \geq 2$. Suppose now that $c \notin C^-$. Then, for any $x, y \in X$ and any $c' \in C^{-xy}$ there exists a $B' \subseteq X$ such that $c'(B') \subset c(B')$: by definition of $C^{-xy}$, it must be the case that $\{x, y\} \subseteq B'$, $c'(B') \in \{B' \smallsetminus \{x\}, B' \smallsetminus \{y\}\}$ hence $c(B') = B'$. Now, consider any $y \in A$ such that $y \notin c(A)$: by WBCC, there exists $x \in c(A)$ such that $y \notin c(\{x, y\})$ whenever $x \in c(\{x, y\})$. But then, by BNC, there is no $B \subseteq X$ such that $\{x, y\} \subseteq c(B)$, a contradiction since $\{x, y\} \subseteq B' = c(B')$. Therefore, any coatom of $(C_X^{s^\circ}, \leqslant'')$ belongs to $C^-$.

Furthermore, recall that *any* single-valued choice function $c \in C_X^1$ is VNMS-rationalizable by the complete dominance digraph $(X, (X \times X)^{-diag})$: see Example 3 above. Then, observe that any $c \in C_X^1$ is $\leqslant$-minimal in $C^\circ$, hence in $C_X^{s^\circ}$ as well. Moreover, notice that for any two *distinct* $c_1, c_2 \in C_X^1$, $c_1 \cap c_2 \notin C_X^\circ$ hence in particular $c_1 \cap c_2 \notin C_X^{s^\circ}$. Also, to check that $C_X^{s^\circ}$ is not $\cup$-closed, consider $c_1, c_2 \in C_X^1$ and three *distinct* $x, y, z \in X$ such that $c_1(\{x, y, z\}) = \{x\}$, $c_2(\{x, y, z\}) = \{y\}$ and $c_1(A) = c_2(A)$ for any $A \subseteq X$, $A \neq \{x, y, z\}$. Then, $(c_1 \cup c_2)(\{x, y, z\}) = \{x, y\}$, while $\#(c_1 \cup c_2)(A) = 1$ for any $A \subseteq X$, $A \neq \{x, y, z\}$. It follows that $c_1 \cup c_2$ violates BNS, hence $c_1 \cup c_2 \notin C_X^s$ (and therefore $c_1 \cup c_2 \notin C_X^{s^\circ}$).

(ii) The proof of point (i) also establishes that $c^{id_X}$ is the maximum of $(C_X^s, \leqslant')$, and that any $c \in C^-$ is a coatom of $(C_X^s, \leqslant')$. Now, suppose that there exists a coatom $c^*$ of $(C_X^s, \leqslant')$ such that $c^* \notin C_X^{s^\circ}$ (thus in particular $c^* \notin C^-$). Hence, by definition, there exists a nonempty $A \subseteq X$ such that $c^*(A) = \varnothing$ and, for any $x, y \in X$, $c^* \notin C^{-xy}$ i.e. there exists a $B_{xy} \subseteq X$ such that $\{x, y\} \subseteq c^*(B_{xy})$. Since $c^* \in C_X^s$, there exists a dominance digraph $(X, \Delta)$ such that $c^*(B_{xy}) \in \mathcal{S}(B_{xy}, \Delta_{B_{xy}})$: thus by internal stability $\Delta_{B_{xy}} \cap \{(x,y),(y,x)\} = \Delta \cap \{(x,y),(y,x)\} = \varnothing$. But then, $\Delta = \varnothing$: it follows that $c^* = c^{id_X}$, a contradiction.

Moreover, notice that the proof of point (i) establishes that $C_X^s$ is not $\cup$-closed. To check that $C_X^s$ is not $\cap$-closed either, take any $c, c' \in C_X^1$ such that there exist two *distinct* $x, y \in X$ with $c(\{x,y\}) = \{x\}$ and $c'(\{x,y\}) = \{y\}$, and notice that, by construction, both $c$ and $c'$ do satisfy ND, 2-PR, BNC, WBCC and RVNMF. However, $(c \cap c')(\{x,y\}) = \varnothing$, hence $c \cap c'$ violates 2-PR and therefore $c \cap c' \notin C_X^s$. ■

**Remark 22** *Notice that the foregoing Theorem only mentions a subclass of minimal elements of $(C_X^s, \leqslant')$, but does not provide a general, explicit description of the set of* all *minimal elements of that poset. To be sure, some general statement about the latter can be easily established. For instance, it is certainly a minimal element of $C_X^s$ any $c \in C_X^s$ such that i) its improper subdomain $D_c^\varnothing = \{Y \subseteq X : c(Y) = \varnothing\}$ is set-inclusion maximal among the improper subdomains of its fellow members of $C_X^s$ and ii) $c(Y)$ is single-valued whenever it is nonempty.*

*However, there may exist minimal elements of $(C_X^s, \leqslant')$ other than those described above. To check this statement, observe that if $\#X = 3$ then the set of minimal elements of the latter poset does indeed consist of those $c \in C_X$ such that $c(X) = \varnothing$, $\#c(Y) = 1$ for any nonempty $Y \subset X$, and $\{c(Y) : Y \subseteq X, \#Y = 2\} = X$, which are VNMS-rationalizable by 3-cyclic dominance digraphs. On the other hand, if $\#X = 4, X = \{x,y,z,w\}$ then the set of minimal elements of $(C_X^s, \leqslant')$ includes among others any* single-valued *$c^* \in C_X$ such that $c^*(Y) = \{w\}$ whenever $w \in Y \subseteq X$, $c^*(\{x,y,z\}) = \varnothing$, and $\{c^*(Y) : Y \subseteq X, \#Y = 2\} = X$ , which is VNMS-rationalizable via*

$$\Delta = \left\{ \begin{array}{c} (w,x),(w,y),(w,z), \\ (c^*(\{x,y\}), c^{*c}(\{x,y\})), (c^*(\{y,z\}), c^{*c}(\{y,z\}), \\ (c^*(\{x,z\}), c^{*c}(\{x,z\})) \end{array} \right\},$$

*as well as some other single-valued $c \in C_X$, such that $c(X) = c(\{x,y,z\}) = \varnothing$.*

*Consider for instance c defined as follows:* $c(X) = c(\{x, y, z\}) = c(\{x, y, w\}) = \varnothing$, $c(\{x, z, w\}) = \{w\}$, $c(\{y, z, w\}) = \{y\}$, $c(\{x, y\}) = \{x\}$, $c(\{x, z\}) = \{z\}$, $c(\{x, w\}) = \{w\}$, $c(\{y, z\}) = \{y\}$, $c(\{y, w\}) = \{y\}$, $c(\{z, w\}) = \{w\}$ *which is VNMS-rationalizable via* $\Delta = \{(x, y), (y, z), (z, x), (w, x), (w, z), (y, w)\}$. *Clearly,* $\{Y \subseteq X : c^*(Y) = \varnothing\} \subset \{Y \subseteq X : c(Y) = \varnothing\}$.

# 6 Related literature

As mentioned in the Introduction, and to the best of the author's knowledge, the extant literature on revealed VNM-solutions essentially reduces to Wilson (1970) and Suzumura (1983).

In order to characterize revealed VNM-solutions within the class of *proper partial* choice functions, Wilson (1970) introduces a condition that we may reformulate for arbitrary (total) choice functions as follows

*Closure with respect to Full Choice-Set Revealed Conjugation (CFCRC):* for any $A \subseteq X$ such that $c(A) \neq \varnothing$,

$$\left\{x \in A : x\widehat{R}(c)y \text{ for all } y \in c(A)\right\} \subseteq c(A).$$

Relying on Wilson's seminal ideas, and in a similar vein, Suzumura (1983) considers a related but *stronger* property, namely

*Closure with respect to Full Choice-Set Revealed Dominance (CFCRD):* for any $A \subseteq X$ such that $c(A) \neq \varnothing$,

$$\{x \in A : xR(c)y \text{ for all } y \in c(A)\} \subseteq c(A).$$

Notice that CFCRC and CFCRD are somewhat opaque and considerably more convoluted than the properties of choice functions typically described and used in the previous sections (except for the 'boundary' condition RVNMF). Moreover, they are apparently more ad hoc than the latter when it comes to characterizing revealed VNM-solutions, since both of them make an *explicit* mention of 'revealed' binary relations, imposing restrictions on their relationships to choice sets.

Concerning CFCRC, the following Claim establishes its equivalence - under 2-PR- to a combination of two simpler properties already considered in Section 2, namely

**Claim 23** *Let $c \in C_X$ satisfy 2-PR. Then $c$ satisfies CFCRC iff it satisfies both BNC and WBCC.*

**Proof.** Let $c \in C_X$ satisfy 2-PR and CFCRC, and take any $A \subseteq X$ and $x, y \in A$ such that $\{x, y\} \subseteq c(A)$. Then, by definition, $\widehat{R}(c) = \{x, y\} \times \{x, y\}$ whence, by CFCRC, $c(\{x, y\}) = \{x, y\}$, and BNC holds. Moreover, let $B \subseteq X$ such that $c(B) \neq \varnothing$, and take any $x \in c^c(B)$ (if $c(B) = B$ there is nothing to prove). Then, by CFCRC, there exists $y_x \in c(B)$ such that *not* $x\widehat{R}(c)y_x$ i.e. there is no $A \subseteq X$ such that $\{x, y_x\} \subseteq c(A)$. Thus, in particular *not* $\{x, y_x\} \subseteq c(\{x, y_x\})$ whence, by 2-PR, $x \in c^c(\{x, y_x\})$ whenever $y_x \in c(\{x, y_x\})$, and WBCC follows.

Conversely, let $c$ satisfy both 2-PR, BNC and WBCC, $A \subseteq X$, and $x \in A$ such that $x\widehat{R}(c)y$ for any $y \in c(A)$ i.e. for any $y \in A$ there exists $A_{xy} \subseteq X$ with $\{x, y\} \subseteq c(A_{xy})$. Then, by BNC and 2-PR, $c(\{x, y\}) = \{x, y\}$ for any $y \in c(A)$. Now, suppose that $x \notin c(A)$. Then by WBCC there exists $y \in c(A)$ such that $x \in c^c(\{x, y\})$ whenever $y \in c(\{x, y\})$, a contradiction. It follows that $x \in c(A)$, and CFCRC holds. ■

Thus, we obtain as an immediate Corollary to Theorem 17 and Claims 2 and 23 the next Proposition, that includes (a restatement in $C_X$ of) Wilson's characterization of nonempty-valued revealed VNM-solutions:

**Proposition 24** *(see also Wilson (1970)) Let $c \in C_X$. Then, the following statements are equivalent:*

*(i) $c$ satisfies PR and is VNMS-rationalizable by a dominance digraph $(X, \Delta)$;*

*(ii) $c$ satisfies PR and is VNMS-rationalizable by $(X, \Delta^{\widehat{R}(c)})$;*

*(iii) $c$ satisfies PR and is a fixed point solution of $\forall$-type for the (reflexive) digraph $(X, \widehat{R}(c))$;*

*(iv) $c$ satisfies PR and is a fixed point solution of $\forall$-type;*

*(v) $c$ satisfies PR and CFCRC;*

*(vi) $c$ satisfies PR, BNC and WBCC.*

**Proof.** (i)$\Longleftrightarrow$(ii): See Lemma 14 and Theorem 17 above;

(ii)$\Longleftrightarrow$(iii): See Claim 2 above;

(iii)$\Longleftrightarrow$(iv): If $c$ satisfies PR and is a fixed point solution of $\forall$-type then, by Claim 2 and the equivalence between points (i) and (ii) above, it must be the case that $c$ is also VNMS-rationalizable by $(X, \Delta^{\widehat{R}(c)})$. Thus, by Claim 2 again, $c$ is a fixed point solution of $\forall$-type for the reflexive digraph $(X, \widehat{R}(c))$. The converse implication is trivial;

(iv)$\Longleftrightarrow$(v): If $c$ satisfies PR and is a fixed point solution of $\forall$-type then by definition there exists an $R \subseteq X \times X$ such that $c(Y) = \{x \in Y : xRy \text{ for all } y \in c(Y)\}$ hence in particular $\{x \in Y : xRy \text{ for all } y \in c(Y)\} \subseteq c(Y)$ for all nonempty $Y \subseteq X$, and conversely. In particular, $R$ must be reflexive, otherwise there exists $x \in X$ such that $c(\{x\}) = \varnothing$, a contradiction since $c$ satisfies PR;

(v)$\Longleftrightarrow$(vi): See Claim 23 above;

(i)$\Longleftrightarrow$(vi): See Theorem 17 above. ■

Notice that Wilson's own original result on nonempty-valued revealed VNM- solutions -just sketched with no explicit proof in Wilson (1970), and scarcely mentioned in the subsequent, otherwise very comprehensive, Suzumura's monograph (Suzumura (1983))- consists of the equivalence between statements (iii), (iv), (v): the equivalence between (i), (ii), and (iii) is just taken for granted - with $(X, \widehat{R}(c))$ regarded as an 'undominance' (i.e. reflexive) digraph- but not spelled out in any detail. Needless to say, the equivalence between (vi) and the former statements is not considered at all in any of those works.

Another immediate consequence of Claim 23 is an alternative, equivalent formulation of Theorem 19, namely

**Proposition 25** *Let $c \in C_X$ . Then $c$ satisfies ND, 2-PR, CFCRC and RVNMF iff $c$ is VNMS-rationalizable by a dominance digraph $(X, \Delta)$.*

**Proof.** Straightforward, by Theorem 19 and Claim 23 above. ■

The implications of all of the above are quite clear: essentially, Wilson's characterization may be lifted to *non-proper* choice functions just *introducing two supplementary local-nonemptiness conditions for choice sets -namely ND and 2-PR- and the boundary condition RVNMF for empty choice sets.*

Concerning CFCRD, Suzumura himself proceeds to unpack that condition in terms of simpler, more transparent properties (see Suzumura (1983)). We find it useful to restate and slightly enrich Suzumura's result in our own setting, offering a further characterization of the subclass of revealed VNM-solutions it covers. This allows us to gain another perspective on -and perhaps a more immediate appreciation of - the boundaries of Suzumura's theorem, which amounts to the equivalence between statements (iv),(v),(vi) and (vii) of the following

**Proposition 26** *(see also Suzumura (1983)) Let $c \in C_X$. Then, the following statements are equivalent:*

*(i) c satisfies PR and is VNMS-rationalizable by an* asymmetric *dominance digraph $(X, \Delta)$ such that $\mathbb{C}(Y, \Delta_Y) \in \mathcal{S}(Y, \Delta_Y)$ for any $Y \subseteq X$;*

*(ii) c satisfies PR and is a fixed point solution of $\forall$-type for the (total) digraph $(X, R^{\Delta})$;*

*(iii) c satisfies PR and is a fixed point solution of $\forall$-type for a total digraph;*

*(iv) c satisfies PR, and there exists a quasitransitive $R \subseteq X \times X$ -i.e. $R^a$ is transitive- such that $c(Y) = \max R_Y$ for any $Y \subseteq X$;*

*(v) c satisfies PR and is VNMS-rationalizable by $(X, (R(c))^a)$;*

*(vi) c satisfies PR and CFCRD;*

*(vii) c satisfies PR, C, SS and CO.*

**Proof.** (i)$\Longleftrightarrow$(ii) It follows from Claim 2, noticing that $\Delta$ is asymmetric iff for any $x, y \in X$, *not* $[x\Delta y$ and $y\Delta x]$ i.e. either *not* $x\Delta y$ or *not* $y\Delta x$, namely iff $R^{\Delta}$ is total;

(ii)$\Longrightarrow$(iii) This implication is trivial;

(iii)$\Longrightarrow$(i) Again from Claim 2, and by the same argument of the previous equivalence between points (i) and (ii), applied to a total $R$ and the corresponding asymmetric $\Delta^R$;

(i)$\Rightarrow$(iv) Since, as it is well-known, whenever the core of $(Y, \Delta_Y)$ is a VNM-stable set of $(Y, \Delta_Y)$ it is also its *unique* VNM-stable set, it follows that $c(Y) = \mathbb{C}(Y, \Delta_Y)$ for any $Y \subseteq X$, hence $c$ is in particular the core-solution of $(X, \Delta)$ (and, by hypothesis, nonempty-valued). Then, consider $R^{\Delta}$ i.e. for any $x, y \in X$ posit $xR^{\Delta}y$ iff *not* $y\Delta x$. Clearly, by definition, $c(Y) = \max R^{\Delta}_Y$ for any $Y \subseteq X$. Now, suppose $(R^{\Delta})^a$ is not transitive: then, there exist $x, y, z \in X$ such that $x(R^{\Delta})^a y$, $y(R^{\Delta})^a z$, and *not* $x(R^{\Delta})^a z$ i.e. by definition again $x\Delta y$, *not* $y\Delta x$, $y\Delta z$, *not* $z\Delta y$, and -since $x\Delta z$ and $z\Delta x$ would imply $c(\{x, z\}) = \mathbb{C}(\{x, z\}, \Delta_{\{x,z\}}) = \varnothing$, a contradiction- both *not* $z\Delta x$ and *not* $x\Delta z$. But then, consider $c(\{x, y, z\})$. Clearly, $c(\{x, y, z\}) = \mathbb{C}(\{x, y, z\}, \Delta_{\{x,y,z\}}) = \{x\}$. However, $\mathcal{S}(\{x, y, z\}, \Delta_{\{x,y,z\}}) = \{\{x, z\}\}$ hence $c$ is not VNMS-rationalizable, a contradiction.

(iv) $\Longleftrightarrow$ (v) This is part of the original Suzumura's characterization result (see e.g. Wilson (1970) and Suzumura (1983), Theorem 2.10 (b), pp.36-38).

(v)$\Rightarrow$(i) Suppose that $c$ is VNMS-rationalizable by $(X, (R(c))^a)$, that is indeed an asymmetric dominance digraph. Let us now assume that $c$ is not a core-solution of $(X, (R(c))^a)$, i.e. there exists a $Y \subseteq X$ such that $c(Y) \neq \mathbb{C}(Y, (R(c))^a_Y)$. Since by hypothesis $\varnothing \neq c(Y) \in \mathcal{S}(Y, (R(c))^a_Y)$, and -by a well-known fact- the core of any dominance digraph $(Y, \Delta_Y)$ is a subset of any VNM-stable set of $(Y, \Delta_Y)$, it must then be the case that there exists $y \in Y \setminus c(Y) \subseteq Y \setminus \mathbb{C}(Y, (R(c))^a_Y)$ such that *not* $x(R(c))^a_Y y$, for any $x \in c(Y)$. Therefore, there exists $z \in Y \setminus c(Y)$ (hence $z \notin \mathbb{C}(Y, (R(c))^a_Y)$) such that $z(R(c))^a_Y y$. Moreover, since $z \notin \mathbb{C}(Y, (R(c))^a_Y)$, there also exists $u \in Y$ such that $u(R(c))^a z$. It follows that $u(R(c))^a y$, by transitivity of $(R)^a$, hence $u \notin c(Y)$ in view of our previous assumption about $y$. Thus, again, $u \notin \mathbb{C}(Y, (R(c))^a_Y)$ as well, and of course $u \neq z$ by construction. But then, we may apply the very same argument starting from $u$ to obtain a further $v \notin \{z, u\} \cup \mathbb{C}(Y, (R(c))^a_Y)$ and so on indefinitely to the effect of obtaining a contradiction, in view of finiteness of $X$.

(v)$\Longleftrightarrow$(vi) See Suzumura (1983), Theorem 2.10 (b).

(vi)$\Longleftrightarrow$(vii) This is also part of the original Suzumura's characterization result (see Suzumura (1983), Theorems 2.5, 2.6 and 2.7 pp. 32-34, and Theorem 2.10 (b), pp. 36-38). ■

It should be remarked here that, quite unsurprisingly, C, SS and CO amount to a considerably higher degree of choice-set-consistency than that exhibited by general VNMS-rationalizable choice functions. Indeed, as noticed above, the Suzumura-Wilson subclass of VNM-solutions does essentially correspond to the special case of revealed dominance digraphs inducing VNM-stable sets that are unique *because they are precisely the cores* (i.e. the sets of all *undominated* outcomes) of the relevant subdigraphs.

# 7 Concluding remarks

Choice functions which may be regarded as VNM-solutions of an underlying dominance digraph $(X, \Delta)$ have been characterized both under VNM-perfection of $(X, \Delta)$ and in the general case. Both characterizations combine restrictions on nonemptiness of the choice sets and weak contraction consistency requirements on choice sets and rejection sets. The characterization provided here for the general case makes use of a property that explicitly forbids the existence of VNM-stable sets of certain digraphs and is to that extent somewhat opaque. Finding a more transparent and elegant one is an open problem, whose solution might have to wait for further progress on the related characterization problem for kernel-perfect digraphs.

# 8 Appendix

**On a representation of dominance digraphs by games in coalitional and in strategic form**

A *coalitional game form* is a triple $\mathbf{G} = (N, X, E)$ where $N$ and $X$ are non-empty sets denoting the sets of players and outcomes, respectively, and $E : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}(X))$ is the coalitional power function: the 'power-value' $E(S)$ of coalition $S \subseteq N$ is the collection of all events $A \subseteq X$ coalition $S \subseteq N$ is able to 'force' (under some suitable interpretation of the latter notion). We also assume $\#N \geq 2$ and $\#X \geq 2$ in order to avoid trivialities. A coalitional game form $\mathbf{G} = (N, X, E)$ is a (standard) *effectivity function* if $E$ satisfies the following boundary conditions:

$EF1$) Souvereignty: $E(N) \supseteq \mathcal{P}(X) \setminus \{\emptyset\}$ ;

$EF2$) Null Set Normalization: $E(\emptyset) = \emptyset$;

$EF3$) Exhaustiveness: $X \in E(S)$ for any $S$, $\emptyset \neq S \subseteq N$.

$EF4$) Null Event Unenforceability: $\emptyset \notin E(S)$ for any $S$ , $\emptyset \subset S \subseteq N$ .

A CGF is $N$-*monotonic* iff for any $S, T \subseteq N$ and any $A \subseteq X$

$[A \in E(S)$ and $S \subseteq T$ entail $A \in E(T)]$,

$X$-*monotonic* iff for any $S \subseteq N$ and any $A, B \subseteq X$

$[A \in E(S)$ and $A \subseteq B$ entail $B \in E(S)]$, and *monotonic* iff it is both $N$-monotonic and $X$-monotonic. Moreover, it is *superadditive* iff for any $S, T \subseteq N$ and $A, B \subseteq X$, $A \in E(S)$, $B \in E(T)$ and $S \cap T = \emptyset$ entail $A \cap B \in E(S \cup T)$.

A *coalitional game* (with preference preorders) induced by coalitional game form $\mathbf{G} = (N, X, E)$ is a tuple $g = (N, X, E, (\succcurlyeq_i)_{i \in N})$ where for any $i \in N$, $\succcurlyeq_i \subseteq X \times X$ is a preorder i.e. a transitive and reflexive binary relation (with symmetric and asymmetric components denoted $\sim_i$and $\succ_i$, respectively).

A *strategic game* (with preference preorders) is a tuple

$\Gamma = (N, X, (S_i)_{i \in N}, h, (\succcurlyeq_i)_{i \in N})$ where $S_i$ denotes player $i$'s strategy set, and $h \in X^{\Pi_i S_i}$ denotes the strategic outcome function.

The $\alpha$-coalitional game attached to strategic game

$\Gamma = (N, X, (S_i)_{i\in N}, h, (\succcurlyeq_i)_{i\in N})$ is the tuple $g^\alpha(\Gamma) = (N, X, E^\alpha_\Gamma, (\succcurlyeq_i)_{i\in N})$ as defined by the following rule: for any $A \subseteq X$, $S \subseteq X$,

$$E^\alpha_\Gamma(S) = \left\{ \begin{array}{c} A \subseteq X\text{: a } t^S \in \prod_{i\in S} S_i \text{ exists such that} \\ h(t^S, s^{N\setminus S}) \in A \\ \text{for any } s^{N\setminus S} \in \prod_{i\in N\setminus S} S_i, \end{array} \right\}.$$

A coalitional game $g$ is *$\alpha$-playable* if there exists a strategic game such that $g = g^\alpha(\Gamma)$.

The following dominance digraphs can be attached in a natural way to any coalitional game $g = (N, X, E, (\succcurlyeq_i)_{i\in N})$:

(i) $(X, \Delta_g^{\alpha^*})$, where for any $x, y \in X$, $x, y \in X$, $x\Delta_g^{\alpha^*}y$ iff there exist $A \subseteq X$ and $S \subseteq N$ such that

$$x \in A \in E(S) \text{ and } z \succ_i y \text{ for all } i \in S \text{ and } z \in A$$

(ii) $(X, \Delta_g^{\beta^*})$, where for any $x, y \in X$, $x\Delta_g^{\beta^*}y$ iff there exist $A \subseteq X$ and $S \subseteq N$ such that

$$x \in A \in E(S) \text{ and } x \succ_i y \text{ for all } i \in S\ .$$

The following representation results can be easily established:

**Proposition 27** *(i) Let $g = (N, X, E, (\succcurlyeq_i)_{i\in N})$ be a coalitional game as defined above. Then both $(X, \Delta_g^{\alpha^*})$ and $(X, \Delta_g^{\beta^*})$ are dominance digraphs; (ii) let $(X, \Delta)$ be an asymmetric dominance digraph. Then, there exists a coalitional game with preference preorders $g = (N, X, E, (\succcurlyeq_i)_{i\in N})$ such that $\Delta = \Delta_g^{\alpha^*} = \Delta_g^{\beta^*}$. Moreover, $g$ is $\alpha$-playable .*

**Proof.** (i) Indeed, suppose there exists $x \in X$ with $x\Delta_g^\gamma x$ (where $\gamma \in \{a^*, \beta^*\}$). Then, in both cases there exist $A \subseteq X$ and $S \subseteq N$ such that $x \in A \in E(S)$, and $x \succ_i x$ for all $i \in S$, a contradiction.

(ii) Let $x, y \in X$, and $x\Delta y$. Then, fix a bijection $\sigma_{xy}$ on $X$ such that $\sigma_{xy}(x) = x, \sigma_{xy}(y) = y$, and take two distinct players $i_{xy}$, $j_{xy}$ with preference preorders $\succ_{i_{xy}}$and $\succ_{j_{xy}}$such that $x \succ_{i_{xy}} y \succ_{i_{xy}} [X \setminus \{x, y\}]$ and $[X \setminus \{x, y\}]^{-1} \succ_{i_{yx}} x \succ_{i_{yx}} y$ (where $[X \setminus \{x, y\}]$ denotes some fixed linear order on $X \setminus \{x, y\}$ and $[X \setminus \{x, y\}]^{-1}$its reverse order).

Next, posit $N^* = \cup_{(x,y)\in\Delta} \{i_{xy}, j_{xy}\}$, and define $E^* : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}(X))$ by the following rule: for any $A \subseteq X$ and $S \subseteq N^*$ , $A \in E^*(S)$ iff $S \neq \varnothing$ and $A = X$ or $\#S > \frac{\#N^*}{2}$ and $A \neq \varnothing$. Now, consider $g^* = (N^*, X, E^*, (\succcurlyeq_i)_{i\in N})$ (this is essentially a slightly generalized version of the standard McGarvey construction: see e.g. Laslier (1997), chpt.2). It is easily checked that, by construction, for each $x, y \in X$, if $x\Delta y$ and *not* $y\Delta x$ then $\#\{i \in N^* : x \succ_i y\} > \#\frac{N^*}{2} > \#\{i \in N^* : y \succ_i x\}$, while $\#\{i \in N^* : x \succ_i y\} = \#\frac{N^*}{2} = \#\{i \in N^* : y \succ_i x\}$ if either (*not* $x\Delta y$ and *not* $y\Delta x$) or ($x\Delta y$ and $y\Delta x$).

Therefore, if $x\Delta y$ then $\{x\} \in E^*(\{i \in N^* : x \succ_i y\})$ whence $x\Delta_{g^*}^{\alpha^*} y$ and $x\Delta_{g^*}^{\beta^*} y$. Conversely, if $x\Delta_{g^*}^{\alpha^*} y$ ($x\Delta_{g^*}^{\beta^*} y$, respectively) then by definition there exist $S \subseteq N^*$, $A \subseteq X$ such that $\#S > \#\frac{N^*}{2}$, $x \in A \in E(S)$, $u \succ_i y$ for all $i \in S$ and all $u \in A$ ($x \succ_i y$ for all $i \in S$, respectively), whence in any case $x\Delta y$ by definition of $E^*$.

Moreover, it is easily checked that $E^*$ is in fact a monotonic and superadditive effectivity function. Hence $g^*$ is $\alpha$-playable by Theorem 5.6 in Otten, Borm, Storcken and Tijs (1995). ■

# References


[1] Aizerman M., Aleskerov F. (1995): *Theory of Choice.* North Holland: Amsterdam.

[2] Danilov V., Koshevoy G. (2009): Choice functions and extensive operators, *Order* 26, 69-94.

[3] Echenique F. (2007): Counting combinatorial choice rules, *Games and Economic Behavior* 58, 230-245.

[4] Ehlers L., Sprumont Y. (2008): Weakened WARP and top-cycle choice rules, *Journal of Mathematical Economics* 44, 87-94.

[5] Galambos A. (2005): Revealed preference in game theory, mimeo Kellogg School of Management, Northwestern University.

[6] Ghoshal J., Laskar R., Pillone D. (1998): Topics on domination in directed graphs, in T.W. Haynes, S.T. Hedetniemi, P.J. Slater (eds.): *Domination in Graphs. Advanced Topics.* M.Dekker: New York.

[7] Greenberg J. (1990): *The Theory of Social Situations. An Alternative Game-Theoretic Approach.* Cambridge University Press: Cambridge UK.

[8] Harary F., Norman R.Z., Cartwright D. (1965): *Structural Models. An Introduction to the Theory of Directed Graphs.* J.Wiley: New York.

[9] Laslier J.-F. (1997): *Tournament Solutions and Majority Voting.* Springer: Berlin.

[10] Lombardi M. (2008): Uncovered set choice rules, *Social Choice and Welfare* 31, 271-279.

[11] Lombardi M. (2009): Minimal covering set solutions, *Social Choice and Welfare* 32, 687-695.

[12] McGarvey D.C. (1953): A theorem on the costruction of voting paradoxes, *Econometrica* 21, 147-160.

[13] Monjardet B., Raderanirina V. (2004): Lattices of choice functions and consensus problems, *Social Choice and Welfare* 23, 349-382.

[14] Moulin H. (1985): Choice functions over a finite set: a summary, *Social Choice and Welfare* 2, 147-160.

[15] Moulin H. (1986): Choosing from a tournament, *Social Choice and Welfare* 3, 271-291.

[16] Otten G.-J., Borm P., Storcken T., Tijs S. (1995): Effectivity functions and associated claim game correspondences, *Games and Economic Behavior* 9, 172-190.

[17] Ray I., Zhou L. (2001): Game theory via revealed preferences, *Games and Economic Behavior* 37, 415-424.

[18] Schmidt G., Ströhlein T. (1985): On kernels of graphs and solutions of games: a synopsis based on relations and fixpoints, *SIAM Journal of Algebraic and Discrete Methods* 6, 54-65.

[19] Sen A.K. (1971): Choice functions and revealed preference, *Review of Economic Studies* 38, 307-317.

[20] Sen A.K. (1977): Social choice theory: a re-examination, *Econometrica* 45, 53-89.

[21] Suzumura K. (1983): *Rational Choice, Collective Decisions, and Social Welfare*. Cambridge University Press: Cambridge UK.

[22] Von Neumann J., Morgenstern O. (1953): *Theory of Games and Economic Behavior* (third edition). Princeton University Press: Princeton.

[23] Wilson R.B. (1970): The finer structure of revealed preference, *Journal of Economic Theory* 2, 348-353.

[24] Yanovskaya E. (1980): Revealed preference in non-cooperative games (in Russian), *Mathematical Methods in the Social Sciences* 13, 73-81.