UNIVERSITÀ DEGLI STUDI DI SIENA

QUADERNI DEL DIPARTIMENTO
DI ECONOMIA POLITICA

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Revealed Cores:
Characterizations and Structure

n. 591 – Aprile 2010

Abstract - Characterizations of the choice functions that select the cores or the externally stable cores induced by an underlying revealed dominance digraph are provided. Relying on such characterizations, the basic order-theoretic structure of the corresponding sets of revealed cores is also analyzed.

JEL classification: C70, C71, D01, D03

Keywords: Core, choice functions, dominance digraphs, revealed preference

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1 Introduction

The *core* of a game is the set of its *undominated* outcomes, with respect to a suitably defined *dominance* relation, or digraph. Now, consider the ongoing operation of an interaction system, e.g. an organization or indeed any suitably complex decision-making unit. Let us then assume that the set of available options does in fact change at a faster pace than the behavioural attitudes of the relevant players *and* the latter interact as predicted by the core. It follows that the corresponding choice behaviour of the given interaction system as recorded by its choice function should be constrained in some way by its game-theoretic structure and thus somehow *reveal* it. But then, what are the characteristic 'fingerprints' of such a choice function, namely the *testable* behavioural predictions of the core as a solution concept? Or more simply, *which choice functions may be regarded as revealed cores*? Let us call that issue, for ease of reference, the *core revelation problem.*

Apparently, such a problem has never been addressed in its *full generality* in the extant literature. To be sure, parts of the massive body of literature on 'revealed preference' provide partial answers covering the case of *nonempty cores*, i.e. of *acyclic* revealed dominance digraphs (see e.g. Wilson (1970), Sen (1971), Suzumura (1983), Moulin (1985), Aizerman and Aleskerov (1995) among many others). But of course the core of a game may be empty, and its *revealed dominance digraph may have cycles.* Here, we are interested precisely in the *general* version of the core revelation problem, namely in *a characterization of all revealed cores as solutions for a certain 'universal' outcome set and its subsets, including (locally) empty-valued cores.*

The present paper is aimed at filling this gap in the literature by addressing the general core revelation problem as formulated above, under *several* variants of the notion of core. A study of the basic order-theoretic properties of the corresponding classes of revealed core-solutions is also provided.

The paper is organized as follows: section 2 includes a presentation of the model and the main characterization results; section 3 provides some basic results concerning the order-theoretic properties of the classes of revealed core-solutions previously characterized; section 4 consists of a few concluding remarks.

2 Choice functions and revealed cores

Let X be a set denoting the 'universal' outcome set, with cardinality $\#X \geq 3$, and $\mathcal{P}(X)$ its power set. It is also assumed for the sake of convenience that X is *finite* (but it should be remarked that the bulk of the ensuing analysis is easily lifted with suitable minor adaptations to the case of an infinite outcome set). A *choice function on* X is a *deflationary* operator on $\mathcal{P}(X)$ i.e. a function $c : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ such that $c(A) \subseteq A$ for any $A \subseteq X$ (empty choice sets are allowed). A choice function c is *proper* if $c(A) \neq \varnothing$ whenever $\varnothing \neq A \subseteq X$. We denote C_X the set of all choice functions on X, and C_X° the subset of all *proper* choice functions on X. The *proper subdomain* of $c \in C_X$ -written D_c- is the set of all subsets of X with a nonempty-valued choice set i.e. $D_c = \{A \subseteq X : c(A) \neq \varnothing\}$. Notice that any $c \in C_X$ fully determines its *dual choice* or *rejection function* $\overline{c}$ by the rule: for any $A \subseteq X$, $\overline{c}(A) = A \setminus c(A)$; clearly, $\overline{c} \in C_X$ as well, and $c = \overline{(\overline{c})}$. For any binary relation $\mathcal{B} \subseteq X \times X$, and any $Y \subseteq X$, $\mathcal{B}^a$ and $\mathcal{B}^s$ denote the asymmetric and symmetric components of $\mathcal{B}$, respectively, while $\mathcal{B}_Y = \mathcal{B} \cap (Y \times Y)$ and $\overline{\mathcal{B}} = (X \times X) \setminus \mathcal{B}$. Recall that $\mathcal{B} \subseteq X \times X$ is *reflexive* iff $x\mathcal{B}x$ for all $x \in X$, *irreflexive* iff *not* $x\mathcal{B}x$ for all $x \in X$, *total* iff $x\mathcal{B}y$ or $y\mathcal{B}x$ for any $x, y \in X$, *asymmetric* iff $x\mathcal{B}y$ entails *not* $y\mathcal{B}x$ for any $x, y \in X$, *transitive* iff $x\mathcal{B}y$ and $y\mathcal{B}z$ entail $x\mathcal{B}z$ for any $x, y, z \in X$, *quasi-transitive* if $\mathcal{B}^a$ is transitive, *negatively transitive* if $\overline{\mathcal{B}}$ is transitive. Moreover, $\mathcal{B}$ is a *strict partial order* iff it is both asymmetric and transitive.

Let $\Delta \subseteq X \times X$ be an *irreflexive* binary relation on X, denoting a suitably defined *dominance* relation: (X, Δ) is the corresponding *dominance digraph* (in graph-theoretic parlance, (X, Δ) is in particular a simple, loopless digraph i.e. a directed graph with at most one arc between any ordered pair of vertices, and with no arc from any vertex to itself). In particular, (X, Δ) is an *asymmetric dominance digraph* if $\Delta = \Delta^a$.

For any $Y \subseteq X$, $\Delta_Y = \Delta \cap (Y \times Y)$ denotes the dominance relation induced by Δ on Y (of course $\Delta_X = \Delta$), and (Y, Δ_Y) is the induced *dominance subdigraph* on Y. Broadly speaking, the *core* of (Y, Δ_Y) is the set of Δ_Y-undominated outcomes in Y, namely

$\mathbb{C}(Y, \Delta_Y) = \{y \in Y : \text{ } not \text{ } z\Delta_Y y \text{ for all } z \in Y\}$.

The *a-core* of (Y, Δ_Y) is the set of Δ_Y^a-undominated outcomes in Y, namely $\mathbb{C}^a(Y, \Delta_Y) = \mathbb{C}(Y, \Delta_Y^a) = \{y \in Y : \text{ } not \text{ } z(\Delta_Y)^a y \text{ for all } z \in Y\}$.

The *core (a-core)* of (Y, Δ_Y) is *externally stable* iff for any $z \in Y \setminus \mathbb{C}(Y, \Delta_Y)$ there exists $y \in \mathbb{C}(Y, \Delta_Y)$ such that $y\Delta_Y z$ (for any $z \in Y \setminus \mathbb{C}^a(Y, \Delta_Y)$ there exists $y \in \mathbb{C}^a(Y, \Delta_Y)$ such that $y(\Delta_Y)^a z$, respectively).

A dominance digraph (X, Δ) is also said to be *core-perfect* or *strictly acyclic (acyclic*, respectively*)* if $\mathbb{C}(Y, \Delta_Y) \neq \varnothing$ ($\mathbb{C}^a(Y, \Delta_Y) \neq \varnothing$, respectively) for any $Y \subseteq X$.

Remark 1 *It should be emphasized here that* any *dominance digraph may arise in a natural way from an underlying game in coalitional form. Moreover, any* asymmetric *dominance digraph may arise in a natural way from a game in coalitional form* and *from a related game in strategic form (see Vannucci (2009) for further details).*

The *(asymmetric) basic revealed dominance digraph* $(X, \Delta(c))$ of a choice function $c \in C_X$ is defined by the following rule: for any $x, y \in X$, $x\Delta(c)y$ if and only if $x \neq y$ and $c(\{x, y\}) = \{x\}$. Clearly enough, $\Delta(c)$ is *asymmetric* hence in particular *irreflexive* by definition. Two further binary relations $R(c)$, R_c induced by c on X and defined as follows will also be considered below: for any $x, y \in X$, $xR_c y$ if and only if $x \in c(\{x, y\})$, while $xR(c)y$ if and only if there exists $Y \subseteq X$ such that $x \in c(Y)$ and $y \in Y$.

A choice function $c \in C_X$ is a *revealed core-solution* if there exists an *irreflexive* relation $\Delta \subseteq X \times X$ such that $c(Y) = \mathbb{C}(Y, \Delta_Y)$ for any $Y \subseteq X$. Similarly, $c \in C_X$ is a *revealed a-core-solution (ES core-solution, ES a-core-solution,* respectively) if there exists an *irreflexive* relation $\Delta \subseteq X \times X$ such that $c(Y) = \mathbb{C}^a(Y, \Delta_Y)$ ($c(Y) = \mathbb{C}(Y, \Delta_Y)$ with $\mathbb{C}(Y, \Delta_Y)$ externally stable, $c(Y) = \mathbb{C}_u(Y, \Delta_Y)$, $c(Y) = \mathbb{C}_u^a(Y, \Delta_Y)$, respectively) for any $Y \subseteq X$. Then, we also say that c is *core-rationalizable (a-core-rationalizable, ES-core-rationalizable, ES-a-core-rationalizable respectively) by the dominance digraph* (X, Δ). Notice that *ES (a-)core-solutions are refinements of (a-)core solutions. Revealed cores* will also be used as a generic label to denote the foregoing choice functions.

Example 2 *Notice that the digraph $(X, \varnothing)$ is also a dominance digraph, and $\mathbb{C}(A, \varnothing_A) = \mathbb{C}^a(A, \varnothing_A) = A$ for any $A \subseteq X$ (hence is trivially externally stable). Therefore, the* identity *operator $c^{id} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is a revealed core-solution (a-core-solution, ES core-solution,).*

Example 3 *By way of contrast, take $\varnothing \subseteq G \subset X$ and consider the dichotomic choice function $c_-^G : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ as defined by the 'strict'* satisficing *rule $c_-^G(A) = A \cap G$ for any $A \subseteq X$. It is easily checked that c_-^G is* not *a revealed core: to see this, take any $x \in X \setminus G$. Then, $c_-^G(\{x\}) = \varnothing$ while for any dominance digraph (X, Δ) and any $x \in X$, it cannot be the case that $x \Delta x$ hence $\mathbb{C}(\{x\}, \Delta_{\{x\}}) = \mathbb{C}^a(\{x\}, \Delta_{\{x\}}) = \{x\}$ (which is also trivially externally stable).*

Example 4 *Next, take again $\varnothing \subseteq G \subset X$ and consider the nonempty valued dichotomic choice function $c_+^G : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ as defined by the 'lax'* satisficing *rule $c_+^G(A) = A \cap G$ for any $A \subseteq X$ if $A \cap G \neq \varnothing$, and $c(A) = A$ otherwise. Now, posit $\Delta = G \times (X \setminus G)$ i.e. $x \Delta y$ iff $x \in G$ and $y \in X \setminus G$. It is easily checked that for any $Y \subseteq X$, $c_+^G(Y) = \mathbb{C}(Y, \Delta_Y) = \mathbb{C}^a(Y, \Delta_Y)$ (which is also externally stable).*

The main objective of this article is precisely to provide a *characterization of all revealed cores* in C_X.

To begin with, let us consider two requirements concerning local existence of nonempty solution sets.

No-dummy property (ND): $c(\{x\}) = \{x\}$ for any $x \in X$.

2- Properness (2-PR): $c(A) \neq \varnothing$ for any $A \subseteq X$ such that $\#A = 2$.

It is easily checked that ND is satisfied by all revealed cores, while 2-PR is only violated by core solutions when the underlying dominance digraph is *not* asymmetric. A stronger property that obviously entails both ND and 2-PR is

Properness (PR): $c(A) \neq \varnothing$ for any nonempty $A \subseteq X$.

The following properties of a choice function $c \in C_X$ play a prominent role, under various labels, in the extant literature:

Chernoff Contraction-consistency (C): for any $A, B \subseteq X$ such that $A \subseteq B$, $c(B) \cap A \subseteq c(A)$.

Concordance (CO): for any $A, B \subseteq X$, $c(A) \cap c(B) \subseteq c(A \cup B)$.

Superset consistency (SS): for any $A, B \subseteq X$, if $A \subseteq B$ and $\varnothing \neq c(B) \subseteq c(A)$ then $c(A) \subseteq c(B)$.

Property C is a contraction-consistency condition for choice sets in that it requires that any outcome chosen out of a certain set should also be chosen out of any subset of the former: essentially, it says that *any good reason to choose a certain option out of a given menu should retain its strenght in every submenu of the former containing that option.* An alternative, equivalent interpretation of CO is as follows: *if there exists a 'fuzzy' classification of options of a given menu in two subclasses with one option belonging to both that is not rejected in either of the two corresponding submenus, then that option should not be rejected in the comprehensive menu under consideration.*

Conversely, property CO (also variously denoted as γ or Generalized Condorcet-consistency) is an expansion-consistency condition for choice sets, requiring that an outcome chosen out of a certain set and of a second one should also be chosen out of the larger set given by the union of those two sets: it says *any good reason to choose a certain option out of two given menus should retain its strenght in the larger menu obtained by merging those two menus.*

Property SS is also an expansion-consistency requirement for choice sets: it rules out the possibility that the choice set of a certain menu be nonempty *and* strictly included in the choice sets of a smaller menu.

We are now ready to prove the main results of this paper. Let us start from the following simple

Claim 5 *Let $R \subseteq X \times X$ be any (binary) relation on X, and define $\Delta^R \subseteq X \times X$ by the following rule: for any $x, y \in X$, $x\Delta^R y$ iff* not *yRx. Then,*

(i) $R^{\Delta^R} = R$;

(ii) for any $Y \subseteq X$, $\max R_Y = \{x \in Y : \text{not } y\Delta^R x \text{ for all } y \in X\}$, and $\max \Delta^R_Y = \{x \in Y : \text{not } yRx \text{ for all } y \in X\}$;

(iii) R is reflexive iff Δ^R is irreflexive, and irreflexive iff Δ^R is reflexive;

(iv) R is total iff Δ^R is asymmetric, and asymmetric iff Δ^R is total;

(v) R is quasi-transitive iff Δ^R is quasi-transitive.

Proof. (i) For any $x, y \in X$, by definition $xR^{\Delta^R}y$ iff *not* $y\Delta^R x$ iff *not* (*not* xRy) iff xRy.

(ii) Let $x \in Y$, and xRy for all $y \in Y$: then, by definition, *not* $y\Delta^R x$ for all $y \in Y$, and conversely if *not* $y\Delta^R x$ for all $y \in Y$ then *not (not xRy)* i.e. xRy

for all $y \in Y$. Similarly, $x \in Y$ and *not* yRx for all $y \in Y$: then by definition $x\Delta^R y$ for all $y \in Y$.

(iii) Indeed, by definition for any $x \in X$, *not* $x\Delta^R x$ iff *not(not* xRx*)* i.e. xRx. Similarly, *not* xRx iff $x\Delta^R x$.

(iv) Suppose Δ^R is asymmetric: then, for any $x, y \in X$, it may be the case that *not* $y\Delta^R x$ or *not* $x\Delta^R y$ (or both). Now, if *not* $y\Delta^R x$ then xRy and if *not* $x\Delta^R y$ then yRx, therefore R is total. Conversely, suppose R is total. If xRy then *not(not* xRy*)* hence *not(* $y\Delta^R x$*)* and similarly yRx entails *not(* $x\Delta^R y$*)*, thus in any case Δ^R is asymmetric. Similarly, R is asymmetric iff for any $x, y \in X$ it cannot be the case that xRy *and* yRx, i.e. by definition iff it is not the case that *not* $y\Delta^R x$ and *not* $x\Delta^R y$, namely Δ^R is total.

(v) Suppose that R is quasi-transitive, and that both $x(\Delta^R)^a y$ and $y(\Delta^R)^a z$. Then, by definition (*not* yRx and xRy), and (*not* zRy and yRz) i.e. $xR^a y$ and $yR^a z$, hence $xR^a z$. Therefore, xRz and *not* zRx i.e. *not* $z\Delta^R x$ and $x\Delta^R z$, namely $x(\Delta^R)^a z$. Conversely, suppose that Δ^R is quasi-transitive, and that both $xR^a y$ and $yR^a z$. Then, by definition (xRy and *not* yRx), and (yRz and *not* zRy) i.e. by definition (*not* $y\Delta^R x$ and $x\Delta^R y$) and (*not* $z\Delta^R y$ and $y\Delta^R z$), i.e. $x(\Delta^R)^a y$ and $y(\Delta^R)^a z$, hence $x(\Delta^R)^a z$. Therefore, $x\Delta^R z$ and *not* $z\Delta^R x$ i.e. *not* zRx and xRz, namely $xR^a z$. ∎

Remark 6 *The content of the previous Claim is certainly not unknown, but I have been unable to find a reference in print except for the statement of point (iv) in Monjardet (2007).*

Theorem 7 *(see also Wilson (1970), Sen(1971)) Let* $c \in C_X$*. Then, the following statements are equivalent:*

(i) c satisfies ND, C and CO;

(ii) there exists a dominance digraph (X, Δ) *such that* $c(Y) = \mathbb{C}(Y, \Delta_Y)$ *for any* $Y \subseteq X$*;*

(iii) there exists a reflexive *relation* $R \subseteq X \times X$ *such that* $c(Y) = \max R_Y$ *for any* $Y \subseteq X$*.*

(iv) $R(c) = R_c$*,* $R(c)$ *is reflexive and* $c(Y) = \max R(c)_Y$ *for any* $Y \subseteq X$*.*

Proof. (i)$\Longrightarrow$(iv): Let $c \in C_X$. Now, for each $Y \subseteq X$ and $x \in c(Y)$, $xR(c)y$ for any $y \in Y$, by definition of $R(c)$. Hence $c(Y) \subseteq \max R(c)_Y$. Now, let $c \in C_X$ also satisfy ND, C and CO, and $x \in \max R(c)_Y$. Then, by definition, for any $y \in Y$ there exists Y_y such that $y \in Y_y$ and $x \in c(Y_y)$. It follows, by C, that $x \in c(\{x, y\})$ for any $y \in Y$ whence, by CO, $x \in c(Y)$. Therefore, $c(Y) = \max R(c)_Y$ (clearly it may be the case that $\max R(c)_Y = \varnothing$). Notice however that, by ND, $x \in c(\{x\})$ i.e. $xR(c)x$ for any $x \in X$. Thus, $R(c)$ is *reflexive*, as required.

(ii)$\Longleftrightarrow$(iii) (see Wilson (1970), Theorem 3): Let $c \in C_X$. Thus, by Claim 5 (ii), for any $Y \subseteq X$, if there exists $R \subseteq X \times X$ such that $c(Y) = \max R_Y$ for any $Y \subseteq X$, then $c(Y) = \left\{x \in Y : \textit{not}\ y\Delta^R x \text{ for all } y \in X\right\}$, for any $Y \subseteq$

X. Moreover, if R is reflexive then by Claim 5(iii) Δ^R is irreflexive hence $c(Y) = \mathbb{C}(Y, \Delta_Y^R)$. Conversely if there exists an irreflexive $\Delta \subseteq X \times X$ such that $c(Y) = \mathbb{C}(Y, \Delta_Y)$ for any $Y \subseteq X$ then by Claim 5 (ii)-(iii) $c(Y) = \max \Delta_Y^R$ for any $Y \subseteq X$, and Δ^R is reflexive.

(iii)$\Longrightarrow$(iv): See Wilson (1970), Theorem 3. Moreover, observe that $R_c \subseteq R(c)$ by definition, and $xR(c)y$ implies $x \in \max R(c)_{\{x,y\}} = c(\{x,y\})$ i.e. xR_cy hence $R_c = R(c)$ (of course, this is an extension to arbitrary choice functions of the proof of the same result for *proper* choice functions due to Sen (1971)).

(iv)$\Longrightarrow$(iii): Trivial.

(iii)$\Longrightarrow$(i): Suppose that there exists a *reflexive* relation $R \subseteq X \times X$ such that $c(Y) = \max R_Y$ for any $Y \subseteq X$. Clearly, by reflexivity of R, $c(\{x\}) = \max R_{\{x\}} = \{x\}$, hence c satisfies ND. Moreover, for any $Y \subseteq Z \subseteq X$ and any $x \in c(Z) = \max R_Z^{\Delta}$, it must also be the case that $x \in \max R_Y^{\Delta} = c(Y)$ hence C is also satisfied by c. Finally, for any $Y, Z \subseteq X$ and $x \in X$, if $x \in c(Y) = \max R_Y^{\Delta}$ and $c(Z) = \max R_Z^{\Delta}$ then clearly $x \in \max R_{Y \cup Z}^{\Delta}$ whence $x \in c(Y \cup Z)$ and CO is satisfied as well. ■

Remark 8 *Notice that the equivalence between statements (ii) and (iii) of Theorem 7 above should in fact be credited to Wilson(1970) because it is strictly related (indeed, essentially equivalent) to Theorem 3 of Wilson (1970), though the latter concerns* nonempty *core-solutions over an* arbitrary *domain* $D \subseteq \mathcal{P}(X) \smallsetminus \{\varnothing\}$ *hence, strictly speaking, is a statement about a class of* partial proper *choice functions.*

Remark 9 *The foregoing characterization result is tight. To check that, consider the following examples.*

(1) Let $c^I \in C_X$ *be defined as follows: for any* $A \subseteq X$, $c^I(A) = \max L_A$ *if* $A \neq \{x^*\}$, *and* $c^I(\{x^*\}) = \varnothing$ *where* L *is a linear order on* X *and* x^* *is its bottom element. Clearly,* c^I *violates ND, but satisfies C and CO;*

(2) Let $X = \{x, y, z\}$, *and* $c^{II} \in C_X$ *be defined as follows:* $c^{II}(\{h\}) = \{h\}$ *for any* $h \in X$, $c^{II}(\{x,y\}) = \{x\}$, $c^{II}(\{y,z\}) = \{y\}$, $c^{II}(\{x,z\}) = \{z\}$, *and* $c^{II}(X) = X$. *It is immediately checked that* c^{III} *satisfies ND and CO, but violates C since e.g.* $y \in c^{II}(X) \cap \{x,y\}$ *but* $y \notin c^{II}(\{x,y\})$;

(3) Let $c^{III} \in C_X$ *be defined as follows: for any* $A \subseteq X$, $c^{III}(A) = \max L_A$ *if* $\#A \leq 2$ *and* $c^{III}(A) = \varnothing$ *otherwise, where* L *is a linear order on* X. *It is easily seen that* c^{III} *satisfies ND and C, but violates CO.*

Theorem 10 *Let* $c \in C_X$. *Then, the following statements are equivalent:*

(i) c *satisfies ND, 2-PR, C and CO;*

(ii) there exists a dominance digraph (X, Δ) *such that* $c(Y) = \mathbb{C}^a(Y, \Delta_Y)$ *for any* $Y \subseteq X$;

(iii) there exists a total *relation* $R \subseteq X \times X$ *such that* $c(Y) = \max R_Y$ *for any* $Y \subseteq X$;

(iv) $R(c) = R_c$, $R(c)$ *is total and* $c(Y) = \max R(c)_Y$ *for any* $Y \subseteq X$.

Proof. (i)$\Longrightarrow$(iii): Let $c \in C_X$ satisfy ND, 2-PR, C, and CO. Then, by ND, C and CO (and in view of Theorem 7 above) there exists a reflexive relation R on X such that $c(Y) = \max R_Y = \{y \in Y : yRz \text{ for all } z \in Y\}$ for each $Y \subseteq X$. Thus, by 2-PR, R is total.

(ii)$\Longleftrightarrow$(iii): Suppose that there exists a total relation $R \subseteq X \times X$ such that $c(Y) = \max R_Y$ for any $Y \subseteq X$. Then, as recorded by Claim 5(ii) $c(Y) = \left\{x \in Y : \textit{not } y\Delta^R x \text{ for all } y \in X\right\}$ for any $Y \subseteq X$. By Claim 5 (iv) Δ^R is asymmetric since R is total, hence in particular $c(Y) = \mathbb{C}(Y, \Delta_Y^R) = \mathbb{C}^a(Y, \Delta_Y^R)$ for any $Y \subseteq X$. Conversely, suppose that there exists a dominance digraph (X, Δ) such that $c(Y) = \mathbb{C}^a(Y, \Delta_Y) = \{x \in Y : \textit{not } y\Delta^a x \text{ for all } y \in X\}$, for any $Y \subseteq X$. Then, as recorded by Claim 5 (ii) $c(Y) = \max(\Delta^a)_Y^R$: by Claim 5 (iv), $(\Delta^a)^R$ is total since Δ^a is asymmetric.

(ii)$\Longrightarrow$(i): Suppose that there exists a dominance digraph (X, Δ) such that $c(Y) = \mathbb{C}^a(Y, \Delta_Y)$ for any $Y \subseteq X$. For any $x \in X$, *not* $x\Delta_{\{x\}}x$ i.e *not* $x\Delta x$ by irreflexivity of Δ whence by definition $c(\{x\}) = \mathbb{C}^a(\{x\}, \Delta_{\{x\}}) = \mathbb{C}(\{x\}, \Delta_{\{x\}}) = \{x\}$ and ND is therefore satisfied by c. Furthermore, for any $x, y \in X$, $\Delta_{\{x,y\}} \subseteq \{(x, y), (y, x)\}$ hence $\Delta^a_{\{x,y\}} \in \{\varnothing, \{(x, y)\}, \{(y, x)\}\}$. If $\Delta^a_{\{x,y\}} = \varnothing$ then $\mathbb{C}^a(\{x, y\}, \Delta_{\{x,y\}}) = \{x, y\}$, otherwise $\mathbb{C}^a(\{x, y\}, \Delta_{\{x,y\}}) = \{x\}$ or $\mathbb{C}^a(\{x, y\}, \Delta_{\{x,y\}}) = \{y\}$, respectively, hence in any case

$c(\{x, y\}) = \mathbb{C}^a(\{x, y\}, \Delta_{\{x,y\}}) \neq \varnothing$ thus c satisfies 2-PR.

Also, for any $Y, Z \subseteq X$ such that $Y \subseteq Z$, and any $x \in c(Z) \cap Y = \mathbb{C}^a(Z, \Delta_Z) \cap Y$, it must be the case that *not* $z\Delta^a_Z x$ for all $z \in Z$ hence in particular *not* $z\Delta^a_Y x$ for all $z \in Y$, i.e. $x \in \mathbb{C}^a(Y, \Delta_Y) = c(Y)$ and c also satisfies C.

Moreover, let $Y, Z \subseteq X$ and $x \in c(Y) \cap c(Z) = \mathbb{C}^a(Y, \Delta_Y) \cap \mathbb{C}^a(Z, \Delta_Z)$. Then, by definition, *not* $y\Delta^a_Y x$ for all $y \in Y$ and *not* $z\Delta^a_Y x$ for all $z \in Z$ hence *not* $u\Delta^a_Y x$ for all $u \in Y \cup Z$ i.e. $x \in \mathbb{C}^a(Y \cup Z, \Delta_{Y\cup Z}) = c(Y \cup Z)$ and CO is satisfied by c.

(iii)$\Longleftrightarrow$(iv): See the proof of Theorem 7 above. ■

Remark 11 *The foregoing characterization result is also tight. To see this, consider the following examples.*

(1) Let $c^I \in C_X$ *as defined above (see Remark 9). Clearly,* c^I *violates ND, but satisfies 2-PR, C and CO;*

(2) Let $c^{II^*} \in C_X$ *be defined as follows:* $c^{II^*}(\{x\}) = \{x\}$ *for any* $x \in X$, *and* $c^{II^*}(A) = \varnothing$ *for any* $A \subseteq X$ *such that* $\#A \geq 2$. *It is easily checked that* c^{II^*} *does indeed satisfy ND, C and CO, but clearly violates 2-PR;*

(3) Let $X = \{x, y, z\}$, *and* $c^{II} \in C_X$ *as defined above (see Remark 9). It is immediately checked that* c^{II} *satisfies ND, 2-PR, and CO, but violates C;*

(4) Let $c^{III} \in C_X$ *as defined above (see Remark 9). It is easily seen that* c^{III} *satisfies ND, 2-PR and C, but violates CO.*

Corollary 12 *(see also Sen (1971), Suzumura (1983)) Let* $c \in \overset{\circ}{C}_X$. *Then, the following statements are equivalent:*

(i) c satisfies C and CO;

(ii) there exists a strictly acyclic *dominance digraph* (X, Δ) *such that* $c(Y) = \mathbb{C}(Y, \Delta_Y) = \mathbb{C}^a(Y, \Delta_Y)$ *for any* $Y \subseteq X$*;*

(iii) there exists a total *relation* $R \subseteq X \times X$ *such that* $c(Y) = \max R_Y$ *for any* $Y \subseteq X$*;*

(iv) there exists a relation $R \subseteq X \times X$ *such that* $c(Y) = \max R_Y$ *for any* $Y \subseteq X$*.*

(v) $R(c) = R_c$*,* $R(c)$ *is total, and* $c(Y) = \max R(c)_Y$ *for any* $Y \subseteq X$*.*

Proof. (i)$\Longrightarrow$(ii): Since $c \in \overset{\circ}{C}_X$, c is proper hence in particular it also satisfies ND and 2-PR. Therefore, by Theorem 8 (ii) above, there exists a dominance digraph (X, Δ) such that $c(Y) = \mathbb{C}^a(Y, \Delta_Y)$ for any $Y \subseteq X$. Moreover, since by hypothesis c is *proper*, $\mathbb{C}^a(Y, \Delta_Y) \neq \varnothing$ for any $Y \subseteq X$ hence (X, Δ) must be *acyclic*. In particular, $\mathbb{C}^a(\{x, y\}, \Delta_{\{x,y\}}) \neq \varnothing$ for any $x, y \in X$, therefore Δ is asymmetric as well. Thus, (X, Δ) is indeed *strictly* acyclic and $\mathbb{C}(Y, \Delta_Y) = \mathbb{C}^a(Y, \Delta_Y) \neq \varnothing$ for any $Y \subseteq X$.

(ii)$\Longrightarrow$(i): See the proof of Theorem 7 above;

(i)$\Longleftrightarrow$(iii): Obvious, by Theorem 10 above, since, again, $c \in \overset{\circ}{C}_X$ entails that c satisfies ND and 2-PR.

(iii)$\Longleftrightarrow$(iv): Suppose there exists $R \subseteq X \times X$ such that $c(Y) = \max R_Y$ for any $Y \subseteq X$. Since $c \in \overset{\circ}{C}_X$, $c(Y) \neq \varnothing$ for any $Y \subseteq X$. Hence, in particular, for any $x, y \in X$, $c(\{x, y\}) \neq \varnothing$. It follows that R is total. The reverse implication is trivial.

(iii)$\Longleftrightarrow$(v): See the proof of Theorem 7 above, and of course Sen (1971). ∎

Remark 13 *Actually, it is well-known that a* proper *c satisfies both C and CO if and only if there exists a binary relation R on X such that* $c(Y) = \max R_Y = \{y \in Y : yRz \text{ for all } z \in Y\}$ *for each* $Y \subseteq X$ *and, moreover,* $R = R(c) = R_c$ *as defined above -indeed,* $R(c) = R_c$ *for any choice function that satisfies C (see e.g. Sen (1971), Suzumura (1983)). Thus, Corollary 12 is -essentially- a restatement of the Sen-Suzumura characterization of revealed 'rational' (proper) choice functions or, equivalently, revealed non-empty core solutions.*

Let us now turn to characterizations of *revealed externally stable core-solutions*. Since externally stable cores (of nonempty sets) are nonempty the corresponding choice functions are proper: thus, given the traditional focus on proper choice functions, this subclass of revealed cores is the most widely studied, and best known. In fact, for the sake of convenience, we collect the main - and *partly well-known*- characterizations of revealed externally stable cores in the following

Theorem 14 *(see also Suzumura (1983)) Let* $c \in C_X$*. Then, the following statements are equivalent:*

(i) c satisfies PR, C, CO and SS;

(ii) there exists a quasi-transitive *relation* $R \subseteq X \times X$ *such that* $c(Y) = \max R_Y \neq \varnothing$ *for any nonempty* $Y \subseteq X$*;*

(iii) there exists a total *and* quasi-transitive *relation* $R \subseteq X \times X$ *such that* $c(Y) = \max R_Y \neq \varnothing$ *for any nonempty* $Y \subseteq X$*;*

(iv) $R(c) = R_c$*,* $R(c)$ *is total and quasi-transitive, and* $c(Y) = \max R(c)_Y \neq \varnothing$ *for any* $Y \subseteq X$*.*

(v) there exists a reflexive *and* negatively transitive *relation* $R \subseteq X \times X$ *such that* $c(Y) = \max R_Y \neq \varnothing$ *for any nonempty* $Y \subseteq X$*;*

(vi) there exists a negatively transitive *relation* $R \subseteq X \times X$ *such that* $c(Y) = \max R_Y \neq \varnothing$ *for any nonempty* $Y \subseteq X$*;*

(vii) there exists a dominance digraph (X, Δ) *such that* $c(Y) = \mathbb{C}(Y, \Delta_Y) = \mathbb{C}^a(Y, \Delta_Y)$ *with* $\mathbb{C}(Y, \Delta_Y)$ externally stable*, for any* $Y \subseteq X$*;*

(viii) there exists a dominance digraph (X, Δ) *such that* Δ *is* transitive *and* $c(Y) = \mathbb{C}(Y, \Delta_Y) = \mathbb{C}^a(Y, \Delta_Y) \neq \varnothing$ *for any nonempty* $Y \subseteq X$*;*

(ix) there exists a dominance digraph (X, Δ) *such that* Δ *is a* strict partial order *and* $c(Y) = \mathbb{C}(Y, \Delta_Y) = \mathbb{C}^a(Y, \Delta_Y) \neq \varnothing$ *for any nonempty* $Y \subseteq X$*.*

Proof. (i)$\Longrightarrow$(ii) (Suzumura (1983)): By Theorem 2.6 of Suzumura (1983), if c satisfies PR, C, CO and SS then there exists a (*reflexive and*) quasi-transitive relation $R \subseteq X \times X$ such that $c(Y) = \max R_Y \neq \varnothing$ for any nonempty $Y \subseteq X$. But of course PR entails that $c(\{x, y\}) = \max R_{\{x,y\}} \neq \varnothing$ for any $x, y \in X$, hence R is *total* as well.

(ii)$\Longrightarrow$(i) (Suzumura (1983)): See again Suzumura (1983), Theorems 2.5, 2.6 and 2.7.

(ii)$\Longleftrightarrow$(iii): Let be $R \subseteq X \times X$ quasi-transitive and such that $c(Y) = \max R_Y \neq \varnothing$ for any nonempty $Y \subseteq X$. Of course, PR entails that in particular $c(\{x, y\}) = \max R_{\{x,y\}} \neq \varnothing$ for any $x, y \in X$, hence R is *total* as well. The reverse implication is trivial.

(iii)$\Longleftrightarrow$(iv): See the proof of Theorem 7 above.

(iii)$\Longleftrightarrow$(v): Let $R \subseteq X \times X$ be total and quasi-transitive, and $x, y, z \in X$ such that *not* xRy and *not* yRz. Hence, yRx and zRy since R is total. Therefore, by definition, $yR^a x$ and $zR^a y$. By quasi-transitivity, it follows that $zR^a x$, whence in particular *not* xRz i.e. R is negatively transitive. Moreover, totality implies reflexivity of R. Conversely, let $R \subseteq X \times X$ be reflexive and negatively transitive. Suppose there exist $x, y \in X$ such that *not* xRy and *not* yRx: then, by negative transitivity, *not* xRx, a contradiction since R is reflexive. Thus, R is also total. Moreover, let $xR^a y$ and $yR^a z$. Then, in particular, *not* yRx and *not* zRy. It follows that, by negative transitivity, *not* zRx whence, by totality, xRz. Thus, $xR^a z$ i.e. R is quasi-transitive as well.

(v)$\Longleftrightarrow$(vi): Let $R \subseteq X \times X$ be a *negatively transitive* relation such that $c(Y) = \max R_Y \neq \varnothing$ for any nonempty $Y \subseteq X$. Then in particular, $c(\{x\}) = \max R_{\{x\}} \neq \varnothing$ for any $x \in X$, hence R is reflexive as well. The reverse implication is trivial.

(iii)$\Longrightarrow$(vii): Let be $R \subseteq X \times X$ total, quasi-transitive and such that $c(Y) = \max R_Y \neq \varnothing$ for any nonempty $Y \subseteq X$. Clearly, by construction, $c(Y) = \{x \in Y : xRy \text{ for all } y \in Y\}$ i.e. $c(Y) = \left\{x \in Y : \textit{not } y\Delta^R x \text{ for all } y \in Y\right\} = \mathbb{C}(Y, \Delta^R_Y)$ for any $Y \subseteq X$ (see Claim 5 (i) above). Moreover, by Claim 5 (iii), Δ^R is asymmetric since R is total, hence $\mathbb{C}(Y, \Delta^R_Y) = \mathbb{C}^a(Y, \Delta^R_Y)$. Now, take any $y_1 \in Y \smallsetminus \mathbb{C}(Y,\Delta_Y^R)$. By definition, there exists $y_2 \in Y$ such that $y_2 \Delta_Y^R y_1$. If $y_2 \in \mathbb{C}(Y,\Delta_Y)$ we are done. Suppose then that $y_2 \in Y \smallsetminus \mathbb{C}(Y,\Delta_Y^R)$ as well: thus, there exists $y_3 \in Y$ such that $y_3 \Delta_Y^R y_2$. It follows, by finiteness of Y and nonemptiness of $\mathbb{C}(Y,\Delta_Y^R)$, that there exists a finite k such that $y_i \Delta_Y^R y_{i-1}$ for any $i = 2,...,k$, and $y_k \in \mathbb{C}(Y,\Delta_Y^R)$. Since Δ^R is asymmetric, it also follows that $y_k \Delta_Y^R y_1$, hence $\mathbb{C}(Y,\Delta_Y^R)$ is externally stable.

(vii)$\Longrightarrow$(i): Suppose that there exists a dominance digraph (X,Δ) such that $c(Y) = \mathbb{C}(Y,\Delta_Y) = \mathbb{C}^a(Y,\Delta_Y)$ with $\mathbb{C}(Y,\Delta_Y)$ externally stable, for any $Y \subseteq X$. By definition of external stability, $c(Y) \neq \varnothing$ for any nonempty $Y \subseteq X$, hence c satisfies PR. Moreover, by Theorem 6 (ii) above (or, for that matter, by Theorem 8 (ii)), it also satisfies C and CO. Finally, consider $Y \subseteq Z \subseteq X$ such that $c(Z) \subseteq c(Y)$, and suppose there exists $y \in c(Y) \smallsetminus c(Z)$ i.e. $y \in \mathbb{C}(Y,\Delta_Y) \smallsetminus \mathbb{C}(Z,\Delta_Z)$. Then, by external stability of $\mathbb{C}(Z,\Delta_Z)$, there exists $z \in \mathbb{C}(Z,\Delta_Z) \subseteq \mathbb{C}(Y,\Delta_Y) \subseteq Y$ such that $z\Delta y$, a contradiction since $y \in \mathbb{C}(Y,\Delta_Y)$. Therefore, c satisfies SS as well.

(viii)$\Longleftrightarrow$(iii): Suppose that there exists a dominance digraph (X,Δ) such that Δ is transitive (hence in particular quasi-transitive) and $c(Y) = \mathbb{C}(Y,\Delta_Y) = \mathbb{C}^a(Y,\Delta_Y) \neq \varnothing$ for any nonempty $Y \subseteq X$. Then, by Claim 5 (i)-(ii) above, $\varnothing \neq c(Y) = \mathbb{C}(Y,\Delta_Y) = \mathbb{C}(Y,\Delta_Y^{R^\Delta}) = \max R_Y^\Delta$ for any nonempty $Y \subseteq X$. Moreover, by Claim 5 (v), $R^\Delta$is quasi-transitive. Also, notice that since by hypothesis Δ is both irreflexive and transitive, it must be *asymmetric* as well. Therefore, by Claim 5 (iv), R^Δ is total. Conversely, suppose that there exists a total and quasi-transitive relation $R \subseteq X \times X$ such that $c(Y) = \max R_Y \neq \varnothing$ for any nonempty $Y \subseteq X$. Then, by Claim 5 (ii) $c(Y) = \max R_Y = \mathbb{C}_u(Y,\Delta_Y^R) \neq \varnothing$ for any nonempty $Y \subseteq X$. Moreover, by Claim 5 (iii),(v), and in view of quasi-transitivity and totality of R, Δ^R is both quasi-transitive and asymmetric, hence *transitive* as well, and such that $\mathbb{C}_u(Y,\Delta_Y^R) = \mathbb{C}_u^a(Y,\Delta_Y^R)$ as required.

(viii)$\Longleftrightarrow$(ix): Suppose that there exists a dominance digraph (X,Δ) such that Δ is *transitive* and $c(Y) = \mathbb{C}(Y,\Delta_Y) = \mathbb{C}^a(Y,\Delta_Y) \neq \varnothing$ for any nonempty $Y \subseteq X$. Again, irreflexivity and transitivity imply *asymmetry* of Δ, which is therefore a *strict partial order*. The reverse implication is trivial. ■

Remark 15 *Observe that the characterization result of revealed externally stable cores in terms of properties of choice functions included in Theorem 14 is also tight. To see this, consider the following examples.*

(1) Let $c^I \in C_X$ *as defined above (see Remark 9). Clearly,* c^I *violates PR,but satisfies C, CO and SS;*

(2) Let $X = \{x,y,z\}$*, and* $c^{II} \in C_X$ *as defined above (see Remark 9). It is immediately checked that* c^{II} *satisfies PR,CO and SS, but violates C;*

(3) Let $X = \{x,y,z\}$*, and* $c^{IV} \in C_X$ *such that* $c^{IV}(\{u\}) = \{u\}$ *for any* $u \in X$, $c^{IV}(\{x,y\}) = \{x,y\}$, $c^{IV}(\{y,z\}) = \{y,z\}$, $c^{IV}(\{x,z\}) = \{x,z\}$ *and* $c^{IV}(\{x,y,z\}) = \{x,y\}$*. Clearly,* c^{IV} *satisfies PR, C and SS. However,* c^{IV} *fails to satisfy CO since* $z \in (c^{IV}(\{x,z\}) \cap c^{IV}(\{y,z\})) \smallsetminus c^{IV}(\{x,y,z\})$*;*

(4) Let $X = \{x,y,z\}$*, and* $c^V \in C_X$ *such that* $c^V(\{u\}) = \{u\}$ *for any* $u \in X$, $c^V(\{x,y\}) = \{x\}$, $c^V(\{y,z\}) = \{y\}$, $c^V(\{x,z\}) = \{x,z\}$ *and* $c^V(\{x,y,z\}) =$

$\{x\}$. Clearly, c^V satisfies PR, C and CO but fails to satisfy SS since $\varnothing \neq c^V(\{x,y,z\}) \subset c^V(\{x,z\})$.

Remark 16 *Notice that Theorem 14 above is a refinement of well-known results due to Suzumura (see e.g. Suzumura (1983), Theorems 2.8 and 2.10). It should also be mentioned here that the conjunction of C and SS turns out to be equivalent (see e.g. Suzumura (1983)) to another well-known and widely used property, namely*

Path Independence (PI): for any $A, B \subseteq X$, $c(A \cup B) = c(c(A) \cup c(B))$

Thus, the equivalent statements of Theorem 14 are also equivalent to the statement '$c \in C_X$ satisfies PR, PI and CO'.

It should be remarked that the characterizations provided above are in general quite straightforward extensions to *arbitrary* choice functions of previously known results concerning *proper* choice functions. Indeed, the gist of the results offered in the present section may be summarized as follows:

(i) remarkably, the characterizations of *general* revealed cores and a-cores considered here consist of the very same properties used to characterize their nonempty-valued counterparts as supplemented with very mild-looking local nonemptiness requirements for choice sets of singleton and two-valued subsets, respectively;

(ii) the exact correspondence between revealed core-solutions and maximizing 'rational' choice functions is fully confirmed to hold within the general space of *arbitrary* choice functions: the alleged extra-generality of the latter subclass that has sometimes been alluded to in the literature (as e.g. in Suzumura (1983), p.21) is definitely confined to the even wider realm of *partial* choice functions.

(iii) Finally, and most notably, the class of general revealed cores turns out to inherit some of the supplementary order-theoretic structure enjoyed by its larger ambient space as compared to the smaller and less regular space of proper choice functions: that is precisely the topic of the next section.

3 Posets and semilattices of revealed cores

Let us now turn to a global description of the order-theoretic structure of the class of all revealed core-solutions (a-core-solutions, nonempty-valued core-solutions, externally stable core-solutions, respectively).

A partially ordered set or *poset* is a pair $\mathbf{P} = (P, \leqslant)$ where P is a set and $\leqslant$ is a reflexive, transitive and antisymmetric binary relation on P (i.e. for any $x \in P$, $x \leqslant x$ and for any $x, y, z \in P$, $x \leqslant z$ whenever $x \leqslant y$ and $y \leqslant z$, and $x = y$ whenever $x \leqslant y$ and $y \leqslant z$). A *coatom* of a poset $\mathbf{P} = (P, \leqslant)$ with a top element or maximum 1_P is any $j \in P$ which is covered by 1_P- written $j \lessdot 1_P$- i.e. $j < 1_P$ and $l = j$ for any $l \in P$ such that $j \leqslant l < 1_P$. The set of all coatoms of $\mathbf{P}$ is denoted A^*_P. Dually, an *atom* of $\mathbf{P}$ is any $j \in P$ which is an upper cover of 0_P- written $0_P \lessdot j$- i.e. $0_P < j$ and $l = j$ for any $l \in P$ such that $0_P < l \leqslant j$. The set of all atoms of $\mathbf{P}$ is denoted A_P.

A poset $\mathbf{P} = (P, \leqslant)$ is a *meet semilattice* (*join semilattice*, respectively) if for any $x, y \in P$ the $\leqslant$-greatest lower bound $x \wedge y$ (the $\leqslant$-least upper bound $x \vee y$, respectively) of $\{x, y\}$ does exist. Moreover, $\mathbf{P}$ is a *lattice* if it is both a meet semilattice and a join semilattice. A *meet irreducible* element of $\mathbf{P}$ is any $j \in P$ such that $j \neq \vee P$ and for any $a, b \in L$ if $j = a \wedge b$ then $j \in \{a, b\}$. The set of all meet irreducible elements of $\mathbf{P}$ is denoted M_P. Similarly, A *join irreducible* element of $\mathbf{P}$ is any $j \in P$ such that $j \neq \wedge P$ and for any $a, b \in P$ if $j = a \vee b$ then $j \in \{a, b\}$. The set of all join irreducible elements of $\mathbf{P}$ is denoted J_P. It is easily checked that in general $A_P \subseteq J_P$ ($A_P^* \subseteq M_P$) while the converse may not hold. $\mathbf{P}$ is *atomistic* iff $A_P = J_P$ i.e. equivalently whenever each element $a \in P$ is the least upper bound of a set of atoms, and *coatomistic* iff $A_P^* = M_P$ i.e. equivalently whenever each element $a \in P$ is the greatest lower bound of a set of atoms.

A lattice $\mathbf{P} = (P, \leqslant)$ is *bounded* if there exist both a bottom element 0_P and a top element 1_P (hence in particular a finite lattice is also bounded), *distributive* iff $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for any $x, y, z \in P$, *complemented* if it is bounded and for any $x \in P$ there exists $x' \in P$ such that $x \vee x' = 1_P$ and $x \wedge x' = 0_P$, and *Boolean* iff it is both distributive and complemented.

The set C_X of all choice functions on X can be endowed in a natural way with the point-wise set inclusion partial order $\leqslant$ by positing, for any $c, c' \in C_X$, $c \leqslant c'$ iff $c(A) \subseteq c'(A)$ for each $A \subseteq X$. Clearly, the identity operator c^{id} is its top element, and the constant empty-valued choice function $c^{\varnothing}$ its bottom element. It is well-known, and easily checked, that $(C_X, \leqslant)$ is in fact a *Boolean lattice* with join $\vee = \cup$ (i.e. set-union) and meet $\wedge = \cap$ (i.e. set-intersection), both defined in the obvious component-wise manner: see e.g. Monjardet, Raderanirina (2004).

For any $x, y \in X$ such that $x \neq y$, $c_{xy}^{+} \in C_X$ and $c_{xy}^{-} \in C_X$ are defined as follows: for all $A \subseteq X$, $c_{xy}^{+}(A) = A \smallsetminus \{y\}$ if $\{x, y\} \subseteq A$, and $c_{xy}^{+}(A) = A$ otherwise, $c_{xy}^{-}(\{x, y\}) = \{y\}$, $c_{xy}^{-}(\{x\}) = \{x\}$, and $c_{xy}^{-}(A) = \varnothing$ for all $A \subseteq X$ such that $A \neq \{x, y\}$ and $\#A \neq 1$. Moreover, $C_{+} = \left\{c_{xy}^{+} : x, y \in X, x \neq y\right\}$, and $C_{-} = \left\{c_{xy}^{-} : x, y \in X, x \neq y\right\}$.

The *minimum ND choice function* $c^{[1]}$ is defined by the following rule: for any $x \in X$, $c^{[1]}(\{x\}) = \{x\}$, and $c^{[1]}(Y) = \varnothing$ for any $Y \subseteq X$ such that $\#Y \neq 1$.

Now, let $C_X^* \subseteq C_X$ denote the set of all revealed core-solutions on X, $C_X^{*a} \subseteq C_X^{*^\circ}$ the set of all revealed asymmetric core-solutions, $C_X^{*^\circ} = C_X^* \cap C_X^\circ$ the set of all revealed nonempty-valued core-solutions, and C_X^{*es} the set of all revealed externally stable core-solutions on X, respectively). We also denote with a slight abuse of notation $(C_X^*, \leqslant)$, $(C_X^{*a}, \leqslant)$, $(C_X^{*^\circ}, \leqslant)$ and $(C_X^{*es}, \leqslant)$ the corresponding subposets of $(C_X, \leqslant)$ (where $\leqslant$ denotes $\leqslant \cap (C_X^* \times C_X^*)$, $\leqslant \cap (C_X^{*a} \times C_X^{*a})$, $\leqslant = \leqslant \cap (C_X^{*^\circ} \times C_X^{*^\circ})$ and $\leqslant = \leqslant \cap (C_X^{*es} \times C_X^{*es})$, respectively). We have the following

Theorem 17 *The poset $(C_X^*, \leqslant)$ of revealed core-solutions is a* sub-meet-semilattice *of $(C_X, \leqslant)$ with c^{id} itself as its top element, but not a sub-join-semilattice of $(C_X, \leqslant)$. The bottom element of $(C_X^*, \leqslant)$ is the minimum ND choice function*

$c^{[1]}$. *Moreover, the set of coatoms of* $(C^*_X, \leqslant)$ *is* C_+, *and the set of its atoms is* C_- .

Proof. Let $c, c' \in C^*_X$, and consider $c \cap c'$. Clearly, for any $x \in X$, $(c \cap c')(\{x\}) = c(\{x\}) \cap c'(\{x\}) = \{x\}$ since c and c' satisfy ND: hence $c \cap c'$ does also satisfy ND.

Moreover, for any $A \subseteq B \subseteq X$, since c and c' both satisfy C, $(c \cap c')(B) \cap A = (c(B) \cap c'(B)) \cap A = c(B) \cap (c'(B)) \cap A) \subseteq c(B) \cap c'(A) \subseteq c(B) \cap A \subseteq A$ hence $c \cap c'$ satisfies C.

Finally, since c and c' satisfy CO, for any $A, B \subseteq X$,

$(c \cap c')(A) \cap (c \cap c')(B) =$

$(c(A) \cap c(B)) \cap (c'(A) \cap c'(B)) \subseteq c(A \cup B) \cap c'(A \cup B) = (c \cap c')(A \cup B)$ and CO also holds for $c \cap c'$. It follows that, by Theorem 7 above, $c \cap c' \in C^*_X$, whence $(C^*_X, \leqslant)$ is a sub-meet-semilattice of $(C_X, \leqslant)$.

It is easily checked that c^{id}, the top element of $(C_X, \leqslant)$, does also satisfy ND, C and CO hence as observed above $c \in C^*_X$ (see Example 2).

Now, consider $c^{[1]}$ as defined above: it satisfies ND, by definition, and, being nonempty-valued precisely on singletons, it trivially satisfies C and CO as well. Thus, $c^{[1]} \in C^*_X$. On the other hand, for any $c \in C^*_X$, c must satisfy ND, hence $c^{[1]} \leqslant c$.

Next, take any $c^+_{xy} \in C_+$. Notice that, by definition, c^+_{xy} satisfies ND. Also, if $A \subseteq B \subseteq X$ then the following cases may be distinguished: (a) $\{x, y\} \subseteq A$; (b) $\{x, y\} \nsubseteq A$ and $\{x, y\} \subseteq B$; (c) $\{x, y\} \nsubseteq B$. If $\{x, y\} \subseteq A$ then $c^+_{xy}(B) \cap A = A \smallsetminus \{y\} = c^+_{xy}(A)$; if $\{x, y\} \nsubseteq A$ and $\{x, y\} \subseteq B$ then $c^+_{xy}(B) \cap A = (B \smallsetminus \{y\}) \cap A = A \smallsetminus \{y\} \subset A = c^+_{xy}(A)$; if $\{x, y\} \nsubseteq B$ then $c^+_{xy}(B) \cap A = A = c^+_{xy}(A)$: thus in any case C holds. Furthermore, let $z \notin c^+_{xy}(A \cup B)$: then by definition $z = y$ and $\{x, y\} \subseteq A \cup B$. Assume now that $y \in c^+_{xy}(A) \cap c^+_{xy}(B)$. Then, $\{x, y\} \nsubseteq A$ and $\{x, y\} \nsubseteq B$ while $y \in A \cap B$. It follows that $x \notin A \cup B$, a contradiction. Thus, CO is also satisfied by c^+_{xy}, Theorem 7 applies, and $c^+_{xy} \in C^*_X$.

Moreover, by definition $c^+_{xy} < c^{id}$ i.e. $c^+_{xy} \leqslant c^{id}$ and $c^+_{xy} \neq c^{id}$. Let $c \in C^*_X$ be such that $c^+_{xy} \leqslant c \leqslant c^{id}$, and assume that there exist $A, B \subseteq X$ with $\{x, y\} \subseteq A \cap B$, $c(A) = A \smallsetminus \{y\}$ and $c(B) = B$. Then, by Theorem 7, c satisfies ND, C and CO: hence in particular, by C, $\{x, y\} = c(B) \cap \{x, y\} \subseteq c(\{x, y\})$ i.e. $c(\{x, y\}) = \{x, y\} \neq c(A)$, and therefore $\{x, y\} \subset A$. But then, take any $z \in A \smallsetminus \{x, y\}$, and consider $A \smallsetminus \{x\}$ and $A \smallsetminus \{z\}$. Clearly, $c(A \smallsetminus \{x\}) \supseteq c^+_{xy}(A \smallsetminus \{x\}) = A \smallsetminus \{x\}$ and $c(A \smallsetminus \{z\}) \supseteq c^+_{xy}(A \smallsetminus \{z\}) = A \smallsetminus \{z\}$, hence $y \in c(A \smallsetminus \{x\})$ and $y \in c(A \smallsetminus \{z\})$. However, $y \notin c(A) = c((A \smallsetminus \{x\}) \cup (A \smallsetminus \{z\}))$, which contradicts CO. It follows that either $c = c^{id}$ or $c = c^+_{xy}$ i.e. c^+_{xy} is indeed a *coatom* of $(C^*_X, \leqslant)$.

Conversely, let c be a coatom of $(C^*_X, \leqslant)$ and suppose $c \notin C_+$. Then, for any pair of distinct $x, y \in X$, neither $c^+_{xy} \leqslant c$ nor $c \leqslant c^+_{xy}$ i.e. there exist $A, B \subseteq X$ such that $c(A) \subset c^+_{xy}(A)$ and $c^+_{xy}(B) \subset c(B)$. Thus, by definition, $c^+_{xy}(B) = B \smallsetminus \{y\}$ and $c(B) = B \supseteq \{x, y\}$, while there exists $z \in A$ such that $z \in c^+_{xy}(A) \smallsetminus c(A)$. Hence, consider any $x \in A \smallsetminus \{z\}$: then, there exists B' such that $\{x, z\} \subseteq B' \subseteq X$ and $c(B') = B'$. By C, $\{x, z\} = c(B') \cap \{x, z\} \subseteq c(\{x, z\})$ i.e. $c(\{x, z\}) = \{x, z\} \subseteq A$ for any $x \in A$ while $z \notin c(A)$, which contradicts CO in view of *finiteness* of X.

To check that each $c^-_{xy} \in C_-$ is an *atom* of $(C^*_X, \leqslant)$, just observe that $c^-_{xy}(A) = c^{[1]}(A)$ for any $A \neq \{x, y\}$, and $c^-_{xy}(\{x, y\}) = \{x\}$ while $c^{[1]}(\{x, y\}) = \varnothing$. Thus, for any $c \in C^*_X$ (indeed, for any $c \in C_X$) if $c^{[1]} \leqslant c \leqslant c^-_{xy}$ then either $c = c^{[1]}$ or $c = c^-_{xy}$. Conversely, assume that c is an atom of $(C^*_X, \leqslant)$ and $c \notin C_-$. Then, by definition of C_-, $c(A) = \varnothing$ for any A such that $\#A = 2$, and there exists $B \subseteq X$ such that $\#B \geq 3$ and $c(B) \neq \varnothing$. It follows that, for any $x \in c(B)$ and any $y \in B \smallsetminus \{x\}$, $c(B) \cap \{x, y\} \nsubseteq \varnothing = c(\{x, y\})$, therefore violating C, a contradiction by Theorem 7.

To check that $(C^*_X, \leqslant)$ is *not* a sub-join-semilattice of $(C_X, \leqslant)$, just consider without loss of generality $X = \{x, y, z\}$,

$R = \{(x, x), (y, y), (z, z), (y, x), (y, z), (x, z)\}$ and

$R^{-1} = \{(x, x), (y, y), (z, z), (x, y), (z, y), (z, x)\}$.

Now, posit $c_I(A) = \mathbb{C}(A, \Delta^R_A)$ and $c_{II}(A) = \mathbb{C}(X, \Delta^{R^{-1}}_A)$ for any $A \subseteq X$. By definition $\mathbb{C}(X, \Delta^R) = \{y\}$, $\mathbb{C}(\{x, z\}, \Delta^R_{\{x,z\}}) = \{x\}$, $\mathbb{C}(X, \Delta^{R^{-1}}) = \{z\}$, and $\mathbb{C}(\{x, y\}, \Delta^{R^{-1}}_{\{x,y\}}) = \{x\}$ hence $(c_I \cup c_{II})(X) = \{y, z\}$, while $x \in (c_I \cup c_{II})(\{x, y\}) \cap (c_I \cup c_{II})(\{x, z\})$, which contradicts CO. ■

Remark 18 *Notice that finiteness of X has been used in the proof above in order to show that the set of coatoms of $(C^*_X, \leqslant)$ is contained in C^+. The latter statement clearly holds for an infinite X as well provided CO is replaced with the following stronger version of 'Concordance'*

CO: for any family $\{A_i\}_{i \in I}$ of subsets of X, $\bigcap_{i \in I} c(A_i) \subseteq c(\bigcup_{i \in I} A_i)$.*

Remark 19 *Since $(C^*_X, \leqslant)$ is a semilattice with a top element (and indeed a finite one, under finiteness of X), it follows that it is also a* lattice *with meet= $\cap$ and join of a pair given by the meet of the (nonempty) set of upper bounds of that pair (see e.g. Davey, Priestley (1990)), which is however not a sublattice of $(C_X, \leqslant)$.*

Thus, the poset of revealed core-solutions enjoys a remarkably regular structure. The posets of revealed a-core-solutions, nonempty-valued core-solutions, and externally stable core-solutions are considerably less regular, as recorded by the following results, namely

Theorem 20 *The poset $(C^{*a}_X, \leqslant)$ of revealed a-core-solutions has a top element, c^{id}, and C_+ is the set of its coatoms, but it is neither a sub-meet-semilattice nor a sub-join-semilattice of $(C_X, \leqslant)$. The minimal elements of $(C^{*a}_X, \leqslant)$ are the choice functions $c \in C_X$ that satisfy ND, 2-PR, C, CO and such that (a) $\#c(A) \leq 1$ for any $A \subseteq X$ and (b)* not *$D_{c'} \subset D_c$ for any c' that satisfies ND, 2-PR, C and CO.*

Proof. To check that c^{id} is indeed the top element of $(C^{*a}_X, \leqslant)$ it is only to be observed -in view of Theorem 7- that c^{id} does in fact also satisfy 2-PR. Similarly -in view of Theorem 7 and of the proof of Theorem 17 provided above- to see that C_+ is the set of coatoms of $(C_X^{*a}, \leqslant)$ it is only to be checked that any $c_{xy}^+ \in C_+$ does also satisfy 2-PR (which is clearly the case, by definition).

The proof of Theorem 17 already establishes that $(C_X^{*a}, \leqslant)$ is not a sub-join-semilattice of $(C_X, \leqslant)$ since, as it is easily checked, c_I and c_{II} as defined there do belong to C_X^{*a}.

Next, consider c_{III} and c_{IV} defined as follows: assume without loss of generality $X = \{x, y, z\}$, and take $\Delta^{III} = \{(x, y), (x, z), (y, z)\}$, $\Delta^{IV} = \{(x, y), (x, z), (z, y)\}$ (notice that both (X, Δ^{III}) and (X, Δ^{IV}) are asymmetric digraphs); then, for any $A \subseteq X$, posit $c_{III}(A) = \mathbb{C}(A, \Delta_A^{III})$ and $c_{IV}(A) = \mathbb{C}(A, \Delta_A^{IV})$. Clearly, by definition, $\left\{c^{III}, c^{IV}\right\} \subseteq C_X^{*a}$.

However, $(c^{III} \cap c^{IV})(\{y, z\}) = \mathbb{C}(\{y, z\}, \Delta_{\{y,z\}}^{III}) \cap \mathbb{C}(\{y, z\}, \Delta_{\{y,z\}}^{IV}) = \{y\} \cap \{z\} = \varnothing$.

Therefore, $c^{III} \cap c^{IV}$ violates 2-PR hence by Theorem 7 $c^{III} \cap c^{IV} \notin C_X^{*a}$. It follows that $(C_X^{*a}, \leqslant)$ is not a sub-meet-semilattice of $(C_X, \leqslant)$.

The last statement about minimal elements of $(C_X^{*a}, \leqslant)$ is a straightforward consequence of Theorem 10. ∎

Theorem 21 *The poset $(C_X^{*^\circ}, \leqslant)$ of nonempty-valued core-solutions has a top element, c^{id}, and C_+ is the set of its coatoms, but it is neither a sub-meet-semilattice nor a sub-join-semilattice of $(C_X, \leqslant)$. The minimal elements of $(C_X^{*a}, \leqslant)$ are the single-valued choice functions that satisfy C and CO.*

Proof. First, notice that by definition c^{id} is proper, hence $c^{id} \in C_X^{*^\circ}$ since as previously shown it is a core-solution. Also, it is immediately checked that, by definition, any c_{xy}^+ is *proper.* Therefore, the proof of Theorem 17 also establishes that C_+ is the set of coatoms of $(C_X^{*^\circ}, \leqslant)$. In the same vein, it is immediately checked that $c_I, c_{II}, c_{III}, c_{IV}$ -as defined above in the proofs of the two previous Theorems- are also proper. It follows, by those proofs, that $(C_X^{*^\circ}, \leqslant)$ is neither a sub-meet-semilattice nor a sub-join-semilattice of $(C_X, \leqslant)$. The final statement about minimal elements of $(C_X^{*^\circ}, \leqslant)$ is an immediate consequence of Corollary 12. ∎

Theorem 22 *The poset $(C_X^{*es}, \leqslant)$ of revealed externally stable core-solutions, has a top element, c^{id}, and C_+ is the set of its coatoms, but it is neither a sub-meet-semilattice nor a sub-join-semilattice of $(C_X, \leqslant)$. The minimal elements of $(C_X^{*a}, \leqslant)$ are the single-valued choice functions that satisfy C, CO and SS.*

Proof. Observe that for any $A \subseteq B \subseteq X$, if $c^{id}(B) \subseteq c^{id}(A)$ then of course $B \subseteq A$ i.e. $B = A$ whence $c^{id}(A) = c^{id}(B)$ and SS is clearly satisfied by c^{id}. In view of Theorem 14, this establishes that c^{id} is also the top element of $(C_X^{*es}, \leqslant)$. Also, it is immediately checked that any c_{xy}^+ satisfies SS: indeed, let $A, B \subseteq X$ be such that $A \subseteq B$ and $\varnothing \neq c_{xy}^+(B) \subseteq c_{xy}^+(A)$. Since $A \subseteq B$, the following jointly exhaustive cases are to be distinguished: (a) $\{x, y\} \subseteq A \cap B$; (b) $\{x, y\} \nsubseteq A \cup B$; (c) $\{x, y\} \subseteq B$ and $\{x, y\} \nsubseteq A$. Under (a), $c_{xy}^+(A) = A \smallsetminus \{y\}$ and $c_{xy}^+(B) = B \smallsetminus \{y\}$ hence $c_{xy}^+(A) \subseteq c_{xy}^+(B)$. Under (b), $c_{xy}^+(A) = A$ and $c_{xy}^+(B) = B$ hence again $c_{xy}^+(A) \subseteq c_{xy}^+(B)$. Under (c), $c_{xy}^+(A) = A$ and $c_{xy}^+(B) = B \smallsetminus \{y\}$ whence

$A \neq B$ i.e. $A \subset B$. By hypothesis, $c^+_{xy}(B) \subseteq c^+_{xy}(A)$ hence $B \setminus \{y\} \subseteq A \subset B$: thus, $y \notin A$ and $B = A \cup \{y\}$ and therefore $c^+_{xy}(B) = B \setminus \{y\} = A = c^+_{xy}(A)$. It follows that c^+_{xy} does in fact satisfy SS. Therefore, the proof of Theorem 17 also establishes that C_+ is the set of coatoms of $(C^{*es}_X, \leqslant)$.

Finally, it is immediately checked by direct inspection that $c_I, c_{II}, c_{III}, c_{IV}$ -as defined above in the proofs of Theorems 19 and 20- do also (trivially) satisfy SS. It follows, by the very same proofs, that $(C^{*es}_X, \leqslant)$ is neither a sub-meet-semilattice nor a sub-join-semilattice of $(C_X, \leqslant)$. The final statement about minimal elements of $(C^{*es}_X, \leqslant)$ is an immediate consequence of Theorem 14. ■

Thus, while only the poset of revealed core-solutions is a (meet) sub-semilattice of $(C_X, \leqslant)$ all the posets of revealed cores defined above share their top element and set of coatoms.

4 Concluding remarks

Choice functions which may be regarded as core-solutions or externally stable core solutions of an underlying dominance digraph (X, Δ) have been characterized both in the general case and for asymmetric dominance digraphs. Both characterizations combine restrictions on local nonemptiness and the usual mix of contraction consistency and expansion consistency conditions for choice sets which is required for proper i.e. nonempty-valued choice functions.

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