UNIVERSITÀ DEGLI STUDI DI SIENA

QUADERNI DEL DIPARTIMENTO
DI ECONOMIA POLITICA E STATISTICA

**Stefano Vannucci**

# Widest Choice

**n. 629 – Dicembre 2011**

**Abstract -** A choice function is (weakly) width-maximizing if there exists a dissimilarity- i.e. an irreflexive symmetric binary relation- on the underlying object set such that the choice sets are (include, respectively) dissimilarity chains of locally maximum size.
Width-maximizing and weakly width-maximizing choice functions on an arbitrary domain are characterized relying on the newly introduced notion of a revealed dissimilarity relation.

**Jell Classification**: D01,D71

**Stefano Vannucci,** Dipartimento di Economia Politica e Statistica, Università di Siena – vannucci@unisi.it

# 1 Introduction

Choice functions may be used to model choices from subsets of jointly available objects- as opposed to alternative options- and that usage is sometimes advertised denoting them as *combinatorial* choice functions (see e.g. Echenique (2007), and Barberà, Bossert, Pattanaik (2004)).

The analysis and assessment of biodiversity preservation policies is a prominent case in point: typically, some diversity index is to be maximized under appropriate constraints, and such a diversity index is usually meant to result from the aggregation of binary *dissimilarity* comparisons between the available objects, combinations of which may be chosen (see e.g. Weitzman (1998), and Nehring, Puppe (2002)).

Furthermore, if dissimilarity comparisons are dichotomous (i.e. any two objects either are dissimilar or are not, with no intermediate degrees of dissimilarity) and unsupplemented by further relevant information, diversity is arguably to be assessed in terms of '*maximum number of mutually dissimilar objects*' or *width* (see Basili, Vannucci (2007)).

Under those circumstances, diversity-maximizing combinatorial choice amounts to '*widest choice*' according to the relevant dissimilarity relation.

But in fact, the range of 'widest choice' is likely to be much larger than that. On digiting "widest choice" on Google one is currently offered a quite impressive list of some $1.7 \times 10^7$ items: at the very least, that fact suggests that very many companies expect that their prospective customers might choose among them relying on a width-maximizing criterion.

But then, how can one tell 'widest choice' from other sorts of combinatorial choice behaviour? Is there any observable prediction following from the hypothesis of width-maximizing choice?

The present note addresses the foregoing issue by providing a characterization of width-maximizing choice on a *general* domain, to the effect of covering the case of *finite* sets of observations (see Bossert, Suzumura (2010) for a recent extensive work on maximizing choice behaviour on general domains according to an arbitrary binary relation).

It will be shown that strict or irredundant width-maximizing choice rules do in fact entail some significant restrictions on combinatorial choice behaviour, while redundant width-maximizing choice only requires nonemptiness of the choice set at any nonempty set.

## 2 Model and results

Let $X$ be a 'universal' set of items, with cardinality $\#X \geq 3$, and $\mathcal{P}(X)$ its power set. A *choice function on* $X$ with domain $\mathcal{D} \subseteq \mathcal{P}(X)$ is a (partial) *contracting* operator on $X$ i.e. a function $f : \mathcal{D} \rightarrow \mathcal{P}(X)$ such that $f(A) \subseteq A$ for any $A \in \mathcal{D}$. We denote by $C_{X(\mathcal{D})}$ the set of all choice functions on $X$ with domain $\mathcal{D}$. We also denote by $C_X$ the set of all choice functions on $X$ and define a partial order $\leqslant$ on $C_X$ by pointwise set-inclusion as follows: for any $f : \mathcal{D} \rightarrow \mathcal{P}(X)$ and $g : \mathcal{D}' \rightarrow \mathcal{P}(X)$, $f, g \in C_X$, $f \leqslant g$ iff $\mathcal{D} \subseteq \mathcal{D}'$ and $f(A) \subseteq g(A)$ for each $A \in \mathcal{D}$. A binary relation $R \subseteq X \times X$ is *irreflexive* iff $(x, x) \notin R$ for any $x \in X$, and *symmetric* iff for any $x, y \in X$, $(x, y) \in R$ entails $(y, x) \in R$, and is said to be a *dissimilarity* (or *orthogonality*) relation on $X$ iff it is both irreflexive and symmetric. We also denote -for any $A \subseteq X$- $R_{|A} = R \cap (A \times A)$ and $\Delta_A = \{(x, x) : x \in A\}$.

Let $D \subseteq X \times X$ be a *dissimilarity* relation on $X$. Then, for any $A \subseteq X$, let
$$\mathcal{MC}(D_{|A}) =: \left\{ \begin{array}{c} B \subseteq A : D_{|B} = B \times B \smallsetminus \Delta_B \text{ and} \\ \#B \geq \#B' \text{ for any } B' \subseteq A \text{ such that} \\ D_{|B'} = B' \times B' \smallsetminus \Delta_{B'} \end{array} \right\}$$
the set of all *maximal* $D$*-chains* in $A$.

A choice function $f \in C_{X(\mathcal{D})}$ is *rationalizable by width-maximization through* $D$ iff $f(A) \in \mathcal{MC}(D_{|A})$ for all $A \in \mathcal{D}$, and *weakly rationalizable by width-maximization through* $D$ iff for any $A \in \mathcal{D}$ there exists $B \in \mathcal{MC}(D_{|A})$ such that $B \subseteq f(A)$.

A choice function $f \in C_{X(\mathcal{D})}$ is *rationalizable by width-maximization (WM-rationalizable)* iff there exists a dissimilarity relation $D \subseteq X \times X$ such that $f$ is rationalizable by width-maximization through $D$. Similarly, a choice function $f \in C_{X(\mathcal{D})}$ is *weakly rationalizable by width-maximization (weakly WM-rationalizable)* iff there exists a dissimilarity relation $D \subseteq X \times X$ such that $f$ is weakly rationalizable by width-maximization through $D$. To put it in other, equivalent, terms WM-rationalizable (weakly WM-rationalizable) choice functions model *irredundant* (*redundant*, respectively) width-maximization. *Notice that, by definition,* $f \in C_{X(\mathcal{D})}$ *is weakly WM if and only if there exists a WM* $f' \in C_{X(\mathcal{D})}$ *such that* $f' \leqslant f$.

For any $f \in C_{X(\mathcal{D})}$ its *revealed dissimilarity* relation $D^f \subseteq X \times X$ may defined by the following rule: for any $x, y \in X$, $xD^f y$ iff $x \neq y$ and there exists $A \in \mathcal{D}$ such that $\{x, y\} \subseteq f(A)$. It is immediately checked that $D^f$ *is indeed irreflexive and symmetric:* it will play a pivotal role in the ensuing analysis.

Let us first address the case of WM-rationalizable choice functions. The following properties are to be introduced

**Definition 1** *(Properness (PR)) A choice function $f \in C_{X(\mathcal{D})}$ is* proper *iff $f(A) \neq \varnothing$ for any $A \subseteq X$, $A \neq \varnothing$.*

PR is a quite natural property for width-maximizing choice. Even in the worst case, when any object appears to be similar to any other objects, *something* -as opposed to *nothing*- should expectedly be chosen among the available objects (possibly a single item, if irredundance is required).

**Definition 2** *(Revealed-dissimilarity coherence (RDC)) A choice function $f \in C_{X(\mathcal{D})}$ is* revealed-dissimilarity coherent *iff for all $A' \in \mathcal{D}$ and $A, B \subseteq X$ with $A \cup B \subseteq A'$, if $\#A < \#B$ and there exist $\{B'_i\}_{i \in I}$ with $B'_i \in \mathcal{D}$ for each $i \in I$, such that $B \times B \subseteq \bigcup_{i \in I} f(B'_i) \times f(B'_i)$ then $f(A') \neq A$.*

RDC is also a natural requirement for width-maximizing choice, at least when its revealed dissimilarity is the underlying dissimilarity relation: indeed, RDC simply dictates that the choice set of any available set of objects cannot be any subset that happens to be of *smaller size* of another available subset consisting of objects that are *revealed to be mutually dissimilar*.

It may not be clear at the outset that *any* width-maximizing choice function should satisfy RDC. That is however one of the consequences of the following

**Lemma 3** *Let $f \in C_{X(\mathcal{D})}$. Then $f$ is WM-rationalizable iff it is WM-rationalizable with respect to $D^f$.*

**Proof.** Let $f \in C_{X(\mathcal{D})}$ be WM-rationalizable through $D$. If $D = \varnothing$ (notice that the empty relation is trivially irreflexive and symmetric) it follows, by definition, that

$$\mathcal{MC}(D_{|A}) =: \left\{ \begin{array}{c} B \subseteq A : \varnothing = B \times B \smallsetminus \Delta_B \\ \text{and } \#B \geq \#B' \text{ for any } B' \subseteq A \\ \text{such that } \ \varnothing = B' \times B' \smallsetminus \Delta_{B'} \end{array} \right\}$$

hence $\mathcal{MC}(\varnothing_{|A}) = \{\{x\} : x \in A\}$ and $\#f(A) = 1$ for all $A \in \mathcal{D}$ i.e. $f$ is *single-valued*. Thus, by definition of $D^f$, $D^f = \varnothing = D$ hence $f$ is trivially WM-rationalizable through $D^f$ as well.

Then, suppose $D \neq \varnothing$, and consider any $A \in \mathcal{D}$. By hypothesis, $f(A) \in \mathcal{MC}(D_{|A})$ hence $D_{|f(A)} = f(A) \times f(A) \smallsetminus \Delta_{f(A)}$ and $\#f(A) \geq \#B$ for any $B \subseteq A$ such that $D_{|B} = B \times B \smallsetminus \Delta_B$. By definition, $xD^f y$ for each $x, y \in f(A)$ such that $x \neq y$ hence $D^f_{|f(A)} = f(A) \times f(A) \smallsetminus \Delta_{f(A)}$.

Now, suppose that there exists some $B \subseteq A$ such that $\#B > \#f(A)$ and $D^f_{|B} = B \times B \smallsetminus \Delta_B$: then, since $f(A) \in \mathcal{MC}(D_{|A})$, $D_{|B} \subset B \times B \smallsetminus \Delta_B$ hence there exist at least two *distinct* $x^*, y^* \in B$ such that *not* $x^* D y^*$.

On the other hand, for all $x, y \in B$ there exists some $B' \in \mathcal{D}$ with $\{x, y\} \subseteq f(B')$, hence in particular there exists $B^* \in \mathcal{D}$ such that $\{x^*, y^*\} \subseteq f(B^*) \in \mathcal{MC}(D_{|B^*})$ whence $x^* D y^*$, a contradiction since by hypothesis *not* $x^* D y^*$.

Therefore, it must be the case that $\#f(A) \geq \#B$ for all $B \subseteq A$ such that $D^f_{|B} = B \times B \smallsetminus \Delta_B$. It follows that $f(A) \in \mathcal{MC}(D_{|A})$ for all $A \in \mathcal{D}$, i.e. $f$ is WM-rationalizable through $D^f$ whenever it is WM-rationalizable at all. ■

We are now ready to state the main characterization result of the present paper, namely

**Theorem 4** *Let $f \in C_{X(\mathcal{D})}$. Then $f$ is WM-rationalizable iff it satisfies PR and RDC.*

**Proof.** Let $f \in C_{X(\mathcal{D})}$ be WM-rationalizable. Then, by Lemma 3 it is WM-rationalizable through $D^f$, i.e. for any $A \subseteq X$,

$$f(A) \in \left\{ \begin{array}{c} B \subseteq A : D^f_{|B} = B \times B \smallsetminus \Delta_B \text{ and } \#B \geq \#B' \text{ for any } B' \subseteq A \\ \text{such that } D^f_{|B'} = B' \times B' \smallsetminus \Delta_{B'} \end{array} \right\}.$$

Now, to see that $f$ satisfies PR, suppose that on the contrary $f(A) = \varnothing$ for some *nonempty* $A \subseteq X$. Thus, for any $B \subseteq A$, $D^f_{|B} = B \times B \smallsetminus \Delta_B$ only if $B = \varnothing$, a contradiction since $A \neq \varnothing$ and, for any $x \in A$, $D^f_{|\{x\}} = \varnothing$ by irreflexivity of $D^f$ whence $\{x\} \times \{x\} \smallsetminus \Delta_{\{x\}} = \varnothing = D^f_{|\{x\}}$ by definition.

To check that RDC also holds, take any $A' \in \mathcal{D}$, $A, B \subseteq X$ such that $A \cup B \subseteq A'$, $\#A < \#B$, and there exist a family $\{B'_i\}_{i \in I}$ with $B'_i \in \mathcal{D}$ for each $i \in I$, and $B \times B \subseteq \bigcup_{i \in I} f(B'_i) \times f(B'_i)$. Now, suppose that $f(A') = A$.

Since $f$ is WM-rationalizable, it is in particular WM-rationalizable through $D^f$, by Lemma 3.

Therefore, $f(A') = A \in \mathcal{MC}(D^f_{|A\cup B})$. However, by definition of $D^f$, $B \times B \subseteq \bigcup_{i\in I} f(B'_i) \times f(B'_i)$ entails $B \times B \setminus \Delta_B \subseteq D^f_{|A\cup B}$, a contradiction since $A \in \mathcal{MC}(D^f_{|A\cup B})$ and $\#A < \#B$.

Conversely, suppose that $f \in C_{X(\mathcal{D})}$ satisfies PR and RDC, and let $A \in \mathcal{D}$. Let us show that $f(A) \in \mathcal{MC}(D^f_{|A})$ whence the thesis follows. First, notice that $f(\varnothing) = \varnothing \in \mathcal{MC}(D^f_{|\varnothing})$ by definition, while $f(A) = \varnothing$ with $A \neq \varnothing$ is impossible by PR. Now, suppose that $A \neq \varnothing$ and $f(A) \notin \mathcal{MC}(D^f_{|A})$: then, since by definition $D^f_{|f(A)} = f(A) \times f(A) \smallsetminus \Delta_{f(A)}$, it must be the case that there exists $B \subseteq A$ such that $D^f_{|B} = B \times B \smallsetminus \Delta_B$ and $\#B > \#f(A)$ (hence $\#B \geqslant 2$). Thus, by definition of $D^f$, for each pair $x, y \in B$ there exists $B'_i \in \mathcal{D}$ such that $\{x, y\} \subseteq f(B'_i)$. But then, by RDC, $f(A) \neq f(A)$, a contradiction. ■

**Remark 5** *It is easily checked that PR and RDC are mutually independent properties. To see this, take* $X = \{x, y, z\}$ *with* $x \neq y \neq z \neq x$, $\mathcal{D} = \{\{x, y\}, X\}$, *and* $f : \mathcal{D} \to \mathcal{P}(X)$ *such that* $f(\{x, y\}) = \{x, y\}$, $f(X) = \{x\}$. *Clearly,* $f$ *satisfies PR. However,* $\{x, y\} \times \{x, y\} \subseteq f(\{x, y\}) \times f(\{x, y\})$ *and* $f(\{x, y, z\}) = \{x\}$ *entail a violation of RDC. Consider now* $f' : \mathcal{D} \to \mathcal{P}(X)$ *defined as follows:* $f'(\{x, y\}) = \varnothing$, $f'(X) = \{x\}$. *Since* $\bigcup_{B\in\mathcal{D}} f(B) \times f(B) = \{(x, x)\}$ *it follows that* $f'$ *trivially satisfies RDC. However,* $f'$ *does not satisfy PR.*

Let us now turn to the characterization of weakly WM-rationalizable choice functions.

**Proposition 6** *Let* $f \in C_{X(\mathcal{D})}$. *Then* $f$ *is weakly WM-rationalizable iff it satisfies PR.*

**Proof.** Let $f \in C_{X(\mathcal{D})}$ be weakly WM-rationalizable. To begin with, notice that by definition there exists a WM-rationalizable $f' \in C_{X(\mathcal{D})}$ such that

$f' \leqslant f$. By Theorem 4, $f'$ satisfies PR hence by construction $f$ also satisfies PR.

Conversely, let $f \in C_{X(\mathcal{D})}$ satisfy PR. Therefore, there exists a single-valued $f' \in C_{X(\mathcal{D})}$ such that $f' \leqslant f$. Now, it is easily checked that *any* single-valued $g \in C_{X(\mathcal{D})}$ is WM-rationalizable: to check this, take $D = \varnothing$ which is trivially irreflexive and symmetric, and observe that for any $A \subseteq X$, $x \in A$ and $B \subseteq A$ such that $\#B \geq 2$:

$\varnothing_{|\{x\}} = \{x\} \times \{x\} \smallsetminus \Delta_{\{x\}} = \varnothing$ while $\varnothing_{|B} \neq B \times B \smallsetminus \Delta_B$.

Thus, $g$ is WM-rationalizable through the empty dissimilarity relation $D = \varnothing$. It follows that in particular $f'$ is WM-rationalizable hence $f' \leqslant f$ implies that $f$ is indeed weakly WM-rationalizable. ■

A remarkable consequence of Proposition 6 is that *if* the available database only collects observations of nonempty choice sets (while empty choice sets are ignored or disallowed), then *anything goes.* Namely, any choice behaviour may be rationalized in terms of weak i.e. redundant width-maximization, under a not uncommon specification of the database.

The foregoing trivializing effect induced by mere choice of database type, however, does never hold for strict or irredundant width-maximizing choice behaviour that- by Theorem 4- is bound to display revealed-dissimilarity coherence as its unmistakable hallmark.

# 3 Concluding remarks

Characterizations of both irredundant and redundant width-maximizing combinatorial choice behaviour on a general domain have been provided. It has been shown that while redundant width-maximizing choice does not entail any restriction on observable choice behaviour except for nonemptiness of the choice set, the hypothesis of irredundant width-maximizing choice provides much more stringent predictions.

Results of this kind parallel and supplement some related work on characterizations of maximizing choice behaviour on general domains under an arbitrary binary relation (see e.g. Bossert, Suzumura (2010)), and of alternative choice rules on more specialized domains of alternative options (see e.g. Sen (1993), Baigent, Gaertner (1996), Brandt, Harrenstein (2011)).

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