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Symmetric revealed cores and pseudocores, and Lawvere-Tierney closure operators

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Abstract - A choice function is a *symmetric revealed* core if there exists a symmetric irreflexive 'dominance' digraph such that choice sets consist precisely of the locally undominated outcomes of the latter. *Symmetric revealed pseudocores* are similarly defined by omitting the irreflexivity requirement on the underlying digraph. *Lawvere-Tierney (LT) closure operators* are those closure operators which are meet-homomorphic: they may be regarded as an algebraic representation of a geometric modality denoting 'locally true', and provide the mathematical backbone of a generalized version of so-called 'Grothendieck topologies' in categories. The classes of symmetric revealed cores and pseudocores are characterized, and their basic order-theoretic structure is studied. In particular, it is shown that their respective posets are sub-meet-semilattices of the canonical lattice of choice functions. An order duality theorem concerning the posets of symmetric revealed pseudocores and LT closure operators on a given ground set is also established.

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1 Introduction

The *core* of a game is the set of its *undominated* outcomes, with respect to a suitably defined irreflexive *dominance* relation, or loopless digraph. In particular, such a dominance relation may well be a *symmetric* one, thus denoting in fact *mutual incompatibilities* between outcomes: then, the core is generated by a symmetric digraph and is therefore denoted here as a *symmetrically-based* or simply *symmetric* core. Indeed, in that case the core may be equivalently regarded as the set of outcomes which are *compatible* with *every* feasible alternative.

Now, consider the ongoing operation of an interaction system, e.g. an organization or indeed any suitably complex decision-making unit that is aptly modelled as a game. Furthermore, let us assume that the set of available options does change at a faster pace than the behavioural attitudes of the relevant players *and* the latter interact as predicted by the (symmetric) core of that game. It follows that the corresponding choice behaviour of the given interaction system as recorded by its choice function should be constrained in some way by its underlying game-theoretic structure, and thus somehow *reveal* it. But then, what are the characteristic features of such a choice function, hence the *testable* behavioural predictions of the core as a solution concept when applied to games with a *symmetric dominance* relation? Namely, how can one tell choice functions induced by cores under a symmetric dominance digraph from non-symmetric cores or other game solutions? Or, to put it in the simplest way, *which choice functions may be regarded as symmetric revealed cores*?

To the best of my knowledge, that problem has never been addressed in the extant literature. To be sure, parts of the massive body of literature on 'revealed preference' provide partial answers addressing the case of *nonempty cores*, i.e. of *acyclic* revealed dominance digraphs (see e.g. Wilson (1970), Sen (1971), Plott (1974), Suzumura (1983)). Moreover, there is also some work covering the case of empty sets of undominated outcomes for an *arbitrary* -possibly *not* irreflexive- binary relation R, hence putting aside the *standard* game-theoretic interpretation of R as a dominance relation (see e.g. Aizerman and Aleskerov (1995), and Danilov and Koshevoy (2009)). But of course the dominance relation of a game as usually construed has to be *irreflexive* (no outcome dominates itself), and the core of a game may well be *empty*, because its *revealed dominance digraph may have cycles.* Here, we are interested precisely in the *general* symmetric version of the core revelation problem: thus, we provide *a characterization of all symmetric revealed cores as solutions for a certain 'universal' outcome set and its subsets, including (locally) empty-valued symmetric cores.* The class of *symmetric revealed pseudocores*, obtained by *dropping irreflexivity* of the underlying revealed digraph, is also considered. Moreover, building on such characterizations, the basic order-theoretic structure of the class of symmetric revealed cores and pseudocores is analyzed. In particular, it is shown that *the posets of symmetric revealed cores and pseudocores are both sub-meet-semilattices (but not sub-lattices) of the canonical lattice of all choice functions* under the component-wise set-inclusion order (see section 4 below).

As clearly exemplified by the foregoing discussion, game-theoretic solution concepts may be typically represented by *choice functions* on a suitably defined outcome set. Choice functions are *deflationary or contracting* operators on a certain ground set: for any feasible subset of the latter a choice function specifies the subset that is selected out of it for acceptance (or perhaps rejection). By contrast, *inflationary or extensive* operators as similarly defined on a certain ground set attach a superset to any subset of the latter. *Closure operators* are those inflationary operators that are also *projections* i.e. satisfy *idempotence and monotonicity.* Amongst them, *Lawvere-Tierney (LT) closure operators* are those enjoying the further property of being *meet-homomorphic* (or *multiplicative*): they may be interpreted as an algebraic representation of a *geometric modality denoting local truth (* e.g. 'it is locally the case that'*)*, and when introduced in a suitable categorial framework they provide a generalized version of so-called 'Grothendieck topologies' (see e.g. Goldblatt (2006)).

In a recent remarkable paper, Danilov and Koshevoy (2009) define both an antitone and an isotone bijection between choice functions (i.e. deflationary or contracting operators) and inflationary (or extensive) operators, and use them in order to establish correspondences between several classes of choice functions and inflationary operators (including monotonic inflationary operators, closure operators, additive inflationary operators). However, they do not take into consideration LT closure operators as discussed above.

The present paper shows that according to the isotone bijection defined by Danilov and Koshevoy *the choice functions that do correspond to LT closure operators are precisely the symmetric revealed pseudocores.* It follows that the posets of symmetric revealed pseudocores and of LT closure operators are *isomorphic* hence our results on the basic order-theoretic structure of the former also apply to the latter.

The paper is organized as follows: section 2 includes a presentation of the model and the main characterization results; section 3 introduces LT closure operators and discusses their isotonic bijection to symmetric revealed pseudocores; section 4 provides some basic results concerning the order-theoretic properties of the classes of symmetric revealed cores and pseudocores as previously characterized; section 5 consists of a few concluding remarks.

2 Choice functions and symmetric revealed cores and pseudocores

Let X be a set denoting the 'universal' outcome set, with cardinality $\#X \geq 3$, and $\mathcal{P}(X)$ its power set. It is also assumed for the sake of convenience that X is *finite* (but it should be remarked that the most part of ensuing analysis is easily lifted with suitable minor adaptations to the case of an infinite outcome set). A *choice function on* X is a *deflationary* operator on X i.e. a function $c : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ such that $c(A) \subseteq A$ for any $A \subseteq X$ (empty choice sets are allowed). A choice function c is *proper* if $c(A) \neq \varnothing$ whenever $\varnothing \neq A \subseteq X$. We denote C_X the set of all choice functions on X. For any binary relation $\mathcal{B} \subseteq X \times X$, and any $Y \subseteq X$, $\mathcal{B}_Y = \mathcal{B} \cap (Y \times Y)$ and $\overline{\mathcal{B}} = (X \times X) \setminus \mathcal{B}$, while $\mathcal{B}^s$ and $\mathcal{B}^a$ denote the symmetric and asymmetric components of $\mathcal{B}$, respectively. Recall that $\mathcal{B} \subseteq X \times X$ is *reflexive* iff $x\mathcal{B}x$ for all $x \in X$, *irreflexive* iff *not* $x\mathcal{B}x$ for all $x \in X$, *symmetric* iff $x\mathcal{B}y$ entails $y\mathcal{B}x$ for any $x, y \in X$. Moreover, $\mathcal{B}$ is a *tolerance* relation iff it is both reflexive and symmetric.

Let $\Delta \subseteq X \times X$ be an *irreflexive* binary relation on X, denoting a suitably defined *dominance* relation: (X, Δ) is the corresponding *dominance digraph* (in graph-theoretic parlance, (X, Δ) is in particular a simple, loopless digraph i.e. a directed graph with at most one arc between any ordered pair of vertices, and with no arc from any vertex to itself). In particular, (X, Δ) is a *symmetric dominance* iff Δ is symmetric i.e. $\Delta = \Delta^s$.

For any $Y \subseteq X$, $\Delta_Y = \Delta \cap (Y \times Y)$ denotes the dominance relation induced by Δ on Y (of course $\Delta_X = \Delta$), and (Y, Δ_Y) is the induced *dominance subdigraph* on Y. The *core* of (Y, Δ_Y) is the set of Δ_Y-undominated outcomes in Y, namely $\mathbb{C}(Y, \Delta_Y) = \{y \in Y : \text{ not } z\Delta_Y y \text{ for all } z \in Y\}$.

Remark 1 *It should be emphasized here that* any (irreflexive) *dominance digraph may arise in a natural way from an underlying game in coalitional form. Moreover, any* symmetric *(irreflexive)dominance digraph may arise* *in a natural way from a game in coalitional form* and *from a related game in strategic form (see Vannucci (2009) for further details).*

The *basic revealed dominance digraph* $(X, \Delta(c))$ of a choice function $c \in C_X$ is defined by the following rule: for any $x, y \in X$, $x\Delta(c)y$ if and only if $x \neq y$ and $y \notin c(\{x, y\})$. Clearly enough, $\Delta(c)$ is *irreflexive* by definition. Two further binary relations $R(c)$, R_c induced by c on X and defined as follows will also be considered below: for any $x, y \in X$, xR_cy if and only if $x \in c(\{x, y\})$, while $xR(c)y$ if and only if there exists $Y \subseteq X$ such that $x \in c(Y)$ and $y \in Y$.

A choice function $c \in C_X$ is a *symmetric revealed core-solution (* or more loosely a *symmetric revealed core)* if there exists a *symmetric* irreflexive relation $\Delta \subseteq X \times X$ such that $c(Y) = \mathbb{C}(Y, \Delta_Y)$. Moreover, if Δ is just *symmetric* (but possibly *not* irreflexive) we shall declare $c \in C_X$ to be a *symmetric revealed pseudocore-solution* (or a *symmetric revealed pseudocore*).

Then, we shall also say that c is *s-core-rationalizable (s-pseudocore-rationalizable*, respectively*)* by *digraph* (X, Δ).

Example 2 *Notice that the digraph $(X, \varnothing)$ is also a symmetric dominance digraph, and $\mathbb{C}(A, \varnothing_A) = A$ for any $A \subseteq X$. Therefore, the* identity *operator $c^{id} : \mathcal{P}(X) \to \mathcal{P}(X)$ is a* symmetric revealed core-solution. *It is worth emphasizing that c^{id} does indeed admit a full-blown game-theoretic implementation. Here is an example: consider a nonempty set N, an effectivity function $E : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}(X))$ such that for any $S \subseteq N$, $A \subseteq X$, $A \in E(S)$ iff $S = N$ and $A \neq \varnothing$ or $S \neq \varnothing$ and $A = X$, and for any $i \in N$, a symmetric binary relation $\succcurlyeq_i$. Then, define the canonical (symmetric) dominance $\Delta^\Gamma \subseteq X \times X$ attached to the coalitional game $\Gamma = (N, X, E, (\succcurlyeq_i)_{i\in N})$ as follows: for any $x, y \in X$, $x\Delta^\Gamma y$ iff $x \succcurlyeq_i y$ and* not *$y \succcurlyeq_i x$ for each $i \in N$. Clearly, $\Delta^\Gamma = \varnothing$.*

Example 3 *By way of contrast, take $\varnothing \subseteq G \subset X$ and consider the dichotomic choice function $c_-^G : \mathcal{P}(X) \to \mathcal{P}(X)$ as defined by the 'strict'* satisficing *rule $c_-^G(A) = A \cap G$ for any $A \subseteq X$. It is easily checked that c_-^G is* not *a symmetric revealed core: to see this, take any $x \in X \setminus G$. Then, $c_-^G(\{x\}) = \varnothing$ while for any dominance digraph (X, Δ) and any $x \in X$, it cannot be the case that $x\Delta x$ hence $\mathbb{C}(\{x\}, \Delta_{\{x\}}) = \{x\}$. However, c_-^G is a* symmetric revealed pseudocore*: to check this, just consider $\Delta^G = (X \setminus G)^2$ i.e. $x\Delta^G y$ iff $\{x, y\} \subseteq X \setminus G$.*

Example 4 *Next, take again $\varnothing \subseteq G \subset X$ and consider the nonempty-valued dichotomic choice function $c_{+}^{G} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ as defined by the 'lax'* satisficing *rule $c_{+}^{G}(A) = A \cap G$ for any $A \subseteq X$ if $A \cap G \neq \varnothing$, and $c(A) = A$ otherwise. Such a choice function is* not *a symmetric revealed pseudocore. Indeed, suppose c_{+}^{G} is s-pseudocore-rationalizable by (symmetric) digraph (X, Δ), and consider $A = X \setminus G$. By definition, $c_{+}^{G}(A) = A$ hence* not *$y \Delta z$, for any $y, z \in A$, while $c_{+}^{G}(X) = G = X \setminus A$, hence for any $y \in A$ there exists $x \in G$ such that $x \Delta y$ whence $y \Delta x$. It follows that $x \notin c_{+}^{G}(X)$, a contradiction.*

The main objective of this article is precisely to provide a *characterization of all symmetric revealed cores* and *pseudocores* in C_X, and study their basic order-theoretic structure.

To begin with, let us consider a requirement concerning local existence of nonempty solution sets.

No-dummy property (ND): $c(\{x\}) = \{x\}$ for any $x \in X$.

It is easily checked that ND is satisfied by all symmetric revealed cores, while it may well be violated by a symmetric pseudocore.

The following properties of a choice function $c \in C_X$ play a prominent role, under various labels, in the extant literature:

Chernoff contraction-consistency (CC): for any $A, B \subseteq X$ such that $A \subseteq B$, $c(B) \cap A \subseteq c(A)$.

Concordance (CO): for any $A, B \subseteq X$, $c(A) \cap c(B) \subseteq c(A \cup B)$.

Consistency under good additions (CGA): for any $A \subseteq X$ and $x \in X$, if $x \in c(A \cup \{x\})$ then $c(A) \subseteq c(A \cup \{x\})$.

Property CC is a contraction-consistency condition for choice sets in that it requires that any outcome chosen out of a certain set should also be chosen out of any subset of the former: essentially, it says that *any good reason to choose a certain option out of a given menu should retain its strenght in every submenu of the former containing that option.*

Conversely, property CO (also variously denoted as γ or Generalized Condorcet-consistency) is an expansion-consistency condition for choice sets, requiring that an outcome chosen out of a certain set and of a second one should also be chosen out of the larger set given by the union of those two sets: it says *any good reason to choose a certain option out of two given menus should retain its strenght in the larger menu obtained by merging those two menus.*

Property CGA (introduced in Danilov and Koshevoy (2009) under the alternative label 'Matroidal axiom') is also an expansion-consistency requirement for choice sets: it rules out the possibility that adding a good option to a certain menu renders 'bad' some 'good' option of the previous menu.

We are now ready to proceed to the statement and proof of the announced characterizations of symmetric revealed cores and pseudocores.

Let us start from the following simple observation, namely

Claim 5 *Let $R \subseteq X \times X$ be any (binary) relation on X, and define $\Delta^R \subseteq X \times X$ by the following rule: for any $x, y \in X$, $x\Delta^R y$ iff* not *yRx. Then,*

(i) $R^{\Delta^R} = R$;

(ii) for any $Y \subseteq X$, $\max R_Y = \{x \in Y : \text{not } y\Delta^R x \text{ for all } y \in X\}$, and $\max \Delta^R_Y = \{x \in Y : \text{not } yRx \text{ for all } y \in X\}$;

(iii) R is reflexive iff Δ^R is irreflexive, and irreflexive iff Δ^R is reflexive;

(iv) R is symmetric iff Δ^R is symmetric.

Proof. (i) For any $x, y \in X$, by definition $xR^{\Delta^R}y$ iff *not* $y\Delta^R x$ iff *not* (*not* xRy) iff xRy.

(ii) Let $x \in Y$, and xRy for all $y \in Y$: then, by definition, *not* $y\Delta^R x$ for all $y \in Y$, and conversely if *not* $y\Delta^R x$ for all $y \in Y$ then *not (not xRy)* i.e. xRy for all $y \in Y$. Similarly, $x \in Y$ and *not* yRx for all $y \in Y$: then by definition $x\Delta^R y$ for all $y \in Y$.

(iii) Indeed, by definition for any $x \in X$, *not* $x\Delta^R x$ iff *not(not xRx)* i.e. xRx. Similarly, *not* xRx iff $x\Delta^R x$.

(iv) Suppose Δ^R is symmetric: then, for any $x, y \in X$, $x\Delta^R y$ entails $y\Delta^R x$. Now, if xRy then *not* $y\Delta^R x$ hence *not* $x\Delta^R y$: thus by definition yRx, as claimed. Conversely, suppose R is symmetric. For any $x, y \in X$, if $x\Delta^R y$ then *not* yRx hence *not* xRy: therefore, by definition, $y\Delta^R x$. ∎

We can now state our characterization of symmetric revealed cores, namely

Theorem 6 *Let $c \in C_X$. Then, the following statements are equivalent:*

(i) c satisfies ND, CC, CO and CGA;

(ii) there exists a symmetric *dominance digraph* (X, Δ) *such that* $c(Y) = \mathbb{C}(Y, \Delta_Y)$ *for any* $Y \subseteq X$*;*

(iii) there exists a tolerance *relation* $R \subseteq X \times X$ *such that* $c(Y) = \max R_Y$ *for any* $Y \subseteq X$.

(iv) $R(c) = R_c$*,* $R(c)$ *is a* tolerance *relation, and* $c(Y) = \max R(c)_Y$ *for any* $Y \subseteq X$.

Proof. (i)$\Longrightarrow$(iv): The equality $R(c) = R_c$ follows immediately from CC: indeed, $R_c \subseteq R(c)$ by definition, and $xR(c)y$ implies by CC that $x \in \max R(c)_{\{x,y\}} = c(\{x, y\})$ i.e. xR_cy hence $R_c = R(c)$ (of course, this is an extension to arbitrary choice functions of the proof of the same result for *nonempty-valued* choice functions due to Sen (1971)).

Now, observe that for each $Y \subseteq X$ and $x \in c(Y)$, $xR(c)y$ for any $y \in Y$, by definition of $R(c)$. Hence $c(Y) \subseteq \max R(c)_Y$. Next, let $x \in \max R(c)_Y$. Then, by definition, for any $y \in Y$ there exists Y_y such that $y \in Y_y$ and $x \in c(Y_y)$. It follows, by CC, that $x \in c(\{x, y\})$ for any $y \in Y$ whence, by CO, $x \in c(Y)$. Therefore, $c(Y) = \max R(c)_Y$ (clearly it may be the case that $\max R(c)_Y = \varnothing$). Notice however that, by ND, $x \in c(\{x\})$ i.e. $xR(c)x$ for any $x \in X$. Thus, $R(c)$ is *reflexive.* Moreover, if $xR(c)y$ then by the equality $R(c) = R_c$, xR_cy i.e. $x \in c(\{x, y\})$. Therefore, by CGA, $c(\{y\}) \subseteq c(\{x, y\})$, and by ND $y \in c(\{y\})$, whence $y \in c(\{x, y\})$. It follows that yR_cx i.e. $R(c) = R_c$ is also symmetric as required.

(iv)$\Longrightarrow$(iii): Trivial.

(ii)$\Longleftrightarrow$(iii): By Claim 5 (ii), if there exists $R \subseteq X \times X$ such that $c(Y) = \max R_Y$ for any $Y \subseteq X$, then $c(Y) = \{x \in Y : \textit{not } y\Delta^R x \text{ for all } y \in X\}$, for any $Y \subseteq X$. Moreover, if R is reflexive and symmetric then by Claim 5(iii)-(iv) Δ^R is irreflexive and symmetric hence $c(Y) = \mathbb{C}(Y, \Delta^R_Y)$ for any $Y \subseteq X$, i.e. c is a symmetric revealed core. Conversely, if there exists an irreflexive and symmetric $R \subseteq X \times X$ such that $c(Y) = \mathbb{C}(Y, R_Y)$ for any $Y \subseteq X$ then by Claim 5 (ii)-(iii)-(iv) $c(Y) = \max R^{\Delta}_Y$ for any $Y \subseteq X$, and R^{Δ} is reflexive and symmetric.

(iii)$\Longrightarrow$(i): Suppose that there exists a *tolerance* relation $R \subseteq X \times X$ such that $c(Y) = \max R_Y$ for any $Y \subseteq X$. Clearly, by reflexivity of R, $c(\{x\}) = \max R_{\{x\}} = \{x\}$, hence c satisfies ND. Moreover, for any $Y \subseteq Z \subseteq X$ and any $x \in c(Z) = \max R_Z$, it must also be the case, by definition, that $x \in \max R_Y = c(Y)$ hence CC is also satisfied by c. Furthermore, for any $Y, Z \subseteq X$ and $x \in X$, if $x \in c(Y) = \max R_Y$ and $c(Z) = \max R_Z$ then clearly $x \in \max R_{Y \cup Z}$ whence $x \in c(Y \cup Z)$ as well and CO is satisfied.

Finally, for any $Y \subseteq X$ and $x, y \in X$, if $x \in c(Y \cup \{x\}) = \max R_{Y \cup \{x\}}$ and $y \in c(Y) = \max R_Y$ then $y \in c(Y \cup \{x\})$: in fact, suppose it does not i.e. $y \notin \max R_{Y \cup \{x\}}$. Then, it must be the case that *not* yRx hence, by symmetry of R, *not* xRy which implies $x \notin \max R_{Y \cup \{x\}}$, a contradiction. Therefore, CGA also holds. ■

Remark 7 *The foregoing characterization result is tight. To check that, consider the following examples.*

(1) Let $c^{\varnothing} \in C_X$ be the empty choice function defined as follows: for any $A \subseteq X$, $c^{\varnothing}(A) = \varnothing$. Clearly, $c^{\varnothing}$ violates ND, but satisfies CC, CO and CGA;

(2) Let $X = \{x, y, z\}$, and $c \in C_X$ be defined as follows: $c(\{x, y\}) = \varnothing$ for any pair of distinct *$x, y \in X$ and $c(Y) = Y$ for any $Y \subseteq X$ such that $\#Y \neq 2$. It is immediately checked that, by construction, c satisfies ND and CO. Moreover, c satisfies CGA: indeed, let $x \in c(Y \cup \{x\})$ and $y \in c(Y) \subseteq Y$ and assume without loss of generality that $x \neq y$ (if $x = y$ there is nothing to prove); by definition $c(Y \cup \{x\}) = Y \cup \{x\}$ since $c(Y \cup \{x\}) \neq \varnothing$, hence $y \in c(Y \cup \{x\})$ as required by CGA. However, c violates CC since e.g. for any $x, y \in X$ with $x \neq y$, $x \in c(X) \cap \{x, y\}$ but $x \notin c(\{x, y\})$.*

(3) Let $c' \in C_X$ be defined as follows: for any $A \subset X$, $c'(A) = A$, and $c'(X) = \varnothing$. It is easily seen that by construction c' satisfies ND and CC. Moreover, let $x \in c'(Y \cup \{x\})$ and $y \in c'(Y)$. Now, suppose that $y \notin c'(Y \cup \{x\})$: then, by definition of c', $Y = X \setminus \{x\}$ and $c'(Y \cup \{x\}) = \varnothing$, a contradiction since $x \in c'(Y \cup \{x\})$. Thus, c' also satisfies CGA. On the other hand, clearly c' violates CO.

(4) Let $X = \{x, y, z\}$, with $x \neq y \neq z \neq x$, and $c'' \in C_X$ be defined as follows: $c''(\{h\}) = \{h\}$ for any $h \in X$, $c''(\{x, y\}) = \{x\}$, $c''(\{y, z\}) = \{y\}$, $c''(\{x, z\}) = \{z\}$, and $c''(X) = \varnothing$. It is immediately checked that c'' satisfies ND, CC and CO, but obviously it violates CGA since e.g. $x \in c''(\{x, y\})$, $y \in c''(\{y\})$ but $y \notin c''(\{x, y\})$.

Notice that, by pointing out that symmetric core outcomes are precisely the maximizers of an underlying tolerance relation, Theorem 6 makes also precise our previous claim that the symmetric core selects precisely those outcomes which are compatible with *any* feasible outcome (see e.g. Schreider (1975) for a thorough discussion of tolerance relations).

By dropping the ND property, the following characterization of symmetric revealed *pseudocores* obtains

Theorem 8 *Let $c \in C_X$. Then, the following statements are equivalent:*
(i) c satisfies CC, CO and CGA;
(ii) there exists a symmetric *digraph* (X, Δ) *such that for any* $Y \subseteq X$ $c(Y) = \{y \in Y :$ *for each* $z \in X$, not $z\Delta y$ $\}$ *i.e. c is a* symmetric revealed pseudocore*;*
(iii) there exists a symmetric *relation* $R \subseteq X \times X$ *such that* $c(Y) = \max R_Y$ *for any* $Y \subseteq X$.
(iv) $R(c) = R_c$, $R(c)$ *is a* symmetric *relation, and* $c(Y) = \max R(c)_Y$ *for any* $Y \subseteq X$.

Proof. It follows at once from the proof of the previous theorem. ■

Remark 9 *Notice that Examples (2),(3),(4) from the previous Remark establish that the foregoing characterization of symmetric revealed pseudocores is also tight.*

It should be remarked that the pseudocore amounts to a slight extension of the core arising from the allowance of some outcomes that are somehow '*self-dominated'*, and in any case are *never* selected. Such a mild extension of the core does in fact disable the *standard* interpretation of its induced 'dominance' digraph as the outcome of an implicit underlying coalitional, strategic or even extensive game and is therefore not inconsequential. However, the foregoing drawback of the pseudocore should not be exaggerated: for instance, one might allow a subset $X^\circ \subseteq X$ of 'dummy' outcomes in the outcome set, and regard the induced digraph (X, Δ) of a pseudocore as an 'extended' dominance digraph where Δ is the set-theoretic join of the standard irreflexive 'dominance' relation (induced on $X \setminus X^\circ$) and $X \times X^\circ$ (i.e. by definition 'dummy' outcomes are 'extendedly' dominated by *any* outcome). Thus, allowance of a locally reflexive induced digraph is not *that* disruptive for a game-theoretic interpretation of the choice function under consideration, if outcomes in the reflexive component of Δ are in fact never selected. Hence, the pseudocore is after all consistent with a suitably adjusted game-theoretic model and is otherwise a rather innocuous generalization of the core as a solution concept.

3 Choice functions and Lawvere-Tierney closure operators

An *inflationary* or *extensive* operator on X is a function $k : \mathcal{P}(X) \to \mathcal{P}(X)$ such that $A \subseteq k(A)$ for any $A \subseteq X$. An operator k on X is *monotonic* if $k(Y) \subseteq k(Z)$ for any $Y, Z \subseteq X$ such that $Y \subseteq Z$, and *idempotent* iff $k(Y) = k(k(Y))$. An inflationary operator k on X is a *closure* operator if it is both monotonic and idempotent. A *Lawvere-Tierney or LT closure operator* on X is a closure operator k that is also *meet-homomorphic* i.e. such that $k(Y \cap Z) = k(Y) \cap k(Z)$ for any $Y, Z \subseteq X$ (notice that a meet-homomorphic operator is bound to be monotonic as well, hence a LT closure operator can also be described as an *inflationary, idempotent and meet-homomorphic* operator).

In a recent paper, Danilov and Koshevoy (2009) define an *isotonic bijection* between choice functions *that satisfy CC* and *monotonic* inflationary operators on X (to a certain extent, that remarkable paper relies on -and extend- some previous related work, including Koshevoy (1999), Monjardet and Raderanirina (2001), Johnson and Dean (2001), Danilov and Koshevoy (2005), Ando (2006), Monjardet (2007)).

Indeed, let $(C_X^*, \leqslant)$ and $(I_X^*, \leqslant')$ denote respectively the class of all choice functions that satisfy CC and the class of monotonic inflationary operators on X, endowed with their component-wise set-inclusion orders: namely, for any $c, c' \in C_X^*$, and any $k, k' \in I_X^*$, $c \leqslant c'$ iff $c(A) \subseteq c'(A)$ for each $A \subseteq X$, and similarly $k \leqslant' k'$ iff $k(A) \subseteq k'(A)$ for each $A \subseteq X$. Also, for any $A \subseteq X$ denote $\overline{A}$ its complement in X. Then, the *isotonic bijection* (namely, the *order-isomorphism* between $(C_X^*, \leqslant)$ and $(I_X^*, \leqslant')$) mentioned above is provided by the function $F^* : C_X^* \to I_X^*$ defined by the following rule: for any choice function $c \in C_X^*$,

$F^*(c) = k_c$ where for any $A \subseteq X$ and $x \in X$: $x \in k_c(A)$ iff $[x \in c(\overline{A} \cup \{x\})$ *or* $x \in A]$,

or by its inverse $G^* : I_X^* \to C_X^*$ defined by the following rule: for any inflationary operator $k \in I_X^*$,

$G^*(k) = c_k$ where for any $A \subseteq X$ and $x \in X$: $x \in c_k(A)$ iff $[x \in A$ *and* $x \in k(\overline{A})]$.

Remark 10 *It is worth noticing at the outset that F^*admits the following equivalent formulation:*

for any $c \in C_X^*$, $A \subseteq X$ and $x \in X$: $k_c \in I_X$ and $x \in k_c(A \setminus \{x\})$ iff $x \in c(\overline{A} \cup \{x\})$.

To check this claim, suppose first $k_c = F^*(c)$. Clearly, $F^*(c) \in I_X$ by construction. Now, assume $x \in k_c(A \setminus \{x\})$. Then, by definition, $x \in c(\overline{A \setminus \{x\}} \cup \{x\}) = c(\overline{A} \cup \{x\})$. On the other hand, if $x \in c(\overline{A} \cup \{x\}) = c(\overline{A \setminus \{x\}} \cup \{x\})$ then by definition $x \in k_c(A \setminus \{x\})$.

Conversely, let $c \in C_X^*$ and $k \in I_X$ be such that for any $A \subseteq X$ and $x \in X$:

$$x \in k(A \setminus \{x\}) \text{ iff } x \in c(\overline{A} \cup \{x\}).$$

Then, $x \in k(A)$ iff $x \in k((A \setminus \{x\}) \cup \{x\})$ iff $x \in c(\overline{A \setminus \{x\}} \cup \{x\}) = c(\overline{A} \cup \{x\})$ hence, by definition of F^*, $x \in k_c(A)$. On the other hand, if $x \in k_c(A)$ then, by definition of F^*, $x \in c(\overline{A} \cup \{x\})$ or $x \in A$. If $x \in c(\overline{A} \cup \{x\}) = c(\overline{A \cup \{x\}} \cup \{x\})$ hence, by definition of k, $x \in k(A)$, while $x \in A$ entails $x \in k(A)$ as well since $k \in I_X$.

It is easy to confirm that $F^*$and G^* are mutually inverse functions and order-homomorphisms (i.e. order-preserving) w.r.t. $(C_X^*, \leqslant)$ and $(I_X^*, \leqslant')$, and do therefore establish an isotonic bijection between C_X^* and I_X^* thus ordered: that fact is noticed and stated by Danilov and Koshevoy (2009) without an explicit proof. We provide here a detailed proof of the foregoing fact for the sake of completeness.

Claim 11 *Let $(C_X^*, \leqslant)$, $(I_X^*, \leqslant')$ and F^*, G^*be as defined above. Then, F^* and G^* are mutually inverse* order-isomorphisms *(from $(C_X^*, \leqslant)$ to $(I_X^*, \leqslant')$ and from $(I_X^*, \leqslant')$ to $(C_X^*, \leqslant)$, respectively).*

Proof. First, notice that $F^*$and G^* are well-defined functions. Then, we are going to check that $F^*$is indeed injective, surjective, and an order-homomorphism from $(C_X^*, \leqslant)$ to $(I_X^*, \leqslant')$.

To see that F^* is injective, take any $c, c' \in C_X^*$ such that $c \neq c'$. Then, there exist $A \subseteq X$ and $x \in A$ such that, say, $x \in c(A)$ and $x \notin c'(A)$. Thus, $x \in A$ i.e. $x \notin \overline{A}$ and $A = A \cup \{x\} = \overline{\overline{A}} \cup \{x\}$: therefore, by definition, $x \in k_c(\overline{A})$ and $x \notin k_{c'}(\overline{A})$ i.e. $F^*(c) \neq F^*(c')$.

To check surjectivity of F^*, take any $k \in I_X^*$ and consider $c_k = G^*(k) \in C_X$, namely for any $A \subseteq X$ and $x \in X$,

$$x \in c_k(A) \text{ iff } x \in A \text{ and } x \in k(\overline{A}).$$

Thus, for any $A \subseteq X$ and $x \in X$,

$x \in F^*(c_k)(A)$ iff $[x \in c_k(\overline{A} \cup \{x\})$ *or* $x \in A]$ iff $[(x \notin A$ and $x \in c_k(\overline{A} \cup \{x\}))$ or $x \in A]$ iff $[x \in c_k(\overline{A})$ or $x \in A]$ iff $[(x \notin A$ and $x \in c_k(\overline{A}))$ or $x \in A]$ iff $[(x \notin A$ and $x \in k(A))$ or $x \in A]$ iff $x \in k(A)$, i.e. $F^*(G^*(k)) = k$.

To check that F^* is order-preserving, consider any $c, c' \in C_X^*$ such that $c \leqslant c'$. Therefore, for any $A \subseteq X$, and $x \in X$, if $x \in F^*(c)(A)$ then by definition $x \in c(\overline{A} \cup \{x\})$ *or* $x \in A$, hence by hypothesis $x \in c'(\overline{A} \cup \{x\})$ *or* $x \in A$ i.e. $x \in F^*(c')(A)$. It follows that $F^*(c) \leqslant' F^*(c')$ as required.

Moreover, take any $c \in C_X^*$: for any $A \subseteq X$ and $x \in X$, $x \in G^*(F^*(c))(A)$ iff $[x \in A$ and $x \in F^*(c)(\overline{A})]$ iff $[x \in A$ and $x \in c(\overline{\overline{A}} \cup \{x\})]$ iff $x \in c(A)$. Thus, $G^*(F^*(c)) = c$. It follows that F^* and G^* are mutually inverse bijections.

Finally, it remains to be checked that G^* is also order-preserving. Indeed, consider any $k, k' \in I_X^*$ such that $k \leqslant' k'$. Then, for any $A \subseteq X$, and $x \in X$, $x \in G^*(k)(A)$ entails $x \in A$ and $x \in k(\overline{A})$ hence by hypothesis $x \in k'(\overline{A})$ and therefore $x \in G^*(k')(A)$ as well, i.e. $G^*(k) \leqslant G^*(k')$ and the thesis follows. ■

Relying on the basic bijection between C_X^* and I_X^* introduced above, a few more specialized bijective correspondences may be established as recorded by the following

Proposition 12 *Let* $F^* : C_X^* \to I_X^*$ *the bijection defined above, and* $c \in C_X^*$. *Then,*

(i) $F^*(c) \in I_X^*$ *is* idempotent *iff* c *satisfies CGA;*

(ii) $F^*(c) \in I_X^*$ *is* meet-homomorphic *iff* c *satisfies CO.*

Proof. (i) To begin with, observe that any inflationary operator k on X is *idempotent* iff for all $A \subseteq X$, $k(k(A)) \subseteq k(A)$. Next, following Danilov and Koshevoy (2009) notice that a *monotonic* inflationary operator $k \in I_X^*$ is idempotent (hence a *closure* operator) iff it satisfies the following property:

$(*)$ for any $A \subseteq X$ and $x \in X$, $x \in k(A)$ entails $k(A \cup \{x\}) \subseteq k(A)$.

(To check the latter statement, suppose first that $k \in I_X^*$ is idempotent, and consider $A \subseteq X$ and $x \in k(A)$. By construction, $A \cup \{x\} \subseteq k(A)$. Hence, by monotonicity $k(A \cup \{x\}) \subseteq k(k(A))$, and by idempotence $k(A \cup \{x\}) \subseteq k(k(A)) = k(A)$. Conversely, let $k \in I_X^*$ satisfy $(*)$. By a simple inductive argument, it follows that for any (finite) $B \subseteq k(A)$, $k(A \cup B) \subseteq k(A)$, hence in particular $k(k(A)) = k(A \cup k(A)) \subseteq k(A)$: see Danilov and Koshevoy (2009)).

Now, suppose that $c \in C_X^*$ and $k_{cx} = F^*(c) \in I_X^*$ is *idempotent.* Then, take any $x \in c(A \cup \{x\})$ and $y \in c(A)$: by definition of F^*, $x \in k_c(\overline{A \cup \{x\}})$ and, since clearly $y \in A$ whence $A = A \cup \{y\}$, $y \in k_c(\overline{A})$. Therefore, by idempotence of k_c, and in view of the foregoing observation, $k_c(\overline{A} \cup \{x\}) = k_c(\overline{A \cup \{x\}} \cup \{x\}) \subseteq k_c(\overline{A \cup \{x\}})$. By monotonicity of k_c, $k_c(\overline{A}) \subseteq k_c(\overline{A} \cup \{x\})$, hence $y \in k_c(\overline{A \cup \{x\}})$ because $y \in k_c(\overline{A})$. Since by construction $y \in A \cup \{x\}$, and by definition of G^* (the inverse bijection of F^* as defined above in the text), $y \in c_{k_c}(A \cup \{x\}) = G^*(F^*(c))(A \cup \{x\}) = c(A \cup \{x\})$ hence c satisfies CGA.

Conversely, let $c \in C_X^*$ satisfy *CGA*. If $A \subseteq X$, $x \in k_c(A)$ and $y \in k_c(A \cup \{x\})$ then, by definition of F^*, $[x \in A$ or $(x \notin A$ and $x \in c(\overline{A} \cup \{x\}))]$ and $[y \in A \cup \{x\}$ or $(y \notin A \cup \{x\}$ and $y \in c(\overline{A \cup \{x\}} \cup \{y\}))]$. Now, if $x \in A$ then $A \cup \{x\} = A$ hence $y \in k_c(A)$. If $x \notin A$, $x \in c(\overline{A} \cup \{x\})$ and $y \in A \cup \{x\}$ then either $y = x$ whence $y \in k_c(A)$ or $y \in A$ hence again $y \in k_c(A)$ since k_c is an inflationary operator. Finally, if $x \notin A$, $x \in c(\overline{A} \cup \{x\})$, $y \notin A \cup \{x\}$ and $y \in c(\overline{A \cup \{x\}} \cup \{y\})$ then $\overline{A \cup \{x\}} \cup \{y\} = \overline{A \cup \{x\}}$ i.e. $y \in c(\overline{A \cup \{x\}})$. Since $x \in c(\overline{A} \cup \{x\}) = c(\overline{A \cup \{x\}} \cup \{x\})$, it follows that, by CGA, $c(\overline{A \cup \{x\}}) \subseteq c(\overline{A} \cup \{x\})$ hence $y \in c(\overline{A} \cup \{x\})$: thus, by definition, $y \in k_c(A)$ again, $(*)$ holds, and idempotence of k_c follows as required.

(ii) Let $F^*(c) = k_c \in I_X^*$ be *meet-homomorphic* i.e. for any $A, B \subseteq X$, $k_c(A) \cap k_c(B) \subseteq k_c(A \cap B)$. Then, take any $x \in c(A) \cap c(B)$ i.e. $x \in c((A \setminus \{x\}) \cup \{x\}) \cap c((B \setminus \{x\}) \cup \{x\})$: in view of Remark 10 above, $x \in k_c((\overline{A} \setminus \{x\}) \setminus \{x\}) = k_c(\overline{A} \setminus \{x\})$ and $x \in k_c((\overline{B} \setminus \{x\}) \cup \{x\}) = k_c(\overline{B} \setminus \{x\})$. Thus, by hypothesis, $x \in k_c(\overline{A} \setminus \{x\}) \cap k_c(\overline{B} \setminus \{x\}) \subseteq k_c((\overline{A} \setminus \{x\}) \cap (\overline{B} \setminus \{x\})) = k_c((\overline{A} \cap \overline{B}) \setminus \{x\}) = k_c((\overline{A \cup B}) \setminus \{x\})$

hence, by Remark 10 again, $x \in c((A \cup B) \cup \{x\})$ i.e. $x \in c(A \cup B)$ since clearly $x \in A \cup B$. It follows that c does satisfy CO.

Conversely, let c satisfy *CO*. Then, take any $x \in k_c(A) \cap k_c(B)$ i.e. $x \in k_c((A \cup \{x\}) \setminus \{x\})$ and $x \in k_c((B \cup \{x\}) \setminus \{x\})$.

Thus, by Remark 10, $x \in c\ ((\overline{A \cup \{x\}}) \cup \{x\}) = c(\overline{A} \cup \{x\})$ and $x \in c((\overline{B \cup \{x\}}) \cup \{x\}) = c(\overline{B} \cup \{x\})$. Therefore, by CO, $x \in c((\overline{A} \cup \{x\}) \cup (\overline{B} \cup \{x\})) = c((\overline{A} \cup \overline{B}) \cup \{x\}) = c(\overline{A \cap B} \cup \{x\})$

whence, by definition of k_c, $x \in k_c(A \cap B)$. It follows that k_c is meet-homomorphic. ■

By combining Proposition 12 and Theorem 8 we may immediately establish a truly remarkable fact, namely

Theorem 13 *Let $k \in I_X$, $c \in C_X$, F^*, G^* the isotonic bijections as defined*

above, $c = c_k = G^*(k)$ and $k = k_c = F^*(c)$. Then, the following statements are equivalent:
(i) k is a LT closure operator;
(ii) c is a symmetric revealed pseudocore.

Proof. Immediate from Theorem 8 and Proposition 12 above. ■

Remark 14 *To be sure, virtually all of the results of this section (apart from Theorem 13) can be shown to follow, in a somewhat roundabout way, from some of the results provided in Danilov and Koshevoy (2009), though -as mentioned previously- the latter work does not cover Lawvere-Tierney closure operators. We have opted here for a direct proof of the relevant order duality for the sake of completeness, and clarity.*

4 The semilattices of symmetric revealed cores and pseudocores

Let us now turn to a global description of the order-theoretic structure of the class of all symmetric revealed cores (pseudocores, respectively). Of course, in view of the duality result embodied in Proposition 11 and Theorem 13 above, *the order-theoretic structure of symmetric revealed pseudocores is precisely the same as that of LT closure operators.*

A partially ordered set or *poset* is a pair $\mathbf{P} = (P, \leqslant)$ where P is a set and $\leqslant$ is a reflexive, transitive and antisymmetric binary relation on P (i.e. for any $x \in P$, $x \leqslant x$ and for any $x, y, z \in P$, $x \leqslant z$ whenever $x \leqslant y$ and $y \leqslant z$, and $x = y$ whenever $x \leqslant y$ and $y \leqslant z$). For any $P' \subseteq P$, and with a slight abuse of language, we shall typically denote by $(P', \leqslant)$ the poset $(P', \leqslant_{P'})$ where $\leqslant_{P'} = (P' \times P') \cap (\leqslant)$. A *coatom* of a poset $\mathbf{P} = (P, \leqslant)$ with a top element or maximum 1_P is any $j \in P$ which is covered by 1_P- written $j \lessdot 1_P$- i.e. $j < 1_P$ and $l = j$ for any $l \in P$ such that $j \leqslant l < 1_P$. The set of all coatoms of $\mathbf{P}$ is denoted A^*_P. Dually, an *atom* of $\mathbf{P}$ is any $j \in P$ which is an upper cover of 0_P- written $0_P \lessdot j$- i.e. $0_P < j$ and $l = j$ for any $l \in P$ such that $0_P < l \leqslant j$. The set of all atoms of $\mathbf{P}$ is denoted A_P.

A poset $\mathbf{P} = (P, \leqslant)$ is a *meet semilattice* (*join semilattice*, respectively) if for any $x, y \in P$ the $\leqslant$-greatest lower bound $x \wedge y$ (the $\leqslant$-least upper bound $x \vee y$, respectively) of $\{x, y\}$ does exist. Moreover, $\mathbf{P}$ is a *lattice* if it is both a meet semilattice and a join semilattice.

A lattice $\mathbf{P} = (P, \leqslant)$ is *bounded* if there exist both a bottom element 0_P and a top element 1_P (hence in particular a finite lattice is also bounded), *distributive* iff $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for any $x, y, z \in P$, *complemented* if it is bounded and for any $x \in P$ there exists $x' \in P$ such that $x \vee x' = 1_P$ and $x \wedge x' = 0_P$, and *Boolean* iff it is both distributive and complemented.

Recall that, as pointed out in Section 2 above, the set C_X of all choice functions on X can be endowed in a natural way with the point-wise set inclusion partial order $\leqslant$ by positing, for any $c, c' \in C_X$, $c \leqslant c'$ iff $c(A) \subseteq c'(A)$ for each $A \subseteq X$. Clearly, the identity operator c^{id} is its top element, and the constant empty-valued choice function $c^{\varnothing}$ its bottom element. It is well-known, and easily checked, that $(C_X, \leqslant)$ is in fact a *Boolean lattice* with join $\vee = \cup$ (i.e. set-union) and meet $\wedge = \cap$ (i.e. set-intersection), both defined in the obvious component-wise manner: see e.g. Monjardet, Raderanirina (2004).

For any $x, y \in X$ such that $x \neq y$, $c_{xy} \in C_X$ and $c^{\circ}_{xy} \in C_X$ are defined as follows: for all $A \subseteq X$, $c_{xy}(A) = A \smallsetminus \{x, y\}$ if $\{x, y\} \subseteq A$, and $c_{xy}(A) = A$ otherwise, $c^{\circ}_{xy}(\{x, y\}) = \{x, y\}$, $c^{\circ}_{xy}(\{z\}) = \{z\}$ for any $z \in X$, and $c^{\circ}_{xy}(A) = \varnothing$ for all $A \subseteq X$ such that $A \neq \{x, y\}$ and $\#A \neq 1$. Moreover, $C^{*}_X = \{c_{xy} : x, y \in X, x \neq y\}$, and $C^{**}_X = \left\{c^{\circ}_{xy} : x, y \in X, x \neq y\right\}$.

The *minimum ND choice function* $c^{[1]}$ is defined by the following rule: for any $x \in X$, $c^{[1]}(\{x\}) = \{x\}$, and $c^{[1]}(Y) = \varnothing$ for any $Y \subseteq X$ such that $\#Y \neq 1$.

Now, let $C^{sc}_X \subseteq C_X$ denote the set of all symmetric revealed cores on X, and $C^{spc}_X \subseteq C_X$ the set of all symmetric revealed pseudocores. We have the following

Theorem 15 *The poset $(C^{sc}_X, \leqslant)$ of symmetric revealed cores is a* sub-meet-semilattice *of $(C_X, \leqslant)$ with c^{id} itself as its top element, but not a sub-join-semilattice of $(C_X, \leqslant)$. The bottom element of $(C^{sc}_X, \leqslant)$ is the minimum ND choice function $c^{[1]}$. Moreover, the set of coatoms of $(C^{sc}_X, \leqslant)$ is C^{*}_X, and the set of its atoms is C^{**}_X .*

Proof. Let $c, c' \in C^{sc}_X$, and consider $c \cap c'$. By Theorem 6 above c and c' satisfy ND, CC, CO and CGA.

Clearly, for any $x \in X$, $(c \cap c')(\{x\}) = c(\{x\}) \cap c'(\{x\}) = \{x\}$ since c and c' satisfy ND: hence $c \cap c'$ does also satisfy ND.

Also, for any $A \subseteq B \subseteq X$, since c and c' both satisfy CC, $(c \cap c')(B) \cap A = (c(B) \cap c'(B)) \cap A = c(B) \cap (c'(B)) \cap A) \subseteq c(B) \cap c'(A) \subseteq c(B) \cap A \subseteq A$ hence $c \cap c'$ satisfies CC.

Moreover, since c and c' satisfy CO, for any $A, B \subseteq X$,
$(c \cap c')(A) \cap (c \cap c')(B) =$
$= (c(A) \cap c(B)) \cap (c'(A) \cap c'(B)) \subseteq c(A \cup B) \cap c'(A \cup B) =$
$= (c \cap c')(A \cup B)$ and CO also holds for $c \cap c'$.

Finally, if $x \in (c \cap c')(A \cup \{x\})$ then $x \in c(A \cup \{x\})$ and $x \in c'(A \cup \{x\})$ hence by CGA $c(A) \subseteq c(A \cup \{x\})$ and $c'(A) \subseteq c'(A \cup \{x\})$. Therefore $(c \cap c')(A) \subseteq (c \cap c')(A \cup \{x\})$ i.e. $c \cap c'$ satisfies CGA.

It follows that, by Theorem 6 again, $c \cap c' \in C_X^{sc}$, whence $(C_X^{sc}, \leqslant)$ is a sub-meet-semilattice of $(C_X, \leqslant)$.

It is easily checked that c^{id}, the top element of $(C_X, \leqslant)$, does also satisfy ND, CC, CO and CGA hence as observed above $c^{id} \in C_X^{sc}$ (see Example 2).

Now, consider $c^{[1]}$ as defined above: by definition, it satisfies ND and, being nonempty-valued precisely on singletons, it trivially satisfies CC, CO and CGA as well. Thus, $c^{[1]} \in C_X^{sc}$. On the other hand, for any $c \in C_X^{sc}$, c must satisfy ND, hence $c^{[1]} \leqslant c$ as claimed.

Next, take any $c_{xy} \in C_X^{*}$. Notice that, by definition, c_{xy} satisfies ND. Also, if $A \subseteq B \subseteq X$ then the following cases may be distinguished: (a) $\{x, y\} \subseteq A$; (b) $\{x, y\} \nsubseteq A$ and $\{x, y\} \subseteq B$; (c) $\{x, y\} \nsubseteq B$. If $\{x, y\} \subseteq A$ then $c_{xy}(B) \cap A = A \smallsetminus \{x, y\} = c_{xy}(A)$; if $\{x, y\} \nsubseteq A$ and $\{x, y\} \subseteq B$ then $c_{xy}(B) \cap A = (B \smallsetminus \{x, y\}) \cap A = A \smallsetminus \{x, y\} \subset A = c_{xy}(A)$; if $\{x, y\} \nsubseteq B$ then $c_{xy}(B) \cap A = A = c_{xy}(A)$: thus in any case CC holds. Next, let $z \notin c_{xy}(A \cup B)$: then by definition $z \in \{x, y\}$ and $\{x, y\} \subseteq A \cup B$. Assume without loss of generality that $z = y$, and suppose that $y \in c_{xy}(A) \cap c_{xy}^{+}(B)$. Then, $\{x, y\} \nsubseteq A$ and $\{x, y\} \nsubseteq B$ while $y \in A \cap B$. It follows that $x \notin A \cup B$, a contradiction. Thus, CO is also satisfied by c_{xy}.

Furthermore, let $z \in c_{xy}(A \cup \{z\})$ and $v \in c_{xy}(A)$. If $v \notin c_{xy}(A \cup \{z\})$ then by construction and definition of c_{xy}, $\{z, v\} = \{x, y\}$: thus $z \notin c_{xy}(A \cup \{z\})$, a contradiction. It follows that c_{xy} satisfies CGA as well. Therefore, Theorem 6 applies, and $c_{xy} \in C_X^{sc}$.

Moreover, by definition $c_{xy} < c^{id}$ i.e. $c_{xy} \leqslant c^{id}$ and $c_{xy} \neq c^{id}$.

Next, let $c \in C_X^{sc}$ be such that $c_{xy} \leqslant c \leqslant c^{id}$, and assume that there exists $A' \subseteq X$ such that $c_{xy}(A') \subset c(A') \subseteq A'$. By definition of c_{xy}, it must be the case that $A' \supseteq \{x, y\}$ and $c_{xy}(A') = A' \smallsetminus \{x, y\} \subset c(A')$. Clearly, if $c = c^{id}$ there is nothing to prove, so suppose that there also exists $B \subseteq X$ such that $c_{xy}(B) \subseteq c(B) \subset B$. By definition of c_{xy}, $B \supseteq \{x, y\}$ and $c_{xy}(B) = B \smallsetminus \{x, y\}$ i.e. $B \smallsetminus \{x, y\} \subseteq c(B) \subset B$. Therefore, $x \notin c(B)$ or $y \notin c(B)$ (or both). Suppose then without loss of generality that $y \notin c(B)$: since $c \in C_X^{sc}$ there exists a symmetric irreflexive $\Delta \subseteq X \times X$ such that $c(A) = \mathbb{C}(A, \Delta_A)$ for any

$A \subseteq X$, hence there exists $z \in B$, $z \neq y$ such that $z\Delta y$ and $y\Delta z$. It follows that $z \notin c(B)$ hence $z = x$ and therefore $\{x,y\} \cap c(A') = \varnothing$, a contradiction since $c_{xy}(A') = A' \smallsetminus \{x,y\} \subset c(A')$.

Thus, either $c = c^{id}$ or $c = c_{xy}$ i.e. c_{xy} is indeed a *coatom* of $(C_X^{sc}, \leqslant)$.

Conversely, let c be a coatom of $(C_X^{sc}, \leqslant)$ and suppose $c \notin C_X^*$. Then, for any pair of distinct $x, y \in X$, neither $c_{xy} \leqslant c$ nor $c \leqslant c_{xy}$ i.e. there exist $A, B \subseteq X$ such that $c(A) \subset c_{xy}(A)$ and $c_{xy}(B) \subset c(B)$. Since $c \in C_X^{sc}$ there exists a symmetric irreflexive Δ such that $c = \mathbb{C}(., \Delta)$. Thus, from $c(A) \subset c_{xy}(A)$ it follows that there exist two distinct $u, v \in A$ such $u\Delta v$ and $v\Delta u$ whence $c(Y) \subseteq Y \smallsetminus \{u,v\}$ for any $Y \subseteq X$ such that $\{u,v\} \subseteq Y$. But then, by definition $c \leqslant c_{uv}$, a contradiction. Thus, it must be the case that $c \in C_X^*$.

To check that each $c_{xy}^{\circ} \in C_X^{**}$ is an *atom* of $(C_X^{sc}, \leqslant)$, first notice that $c_{xy}^{\circ} \in C_X^{sc}$. Indeed, c_{xy}° satisfies ND by construction. Also, if $A \subseteq B$ then $c_{xy}^{\circ}(B) \cap A \neq \varnothing$ entails that either $A = B = \{z\}$ for some $z \in X$, or $A \subseteq B \subseteq \{x,y\}$ i.e. either A is a singleton or $A = B$. Thus, in any case, if $A \subseteq B$ then by definition $c_{xy}^{\circ}(B) \cap A \subseteq c_{xy}^{\circ}(A)$ hence c_{xy}° satisfies CC. Moreover, for any $A, B \subseteq X$, if $x \in c_{xy}^{\circ}(A) \cap c_{xy}^{\circ}(B)$ then by definition of c_{xy}° either $A = B = \{x\}$ or ($A \cup B \in \{A, B\}$ and $A \cup B = \{x,y\}$): thus, in any case, $x \in c_{xy}^{\circ}(A \cup B)$ and CO is also satisfied by c_{xy}°. Finally, suppose that $x \in c_{xy}^{\circ}(A \cup \{x\})$ and $z \in c_{xy}^{\circ}(A)$. Then it must be the case that $A \subseteq \{x,y\}$ whence $z \in A \cap \{x,y\}$: if $z = x$ there is nothing to prove, and if $z = y$ then $A \cup \{x\} = \{x,y\}$ hence by definition $z \in c_{xy}^{\circ}(A \cup \{x\})$. It follows that c_{xy}° also satisfies CGA, as required. Next, observe that $c_{xy}^{\circ}(A) = c^{[1]}(A)$ for any $A \neq \{x,y\}$, and $c_{xy}^{\circ}(\{x,y\}) = \{x,y\}$ while $c^{[1]}(\{x,y\}) = \varnothing$. Thus, $c^{[1]} \leqslant c_{xy}^{\circ}$, $c^{[1]} \neq c_{xy}^{\circ}$ and by construction for any $c \in C_X^*$ (indeed, for any $c \in C_X$) if $c^{[1]} \leqslant c \leqslant c_{xy}^{\circ}$ then either $c = c^{[1]}$ or $c = c_{xy}^{\circ}$.

Conversely, assume that c is an atom of $(C_X^{sc}, \leqslant)$ and $c \notin C_X^{**}$. Then, by definition of C_X^{**}, $c(A) = \varnothing$ for any A such that $\#A = 2$, and there exists $B \subseteq X$ such that $\#B \geq 3$ and $c(B) \neq \varnothing$. It follows that, for any $x \in c(B)$ and any $y \in B \smallsetminus \{x\}$, $c(B) \cap \{x,y\} \neq \varnothing$ while $c(\{x,y\}) = \varnothing$, therefore violating CC, a contradiction by Theorem 6.

To check that $(C_X^{sc}, \leqslant)$ is *not* a sub-join-semilattice of $(C_X, \leqslant)$, just take $x, y, z \in X$, $x \neq y \neq z \neq x$, and consider $c_{xy}, c_{yz} \in C_X^*$ and $c_{xy} \cup c_{yz}$ (the least upper bound of $\{c_{xy}, c_{yz}\}$ in $(C_X, \leqslant)$). Let be $A' \subseteq X$, $\{x,y,z\} \subseteq A'$: clearly, $(c_{xy} \cup c_{yz})(A') = (A' \smallsetminus \{x,y\}) \cup (A' \smallsetminus \{y,z\}) = A' \smallsetminus \{y\}$. Now, if $c_{xy} \cup c_{yz} \in C_X^{sc}$ then there exists a symmetric irreflexive digraph (X, Δ) such that $(c_{xy} \cup c_{yz})(A') = \mathbb{C}(A', \Delta_{A'})$ hence there exists $u \in A' \setminus \{y\}$ with $u\Delta y$ and $y\Delta u$. It follows that $u \notin (c_{xy} \cup c_{yz})(A')$, a contradiction. ■

Remark 16 *Notice that finiteness of X has been used in the proof above in order to show that the set of coatoms of $(C^*_X, \leqslant)$ is contained in C^+. The latter statement clearly holds for an infinite X as well provided CO is replaced with the following stronger version of 'Concordance'*

CO: for any family $\{A_i\}_{i\in I}$ of subsets of X, $\bigcap_{i\in I} c(A_i) \subseteq c(\bigcup_{i\in I} A_i)$.*

Let us now move to the analysis of the poset $(C^{psc}_X, \leqslant)$ of all symmetric revealed pseudocores on X. In order to proceed, let us first introduce two more special classes of choice functions. For any $x \in X$ and $A \subseteq X$ such that $x \in A$, $c_{-x} \in C_X$ and $c_{x,A} \in C_X$ are defined as follows: for all $B \subseteq X$, $c_{-x}(B) = B \setminus \{x\}$, $c_{x,A}(B) = \{x, y\}$ if $B = \{x, y\}$ and $c_{x,A}(B) = \varnothing$ otherwise. Moreover, denote $C_{X-} = \{c_{-x} : x \in X\}$, and $\widehat{C}_X = \{c_{x,A} : x \in A \subseteq X\}$ (see also Monjardet and Raderanirina (2004) for an earlier introduction -and use- of the class $\widehat{C}_X$ in a related context). We are now ready to state the following result on $(C^{psc}_X, \leqslant)$.

Theorem 17 *The poset $(C^{psc}_X, \leqslant)$ of symmetric revealed pseudocores is a* sub-meet-semilattice *of $(C_X, \leqslant)$ with c^{id} itself as its top element, but not a sub-join-semilattice of $(C_X, \leqslant)$. The bottom element of $(C^{psc}_X, \leqslant)$ is the empty choice function $c^{\varnothing}$. Moreover, the set of coatoms of $(C^{psc}_X, \leqslant)$ is $C^*_X \cup C_{X-}$, and the set of its atoms is $\widehat{C}_X$.*

Proof. Let $c, c' \in C^{psc}_X$, and consider $c \cap c'$. By Theorem 8 above c and c' satisfy CC, CO and CGA.

The proof of the previous Theorem also establishes that $c \cap c'$ also satisfies CC, CO and CGA.

It follows that, by Theorem 8 again, $c \cap c' \in C^{psc}_X$, whence $(C^{sc}_X, \leqslant)$ is a sub-meet-semilattice of $(C_X, \leqslant)$.

Clearly, c^{id} and $c^{\varnothing}$ do trivially satisfy CC, CO and CGA hence c^{id},$c^{\varnothing} \in C^{psc}_X$ and are therefore the top and bottom element of $(C^{psc}_X, \leqslant)$, respectively.

Now, take any $c_{xy} \in C^*_X$: the proof of the previous theorem already establishes that c_{xy} is indeed a *coatom* of $(C^{psc}_X, \leqslant)$. Next, take any $c_{-x} \in C_{X-}$: it is immediately checked that, by definition, it trivially satisfies CC, CO and CGA hence by Theorem 8 it belongs to C^{psc}_X. Moreover, let $c \in C^{psc}_X$

be such that $c_{-x} \leqslant c \leqslant c^{id}$, and assume that there exists $A' \subseteq X$ such that $c_{-x}(A') \subset c(A') \subseteq A'$. By definition of c_{-x}, it must be the case that $x \in A'$ and $c_{-x}(A') = A' \smallsetminus \{x\} \subset c(A')$. Clearly, if $c = c^{id}$ there is nothing to prove, so suppose that there also exists $B \subseteq X$ such that $c_{-x}(B) \subseteq c(B) \subset B$. By definition of c_{-x}, $x \in B$ and $c_{-x}(B) = B \smallsetminus \{x\}$ i.e. $B \smallsetminus \{x\} \subseteq c(B) \subset B$. Therefore, $x \notin c(B)$: since $c \in C_X^{psc}$ there exists a symmetric $\Delta \subseteq X \times X$ such that $c(A) = \mathbb{C}(A, \Delta_A)$ for any $A \subseteq X$, hence there exists $z \in B$, such that $z\Delta x$ and $x\Delta z$. It follows that $z \notin c(B)$ hence $z = x$ and therefore $\{x\} \cap c(A') = \varnothing$, a contradiction since $c_{-x}(A') = A' \smallsetminus \{x\} \subset c(A')$.

Conversely, let c be a coatom of $(C_X^{psc}, \leqslant)$ and suppose $c \notin C_X^* \cup C_{X-}$. Then, for any pair of distinct $x, y \in X$, not $c \leqslant c_{-x}$ and neither $c_{xy} \leqslant c$ nor $c \leqslant c_{xy}$ i.e. there exist $A, B \subseteq X$ such that $c(A) \subset c_{xy}(A)$ and $c_{xy}(B) \subset c(B)$. Since $c \in C_X^{psc}$ there exists a symmetric (possibly not irreflexive) Δ such that $c = \mathbb{C}(., \Delta)$. Thus, from $c(A) \subset c_{xy}(A)$ it follows that there exist $u, v \in A$ such $u\Delta v$ and $v\Delta u$ whence $c(Y) \subseteq Y \smallsetminus \{u, v\}$ for any $Y \subseteq X$ such that $\{u, v\} \subseteq Y$. But then, if $u = v$ then $c \leqslant c_{-u}$, a contradiction, and if $u \neq v$ then $c \leqslant c_{uv}$ a contradiction again. Thus, it must be the case that $c \in C_X^* \cup C_{X-}$.

To check that each $c_{x,A} \in \widehat{C}_X$ is an *atom* of $(C_X^{psc}, \leqslant)$, first notice that $c_{x,A} \in C_X^{psc}$. Indeed, if $A \subseteq B$ then $c_{x,A}(B) \cap A \neq \varnothing$ entails that $A = B$ hence $c_{x,A}$ trivially satisfies CC. Also, for any $A', B' \subseteq X$, if $x \in c_{x,A}(A') \cap c_{x,A}(B')$ then by definition of $c_{x,A}$ $A = A' = B$ hence $x \in c_{x,A}(A \cup B)$ and CO is also trivially satisfied by $c_{x,A}$. Finally, suppose that $x \in c_{x,A}(B \cup \{x\})$ and $z \in c_{x,A}(B)$. Then it must be the case that $B \cup \{x\} = A$, $z = x$ and $B = A$ whence obviously $z \in c_{x,A}(B \cup \{x\}$ and CGA is satisfied as well.

On the other hand, it is straightforward to check that for each $c_{x,A} \in \widehat{C}_X$, $c^{\varnothing} \leqslant c_{x,A}$, $c^{\varnothing} \neq c_{x,A}$ and for any $c \in C_X^{psc}$ (indeed, for any $c \in C_X$), if $c^{\varnothing} \leqslant c \leqslant c_{x,A}$ then either $c = c^{\varnothing}$ or $c = c_{x,A}$. Thus, each $c_{x,A} \in \widehat{C}_X$ is indeed an atom of $(C_X^{psc}, \leqslant)$.

Conversely, assume that c is an atom of $(C_X^{psc}, \leqslant)$ and $c \notin \widehat{C}_X$. Then, by definition of $\widehat{C}_X$, either $\#c(A) \geq 2$ for some $A \subseteq X$, or there exist two *distinct* $A, B \subseteq X$ such that $c(A) \neq \varnothing$ and $c(B) \neq \varnothing$. In any case, there clearly exists some $c_{x,A'} \in \widehat{C}_X$ such that by construction $c_{x,A'} \neq c$ and $c_{x,A'} \leqslant c$, a contradiction since c is an atom of $(C_X^{psc}, \leqslant)$.

To check that $(C_X^{psc}, \leqslant)$ is *not* a sub-join-semilattice of $(C_X, \leqslant)$, take $X = \{x, y, z\}$ with $x \neq y \neq z \neq x$, $\Delta = \{(x, y), (y, x)\}$, $\Delta' = \{(x, z), (z, x)\}$, $A = \{x, z\}$, $B = \{x, y\}$. Then, define $c, c' \in C_X$ as follows: for any $Y \subseteq X$,

$c(Y) = \mathbb{C}(Y, \Delta_Y)$ and $c'(Y) = \mathbb{C}(Y, \Delta'_Y)$. By construction, $\{c, c'\} \subseteq C_X^{sc} \subseteq C_X^{psc}$. Next, notice that

$$\begin{aligned}&(c \cup c')(A) \cap (c \cup c')(B) = \\ &= [\mathbb{C}(A, \Delta_A) \cup \mathbb{C}(A, \Delta'_A)] \cap [\mathbb{C}(B, \Delta_B) \cup \mathbb{C}(B, \Delta'_B)] = \\ &= \{x, z\} \cap \{x, y\} = \{x\}.\end{aligned}$$

However,

$$(c \cup c')(A \cup B) = \mathbb{C}(A \cup B, \Delta_{A \cup B}) \cup \mathbb{C}(A \cup B, \Delta'_{A \cup B}) = \{y, z\}.$$

Thus, $c \cup c'$ fails to satisfy CO: it follows, by Theorem 8, that $c \cup c' \notin C_X^{psc}$.

■

Remark 18 *Notice that, since $(C_X^{sc}, \leqslant)$ and $(C_X^{psc}, \leqslant)$ are meet-semilattices with a top element (and finite ones since X is assumed to be finite), it follows that both of them are* lattices *with meet= $\cap$ and join of a pair given by the meet of the (nonempty) set of upper bounds of that pair (see e.g. Davey and Priestley (1990), Monjardet and Raderanirina (2004)). By the previous theorems, however, such lattices are* not *sublattices of $(C_X, \leqslant)$.*

5 Concluding remarks

Choice functions which may be regarded as cores or pseudocores of an underlying *symmetric* digraph (X, Δ) have been characterized. Symmetric pseudocores have been shown to be precisely the *order duals* of Lawvere-Tierney closure operators. The characterizations provided have also been shown to be helpful for a simple analysis of the basic order-theoretic structure of symmetric revealed cores and pseudocores. Moreover, the duality between symmetric revealed pseudocores and Lawvere-Tierney closure operators pointed out above sheds also some light on the order-theoretic structure of the latter.

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