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Strategy-proofness and unimodality in bounded distributive lattices

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Abstract - It is shown that a social choice rule $f : X^N \longrightarrow X$ as defined on a bounded distributive lattice $(X,\leq)$ is strategy-proof on the set of all profiles of unimodal total preorders on X if and only if it can be represented as an iterated median of projections and constants. The equivalence of individual and coalitional strategy-proofness that is known to hold in more specialized unimodal domains fails in such a more general setting.

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1. Introduction

It is well-known that majority-like social choice rules as defined on unimodal preference domains on a line are both median-representable and coalitionally strategy-proof. Some authors have also provided median-based characterizations of some large class of strategy-proof social choice rules on suitably defined unimodal domains, including Moulin (1980), Border and Jordan (1983), Barberà, Gul and Stacchetti (1993), Danilov (1994), Chichilnisky and Heal (1997), Ching (1997), Barberà, Massò and Neme (1997), Peremans, Peters, van der Stel and Storcken (1997), Schummer and Vohra (2002), Nehring and Puppe (2007(a), 2007(b)). Most of those contributions define unimodality with reference to a (bounded) linear order on the set of alternatives (see e.g. Moulin (1980)), or to a profile of metric-induced total preorders on a multidimensional Euclidean space (see e.g. Border and Jordan (1983), Barberà, Gul and Stacchetti (1993), Chichilnisky and Heal (1997)). Danilov (1994) provides a similar median-based characterization of strategy-proof social choice rules on unimodal domains of linear orders in undirected *trees*. Moreover, most of those works also establish *equivalence of individual and coalitional strategy-proofness* on the relevant unimodal domains.

Now, (bounded) linear orders are a quite special subclass of (bounded) distributive lattices.

But then, *what about strategy-proof social social choice rules on unimodal domains in arbitrary bounded distributive lattices*? Are they *median-representable*? Do they also enjoy *equivalence of individual and coalitional strategy-proofness* as well?

The relevance of such a further extension of the study of strategy-proofness properties on unimodal domains in bounded distributive lattices is apparent, and can be easily motivated with a few significant examples including the following:

(i) the alternatives consist of thresholds or systems of thresholds in a partially ordered set e.g. poverty thresholds for a partially ordered population, a profile of which is to be aggregated into a unique consensus threshold;

(ii) the alternatives amount to judgments consisting of deductively closed sets of statements, a profile of which is to be aggregated to produce a unique deductively closed set of acceptable statements;

(iii) the alternatives consist of several distinct dissimilarity relations -or dually of several tolerance (or similarity) relations- a profile of which is to be somehow amalgamated into a unique relation of the same type;

(iv) the alternatives consist of arbitrary choice functions on a fixed space a profile of which is to be suitably aggregated into a unique choice function;

(v) the alternatives consist of graded evaluations of objects of a fixed population, a profile of which is to be aggregated into a unique evaluation (that is the salient kind of exercise recently addressed in Balinski and Laraki (2010)).

Unfortunately, a study of strategy-proof social choice rules on unimodal domains in general bounded distributive lattices is still missing in the literature. To be sure, there is some valuable work by Nehring and Puppe (2007 (a), (b)) on *strict* unimodality and strategy-proofness in *finite* median spaces, and (finite) distributive lattices *are* a prominent instance of (finite) median spaces. However, due to their choice of *linear* preference domains as combined with their *strict* notion of unimodality, it turns out that their results do not in fact address anyway the case of finite distributive lattices except for finite linear orders as explained in Section 3 below.

The present paper purports to fill this significant gap in the literature and provide a study of strategy-proofness and unimodality in bounded distributive lattices. A median-based characterization of strategy-proof social choice rules on unimodal domains in bounded distributive lattices is established. It is also proved that the equivalence between strategy-proofness and coalitional strategy-proofness -that is known to hold in standard, more restricted unimodal domains- fails in general bounded distributive lattices.

2. Model and results

Let $N = \{1, .., n\}$ denote the finite population of voters, and $\mathcal{X} = (X, \leqslant)$ the partially ordered set of alternative outcomes (i.e. $\leqslant$ is a reflexive, transitive and antisymmetric binary relation on X). We assume $n \geq 3$ in order to avoid tedious qualifications. Let us also assume that $\mathcal{X} = (X, \leqslant)$ is a *distributive lattice* and denote by $\vee$ and $\wedge$ the *least-upper-bound* and *greatest-lower-bound* binary operations on X as induced by $\leqslant$, respectively: hence, for all $x, y, z \in X$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (or, equivalently, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$) holds. For any $x, y \in X$, $[x, y| \leqslant] := \{z \in X : x \wedge y \leqslant z \leqslant x \vee y\}$ denotes the *interval* induced by x and y: we shall also write $[x, y]$ for $[x, y| \leqslant]$ whenever the underlying order $\leqslant$ is unambiguously fixed.

The *median* on $\mathcal{X}$ is the ternary operation $\mu : X^3 \to X$ defined as follows: for all $x, y, z \in X$

$$\mu(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (x \wedge z)$$

(see e.g. Birkhoff, Kiss (1947) for an early study of the basic properties of the median in a distributive lattice).

Moreover, $\mathcal{X}$ is said to be *bounded* iff there exist $\perp, \top \in X$ such that $\perp \leqslant x \leqslant \top$ for all $x \in X$.

Now, consider the set T_X of all *topped* total preorders on X (i.e. *connected*, reflexive, and transitive binary relations having a unique maximum in X): for any $\succcurlyeq \in T_X$, $top(\succcurlyeq)$ denotes the unique maximum of $\succcurlyeq$ (while $\succ$ and $\sim$ denote the asymmetric and symmetric components of $\succcurlyeq$). A topped total preorder order $\succcurlyeq \in T_X$ is *unimodal* (with respect to $\mathcal{X} = (X, \leqslant)$) iff for each $x, y, z \in X$, $z \in [x, y]$ implies that either $z \succcurlyeq x$ or $z \succcurlyeq y$ (or both). We denote by $U_{\mathcal{X}} \subseteq T_X$ the set of all unimodal total preorders on $\mathcal{X}$, and by $U_{\mathcal{X}}^N$ the set of all N-*profiles* of unimodal total preorders on X.

A *social choice rule* for (N, X) is a function $f : X^N \to X$: notice that a social choice rule may also be regarded as an n-ary *aggregation* operation on X.

For any profile $(Y_i)_{i\in N}$ (where $Y_i \subseteq X$ for all $i \in N$) a *restricted social choice rule* for $(N, (Y_i)_{i\in N})$ is a function $f : \Pi_{i\in N} Y_i \to X$. A social choice rule $f : \Pi_{i\in N} Y_i \to X$ is *monotonic* with respect to $\mathcal{X}$ iff for all $x_N = (x_j)_{j\in N} \in Y^N, i \in N$ and $x'_i \in Y$: $f(x_N) \in [x_i, f(x'_i, x_{N\smallsetminus\{i\}})]$.

Moreover, for any $i \in N$ let $D_i \subseteq U_{\mathcal{X}}$ such that *top*$(\succcurlyeq) \in Y_i$ for all $\succcurlyeq \in D_i$: then, $f : \Pi_{i\in N} Y_i \to X$ is (individually) *strategy-proof* on $\Pi_{i\in N} D_i \subseteq U_{\mathcal{X}}^N$ iff for all $x_N \in \Pi_{i\in N} Y_i$, $i \in N$ and $x' \in Y_i$, and for all $\succcurlyeq = (\succcurlyeq_j)_{j\in N} \in \Pi_{i\in N} D_i$: $f(\mathit{top}(\succcurlyeq_i), x_{N\smallsetminus\{i\}}) \succcurlyeq_i f(x', x_{N\smallsetminus\{i\}})$. Similarly, $f : \Pi_{i\in N} Y_i \to X$ is *coalitionally strategy-proof* on $\Pi_{i\in N} D_i \subseteq U_{\mathcal{X}}^N$ iff for all $x_N \in \Pi_{i\in N} Y_i$, $C \subseteq N$ and $x'_C \in \Pi_{i\in C} Y_i$, and for all $\succcurlyeq = (\succcurlyeq_j)_{j\in N} \in \Pi_{i\in N} D_i$: there exists $i \in C$ such that $f(x_N) \succcurlyeq_i f(x'_C, x_{N\smallsetminus C})$. Finally, $f : \Pi_{i\in N} Y_i \to X$ is *efficient* iff for all $(\succcurlyeq_j)_{j\in N} \in \Pi_{i\in N} D_i \subseteq U_{\mathcal{X}}^N$ and $y \in X$, $y \notin f((\mathit{top}(\succcurlyeq_j)_{j\in N}))$ if there exists $x \in X$ such that $x \succ_i y$ for all $j \in N$.

A notable class of strategy-proof social choice rules is provided by the family of *projections* (or *dictatorial rules)* $\pi_i : X^N \to X$, $i \in N$ where for all $x_N \in X^N$, $\pi_i(x_N) = x_i$, another one is given by the family of *constant* rules $f_x : X^N \to X$, $x \in X$ where for all $x_N \in X^N$, $f_x(x_N) = x$. Is is also easily checked that both dictatorial and constant rules are *monotonic* (indeed, for all $x_N = (x_j)_{j\in N} \in Y^N, i \in N$ and $x'_i \in Y$: $f(x_N) = x_i \in [x_i, f(x'_i, x_{N\smallsetminus\{i\}})]$ if f is the i-th projection, and $f(x_N) = f(x'_i, x_{N\smallsetminus\{i\}}) \in [x_i, f(x'_i, x_{N\smallsetminus\{i\}})]$ if f is a constant function).

For any $x, y \in X$ we also denote

$\mu(x, ., y) = \{z \in X :$ there exists $u \in X$ such that $z = \mu(x, u, y)\}$ and $B_\mu(\mathcal{X})$ the ternary *betweenness* relation induced by μ, namely

$B_\mu(\mathcal{X}) = \big\{(x, y, z) \in X^3 : \mu(x, y, z) = y\big\}$.

Let $\sigma \in N^N$ be a permutation of N. The *canonical μ-sequence with basis* $(\{f(x^*)\}_{x^*\in\{\bot,\top\}^N}$, $\sigma)$ induced by social choice rule f at $x_N \in X^N$ is the sequence $\boldsymbol{\mu}(\{f(x^*)\}_{x^*\in\{\bot,\top\}^N}, \sigma)(x_N) = \langle \mu_i(x_N)\rangle_{i\in\{1,\ldots,\Sigma_{i=1}^n 2^{n-i}\}}$ of median-terms defined recursively as follows:

$\mu_1(x_N) = \mu(f(\bot, x_{N\smallsetminus\{\sigma(1)\}}), \pi_{\sigma(1)}(x_N), f(\top, x_{N\smallsetminus\{\sigma(1)\}}))$,

$\mu_2(x_N) = \mu(f(\bot, \bot, x_{N\smallsetminus\{\sigma(1),\sigma(2)\}}), \pi_{\sigma(2)}(x_N), f(\bot, \top, x_{N\smallsetminus\{\sigma(1),\sigma(2)\}}))$,

$\mu_3(x_N) = \mu(f(\top, \bot, x_{N\smallsetminus\{\sigma(1),\sigma(2)\}}), \pi_{\sigma(2)}(x_N), f(\top, \top, x_{N\smallsetminus\{\sigma(1),\sigma(2)\}}))$,

$\mu_4(x_N) = \mu(\mu_2(x_N), \pi_1(x_N), \mu_3(x_N))$,

........................,

$\mu_{\Sigma_{i=1}^n 2^{n-i}}(x_N) = \mu(\mu_{2^{n-1}+2^{n-2}+..+2^{n-(n-2)}+1}(x_N), \pi_{\sigma(n)}(x_N), \mu_{2^{n-1}+2^{n-2}+..+2^{n-(n-2)}+2}(x_N))$.

Thus, each term of any sequence $\boldsymbol{\mu}(\{f(x^*)\}_{x^* \in \{\perp,\top\}^N}, \sigma)(x_N)$ is either a median of one projection and two constants, or a (iterated) median of one projection and two (iterated) medians involving projections and constants only. A social choice rule $f : X^N \to X$ is *canonically median-representable* iff there exists a permutation $\sigma \in N^N$ such that $f = \mu_{\Sigma_{i=1}^n 2^{n-i}}$ where $\mu_{\Sigma_{i=1}^n 2^{n-i}} : X^N \to X$ is the social choice rule induced by the last terms of the canonical μ-sequences $\boldsymbol{\mu}(\{f(x^*)\}_{x^* \in \{\perp,\top\}^N}, \sigma)(x_N)$ with basis $(\{f(x^*)\}_{x^* \in \{\perp,\top\}^N}, \sigma)$ induced by f at x_N, for each $x_N \in X^N$.

The ensuing analysis shall be mostly focussed on *bounded* distributive lattices. In order to fully appreciate the remarkably wide scope and relevance of such a setting let us consider just a few prominent examples, namely

Example 1: **Aggregation of points on a bounded subset of the extended real line**.

Let $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$ denote the extended real line, $\leq^*$the extended natural order, $Y \subseteq \mathbb{R}^*$ and $x, y \in Y$ with $x \leq^* y$. Then, take $\mathcal{X} = (X, \leqslant)$ with $X = \{z \in Y : x \leq^* z \leq^* y\}$ and $\leqslant = \leq^*_{|X}$. This is the standard setting employed in a considerable part of the literature on strategy-proofness in unimodal domains, including the seminal Moulin (1980) where $\mathcal{X} = (\mathbb{R}^*, \leq^*)$.

Example 2: **Aggregation of subsets of a fixed set**.

Let Y be a set of items, and $\mathcal{P}(Y)$ its power set. Then, take $\mathcal{X} = (\mathcal{P}(Y), \subseteq)$. This kind of domain obtains in a most natural way whenever a *combinatorial* social choice problem (i.e. a social choice problem among *mutually compatible* objects) or a *judgment aggregation* problem (with an unconstrained 'atomic' agenda) is under consideration.

Example 3: **Aggregation of (arbitrary, full-domain) choice functions on a fixed set**.

Let Y be a set of items, and $\mathcal{P}(Y)$ its power set. A (full-domain) *choice function* on Y is a function $f : \mathcal{P}(Y) \to \mathcal{P}(Y)$ such that $f(A) \subseteq A$ for each $A \subseteq Y$. Now, denote by $\mathcal{C}_Y$ the set of all choice functions on Y, and for any $f, g \in \mathcal{C}_Y$ posit $f \leqslant' g$ iff $f(A) \subseteq g(A)$ for all $A \subseteq Y$. Then, take $\mathcal{X} = (\mathcal{C}_Y, \leqslant')$.

Example 4: **Aggregation of dissimilarity relations on a fixed set**.

Let Y a set of items: a *dissimilarity* (or orthogonality) relation on Y is an irreflexive and symmetric binary relation D on Y i.e. $D \subseteq Y \times Y$ is such that (i) $(y, y) \notin D$ for all $y \in Y$ and (ii) $(y, z) \in D$ implies $(z, y) \in D$ for all $y, z \in Y$. Denote by $\mathcal{D}_Y$ the set of all dissimilarity relations on Y, and take $\mathcal{X} = (\mathcal{D}_Y, \subseteq)$.

Example 5: **Aggregation of tolerance relations on a fixed set**.

Let Y a set of items: a *tolerance* (or similarity) relation on Y is a reflexive and symmetric binary relation D on Y i.e. $D \subseteq Y \times Y$ is such that (i) $(y, y) \in D$ for all $y \in Y$ and (ii) $(y, z) \in D$ implies $(z, y) \in D$ for all $y, z \in Y$. Denote by $\mathcal{T}_Y$ the set of all tolerance relations on Y, and take $\mathcal{X} = (\mathcal{T}_Y, \subseteq)$.

Example 6: **Aggregation of order filters over a partially ordered population**.

Let $\mathcal{Y} = (Y, \leqslant)$ denote a finite partially ordered population. An *order filter* of $\mathcal{Y}$ is a set $F \subseteq Y$ such that for all $y, z \in Y$, $z \in F$ whenever $y \in F$ and $y \leqslant z$. Denote by $\mathcal{F}_{\mathcal{Y}}$ the set of all order filters of $\mathcal{Y}$, and take $\mathcal{X} = (\mathcal{F}_{\mathcal{Y}}, \subseteq)$. Order filters may variously arise in several aggregation problems, including choice of a (system of) threshold(s) for the analysis of opportunity inequality, and judgment aggregation problems with implication-constrained agendas.

Example 7: **Aggregation of order ideals over a partially ordered population.**

Let $\mathcal{Y} = (Y, \leqslant)$ denote a finite partially ordered population. An *order ideal* of $\mathcal{Y}$ is a set $I \subseteq Y$ such that for all $y, z \in Y$, $z \in I$ whenever $y \in I$ and $z \leqslant y$. Denote by $\mathcal{I}_{\mathcal{Y}}$ the set of all order ideals of $\mathcal{Y}$, and take $\mathcal{X} = (\mathcal{I}_{\mathcal{Y}}, \subseteq)$. Order ideals are also relevant to several aggregation problems, including choice of a (system of) threshold(s) in multidimensional poverty analysis.

Example 8: **Aggregation of graded evaluations**.

Let $\Lambda = (L, \leq)$ denote a (bounded) linearly ordered set of grades, X a (finite) population of candidates to be evaluated, and N a (finite) population of evaluators. Then, denote by L^X the set of all possible gradings of X, by $\leqslant$ the point-wise partial order induced by $\leq$, and take $\mathcal{X} = (L^X, \leqslant)$. This is indeed the formal setting recently proposed by Balinski and Laraki (2010) in order to advance their case for *majority judgment*.

We are now ready to state the main result of this paper concerning the characterization of strategy-proof social choice rules on unimodal profiles. Our characterization result relies on the following three lemmas.

The first lemma simply establishes the equivalence between *monotonicity* with respect to an arbitrary distributive lattice $\mathcal{X}$ and *strategy-proofness on the corresponding full unimodal domain* $U_{\mathcal{X}}^N$.

Lemma 1. *Let $\mathcal{X} = (X, \leqslant)$ be a distributive lattice. A social choice rule $f : X^N \to X$ is strategy-proof on $U_{\mathcal{X}}^N$ iff it is monotonic with respect to $\mathcal{X}$.*

Proof. Let us assume that $f : X^N \to X$ is *not* monotonic with respect $\mathcal{X}$: thus, there exist $i \in N$, $x_i' \in X$ and $x_N = (x_i)_{i\in N} \in X^N$ such that $f(x_N) \notin [x_i, f(x_i', x_{N\smallsetminus\{i\}})]$. Then, consider the total preorder $\succcurlyeq^*$ on X defined as follows: $x_i = top(\succcurlyeq^*)$ and for all $y, z \in X \smallsetminus \{x_i\}$, $y \succcurlyeq^* z$ iff (i) $\{y, z\} \subseteq [x_i, f(x_i', x_{N\smallsetminus\{i\}})] \smallsetminus \{x_i\}$ or (ii) $y \in [x_i, f(x_i', x_{N\smallsetminus\{i\}})] \smallsetminus \{x_i\}$ and $z \notin [x_i, f(x_i', x_{N\smallsetminus\{i\}})]$ or (iii) $y \notin [x_i, f(x_i', x_{N\smallsetminus\{i\}})]$ and $z \notin [x_i, f(x_i', x_{N\smallsetminus\{i\}})]$. Clearly, by construction $\succcurlyeq^*$consists of three indifference classes with $\{x_i\}$, $[x_i, f(x_i', x_{N\smallsetminus\{i\}})] \smallsetminus \{x_i\}$ and $X \smallsetminus [x_i, f(x_i', x_{N\smallsetminus\{i\}})]$ as top, medium and bottom indifference classes, respectively. Now, observe that $\succcurlyeq^* \in U_{\mathcal{X}}$. To check this statement, take any $y, z, v \in X$ such that $y \neq z$ and $v \in [y, z]$ (if $y = z$ then $v = y = z$ and there is in fact nothing to prove). If $\{y, z\} \subseteq [x_i, f(x_i', x_{N\smallsetminus\{i\}})]$ then by construction $x_i \wedge f(x_i', x_{N\smallsetminus\{i\}}) \leqslant y \wedge z \leqslant v \leqslant y \vee z \leqslant x_i \vee f(x_i', x_{N\smallsetminus\{i\}})$ i.e. $v \in [x_i, f(x_i', x_{N\smallsetminus\{i\}})]$. Assume without loss of generality that $y \neq x_i$: it follows that $v \succcurlyeq^* y$ by definition of $\succcurlyeq^*$. If on the contrary $\{y, z\} \cap (X \smallsetminus [x_i, f(x_i', x_{N\smallsetminus\{i\}})]) \neq \varnothing$ then clearly by definition of $\succcurlyeq^*$there exists $w \in \{y, z\}$ such that $v \succcurlyeq^* w$. Thus, $\succcurlyeq^* \in U_{\mathcal{X}}$ as claimed. Also, by assumption $f(x_N) \in X \smallsetminus [x_i, f(x_i', x_{N\smallsetminus\{i\}})]$ whence by construction $f(x_i', x_{N\smallsetminus\{i\}}) \succ^* f(x_N)$. But then, f is *not* strategy-proof on $U_{\mathcal{X}}^N$.

Conversely, let f be monotonic with respect to $\mathcal{X}$. Now, consider any $\succcurlyeq = (\succcurlyeq_j)_{j\in N} \in U_{\mathcal{X}}^N$ and any $i \in N$. By definition of monotonicity $f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}}) \in [top(\succcurlyeq_i), f(x_i, x_{N\smallsetminus\{i\}})]$ for all $x_{N\smallsetminus\{i\}} \in X^{N\smallsetminus\{i\}}$ and $x_i \in X$. But then, since clearly $top(\succcurlyeq_i) \succcurlyeq_i f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}})$, either $f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}}) = top(\succcurlyeq_i)$ or $f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}}) \succcurlyeq_i f(x_i, x_{N\smallsetminus\{i\}})$ by unimodality of $\succcurlyeq_i$. Hence, $f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}}) \succcurlyeq_i f(x_i, x_{N\smallsetminus\{i\}})$ in any case. It follows that f is indeed strategy-proof on $U_{\mathcal{X}}^N$. □

Remark 1. *Observe that a restricted social choice rule may be strategy-proof on its restricted unimodal domain while being not monotonic (i.e. the first implication of the previous lemma does not hold in general for restricted social choice rules).*

To see this, consider the following example, adapted from Barberà, Berga and Moreno (2010), and slightly simplified: consider $X = \{a, b, c, d\}$ *with* a, b, c, d *mutually distinct,* $\Delta_X = \{(x, x) : x \in X\}$,

$\leqslant^* = \{(a, b), (a, c), (a, d), (b, c), (b, d), (d, c)\} \cup \Delta_X$ *i.e.* $\mathcal{X}^* = (X, \leqslant^*)$ *is the* 4*-chain.*

Then, posit $\succcurlyeq = \{(a, b), (a, c), (a, d), (b, c), (b, d), (c, d), (d, c)\} \cup \Delta_X$,

$\succcurlyeq' = \{(d, b), (d, c), (d, a), (b, c), (b, a), (c, a), (a, c)\} \cup \Delta_X$, $Y = \{a, d\}$

and define

$f' : Y^2 \times X^{N \smallsetminus \{1,2\}} \to X$ *by the following rule: for all* $x_{N \smallsetminus \{1,2\}} \in X^{N \smallsetminus \{1,2\}}$, $f'(a, a, x_{N \smallsetminus \{1,2\}}) = a$,

$f'(d, d, x_{N \smallsetminus \{1,2\}}) = d$, $f'(a, d, x_{N \smallsetminus \{1,2\}}) = b$, $f'(d, a, x_{N \smallsetminus \{1,2\}}) = c$.

First, observe that both $\succcurlyeq$ *and* $\succcurlyeq'$ *are in* $U^N_{\mathcal{X}}$ *i.e. are unimodal: indeed, top*$(\succcurlyeq) = a$, *top*$(\succcurlyeq') = d$ *and it is immediately seen that*

$$B_\mu(X, \leqslant^*) = \left\{ \begin{array}{c} (a, b, c), (a, b, d), (a, d, c), (b, d, c), \\ (c, b, a), (d, b, a), (c, d, a), (c, d, b) \end{array} \right\}$$

$\cup \left\{(x, y, z) \in X^3 : x = y \text{ or } z = y\right\}$,

But then, since $\{(b, c), (b, d), (d, c)\} \cup \Delta_X$ *is a subrelation of* $\succcurlyeq$ *and* $\{(b, c), (b, a), (d, a), (d, c)\} \cup \Delta_X$ *is a subrelation of* $\succcurlyeq'$ *, it follows that unimodality of* $\succcurlyeq$ *and* $\succcurlyeq'$ *with respect to* $\mathcal{X}^*$ *holds. Moreover,* f' *is by construction strategy-proof on* $D^2 \times U_X^{N \smallsetminus \{1,2\}}$*: to check this, notice that* 1 *and* 2 *are the only nondummy voters, and for all* $x_{N \smallsetminus \{1,2\}} \in X^{N \smallsetminus \{1,2\}}$ $f'(a, a, x_{N \smallsetminus \{1,2\}}) \succcurlyeq f'(d, a, x_{N \smallsetminus \{1,2\}})$, $f'(a, d, x_{N \smallsetminus \{1,2\}}) \succcurlyeq f'(d, d, x_{N \smallsetminus \{1,2\}})$, $f'(a, a, x_{N \smallsetminus \{1,2\}}) \succcurlyeq f'(a, d, x_{N \smallsetminus \{1,2\}})$, $f'(d, a, x_{N \smallsetminus \{1,2\}}) \succcurlyeq f'(d, d, x_{N \smallsetminus \{1,2\}})$,

and similarly $f'(d, a, x_{N \smallsetminus \{1,2\}}) \succcurlyeq' f'(a, a, x_{N \smallsetminus \{1,2\}})$, $f'(d, d, x_{N \smallsetminus \{1,2\}}) \succcurlyeq' f'(a, d, x_{N \smallsetminus \{1,2\}})$, $f'(a, d, x_{N \smallsetminus \{1,2\}}) \succcurlyeq' f'(a, a, x_{N \smallsetminus \{1,2\}})$, $f'(d, d, x_{N \smallsetminus \{1,2\}}) \succcurlyeq f'(d, a, x_{N \smallsetminus \{1,2\}})$, *whence strategy-proofness of* f' *follows. However, observe that* $f'(d, a, x_{N \smallsetminus \{1,2\}}) = c \notin [d, a| \leqslant^*] = [d, f'(a, a, x_{N \smallsetminus \{1,2\}})| \leqslant^*]$ *hence* f' *is not monotonic with respect to* $(X, \leqslant^*)$.

The next lemma ensures that in an arbitrary distributive lattice the median operation as applied to social choice rules does preserve monotonicity.

Lemma 2. *Let* $\mathcal{X} = (X, \leqslant)$ *be a distributive lattice, and* $f : X^N \to X$, $g : X^N \to X$, $h : X^N \to X$ *social choice rules that are monotonic with respect to* $\mathcal{X}$*. Then* $\mu(f, g, h) : X^N \to X$ *(where* $\mu(f, g, h)(x_N) = \mu(f(x_N), g(x_N), h(x_N))$ *for all* $x_N \in X^N$*) is also monotonic with respect to* $\mathcal{X}$*.*

Proof. Take any $x_N \in X^N$. By definition of monotonicity with respect to $\mathcal{X}$, it suffices to show that for any $i \in N$ and $x'_i \in X$, $\mu(f, g, h)(x_N) \in [x_i, \mu(f, g, h)(x'_i, x_{N \smallsetminus \{i\}})]$.

Indeed, by monotonicity of f, g, h with respect to $\mathcal{X}$,

$f(x_N) \in [x_i, f(x'_i, x_{N \smallsetminus \{i\}})]$, $g(x_N) \in [x_i, g(x'_i, x_{N \smallsetminus \{i\}})]$,

and $h(x_N) \in [x_i, h(x'_i, x_{N \smallsetminus \{i\}})]$.

A change of variables is in order here for the sake of convenience, namely

$x_f = f(x_N)$, $x'_f = f(x'_i, x_{N \smallsetminus \{i\}})$, $x_g = g(x_N)$, $x'_g = g(x'_i, x_{N \smallsetminus \{i\}})$, $x_h = h(x_N)$, $x'_h = h(x'_i, x_{N \smallsetminus \{i\}})$

whence

$\mu(f, g, h)(x_N) = \mu(x_f, x_g, x_h)$, and $\mu(f, g, h)(x'_i, x_{N \smallsetminus \{i\}}) = \mu(x'_f, x'_g, x'_h)$.

Thus, $x_i \wedge x'_l \leqslant x_l \leqslant x_i \vee x'_l$, $l = f, g, h$, by hypothesis, while the thesis amounts to

$x_i \wedge \mu(x'_f, x'_g, x'_h) \leqslant \mu(x_f, x_g, x_h) \leqslant x_i \vee \mu(x'_f, x'_g, x'_h)$.

Now, $\mu(x'_f, x'_g, x'_h) = (x'_f \wedge x'_g) \vee (x'_g \wedge x'_h) \vee (x'_f \wedge x'_h)$

hence by distributivity and the basic latticial identities

$x_i \wedge ((x'_f \wedge x'_g) \vee (x'_g \wedge x'_h) \vee (x'_f \wedge x'_h)) =$

$(x_i \wedge (x'_f \wedge x'_g)) \vee (x_i \wedge (x'_g \wedge x'_h)) \vee (x_i \wedge (x'_f \wedge x'_h)) =$

$((x_i \wedge x'_f) \wedge (x_i \wedge x'_g)) \vee ((x_i \wedge x'_g) \wedge (x_i \wedge x'_h)) \vee ((x_i \wedge x'_f) \wedge (x_i \wedge x'_h))$.

However, by hypothesis, distributivity and the basic latticial identities again

$((x_i \wedge x'_f) \wedge (x_i \wedge x'_g)) \vee ((x_i \wedge x'_g) \wedge (x_i \wedge x'_h)) \vee ((x_i \wedge x'_f) \wedge (x_i \wedge x'_h)) \leqslant$

$\leqslant (x_f \wedge x_g) \vee (x_g \wedge x_h) \vee (x_f \wedge x_h) = \mu(x_f, x_g, x_h) \leqslant$

$\leqslant ((x_i \vee x'_f) \wedge (x_i \vee x'_g)) \vee ((x_i \vee x'_g) \wedge (x_i \vee x'_h)) \vee ((x_i \vee x'_f) \wedge (x_i \vee x'_h)) =$

$= (x_i \vee (x'_f \wedge x'_g)) \vee (x_i \vee (x'_g \wedge x'_h)) \vee (x_i \vee (x'_f \wedge x'_h)) =$

$= x_i \vee ((x'_f \wedge x'_g) \vee (x'_g \wedge x'_h) \vee (x'_f \wedge x'_h)) = x_i \vee \mu(x'_f, x'_g, x'_h)$

as required. $\square$

Finally, the next lemma -that only concerns *bounded* distributive lattices- provides a canonical median-based representation of all monotonic social choice rules hence - in view of Lemma 1 above- of all strategy-proof social choice rules on the corresponding full unimodal domain.

Lemma 3. *Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice and $f : X^N \to X$ a social choice rule that is monotonic with respect to $\mathcal{X}$. Then, there exists a permutation $\sigma \in N^N$ such that $f = \mu_{\Sigma_{i=1}^n 2^{n-i}}$ i.e. for all $x_N \in X^N$: $f(x_N) = \mu_{\Sigma_{i=1}^n 2^{n-i}}(x_N)$ where $\mu_{\Sigma_{i=1}^n 2^{n-i}}(x_N)$ is the last term of the canonical μ-sequence with basis $(\{f(x^*)\}_{x^* \in \{\bot, \top\}^N}, \sigma)$ induced by f at x_N.*

Proof. Take any $x_{N\smallsetminus\{1\}} \in X^{N\smallsetminus\{1\}}$ and consider $f_{x_{N\smallsetminus\{1\}}} : X \to X$ as defined by the rule $f_{x_{N\smallsetminus\{1\}}}(x_1) = f(x_1, x_{N\smallsetminus\{1\}})$ for all $x_1 \in X$. Thus, by definition $f_{x_{N\smallsetminus\{1\}}}$ is monotonic with respect to $(X, \leqslant)$ i.e. $f_{x_{N\smallsetminus\{1\}}}(x) \in [x,\ f_{x_{N\smallsetminus\{1\}}}(y)]$, namely $x \wedge f_{x_{N\smallsetminus\{1\}}}(y) \leqslant f_{x_{N\smallsetminus\{1\}}}(x) \leqslant x \vee f_{x_{N\smallsetminus\{1\}}}(y)$ for any $x, y \in X$. In particular, $\bot = \bot \wedge f_{x_{N\smallsetminus\{1\}}}(x_1) \leqslant f_{x_{N\smallsetminus\{1\}}}(\bot) \leqslant \bot \vee f_{x_{N\smallsetminus\{1\}}}(x_1) = f_{x_{N\smallsetminus\{1\}}}(x_1)$, $f_{x_{N\smallsetminus\{1\}}}(x_1) = \top \wedge f_{x_{N\smallsetminus\{1\}}}(x_1) \leqslant f_{x_{N\smallsetminus\{1\}}}(\top) \leqslant \top \vee f_{x_{N\smallsetminus\{1\}}}(x_1) = \top$, $x_1 \wedge f_{x_{N\smallsetminus\{1\}}}(\bot) \leqslant f_{x_{N\smallsetminus\{1\}}}(x_1) \leqslant x_1 \vee f_{x_{N\smallsetminus\{1\}}}(\bot)$, and $x \wedge f_{x_{N\smallsetminus\{1\}}}(\top) \leqslant f_{x_{N\smallsetminus\{1\}}}(x_1) \leqslant x_1 \vee f_{x_{N\smallsetminus\{1\}}}(\top)$, for all $x_1 \in X$. Now, take any $x_1 \in X$ and consider $\mu(f_{x_{N\smallsetminus\{1\}}}(\bot), x_1, f_{x_{N\smallsetminus\{1\}}}(\top))$. By definition, and distributivity of $(X, \leqslant)$, $\mu(f_{x_{N\smallsetminus\{1\}}}(\bot), x_1, f_{x_{N\smallsetminus\{1\}}}(\top)) = (x_1 \wedge f_{x_{N\smallsetminus\{1\}}}(\bot)) \vee (x_1 \wedge f_{x_{N\smallsetminus\{1\}}}(\top)) \vee (f_{x_{N\smallsetminus\{1\}}}(\bot) \wedge f_{x_{N\smallsetminus\{1\}}}(\top)) =$

$(x_1 \vee f_{x_{N\smallsetminus\{1\}}}(\bot)) \wedge (x_1 \vee f_{x_{N\smallsetminus\{1\}}}(\top) \wedge (f_{x_{N\smallsetminus\{1\}}}(\bot) \vee f_{x_{N\smallsetminus\{1\}}}(\top))$. Since by monotonicity -as observed above- $x_1 \wedge f_{x_{N\smallsetminus\{1\}}}(\bot) \leqslant f_{x_{N\smallsetminus\{1\}}}(x_1)$, $x_1 \wedge f_{x_{N\smallsetminus\{1\}}}(\top) \leqslant f_{x_{N\smallsetminus\{1\}}}(x_1)$, and $f_{x_{N\smallsetminus\{1\}}}(\bot) \wedge f_{x_{N\smallsetminus\{1\}}}(\top) = f_{x_{N\smallsetminus\{1\}}}(\bot) \leqslant f_{x_{N\smallsetminus\{1\}}}(x_1)$, it follows that $\mu(f_{x_{N\smallsetminus\{1\}}}(\bot), x_1, f_{x_{N\smallsetminus\{1\}}}(\top)) \leqslant f_{x_{N\smallsetminus\{1\}}}(x_1)$. Similarly, $f_{x_{N\smallsetminus\{1\}}}(x_1) \leqslant x_1 \vee f_{x_{N\smallsetminus\{1\}}}(\bot)$, $f_{x_{N\smallsetminus\{1\}}}(x_1) \leqslant x_1 \vee f_{x_{N\smallsetminus\{1\}}}(\top)$, and $f_{x_{N\smallsetminus\{1\}}}(x_1) \leqslant f_{x_{N\smallsetminus\{1\}}}(\top) = f_{x_{N\smallsetminus\{1\}}}(\bot) \vee f_{x_{N\smallsetminus\{i\}}}(\top)$.

It follows that $f_{x_{N\smallsetminus\{1\}}}(x_1) \leqslant \mu(f_{x_{N\smallsetminus\{1\}}}(\bot), x_1, f_{x_{N\smallsetminus\{1\}}}(\top))$ as well, whence $f_{x_{N\smallsetminus\{1\}}}(x_1) = \mu(f_{x_{N\smallsetminus\{1\}}}(\bot), x_1, f_{x_{N\smallsetminus\{1\}}}(\top)) =$

$\mu(f_{x_{N\smallsetminus\{1\}}}(\bot), \pi_1(x_1), f_{x_{N\smallsetminus\{1\}}}(\top)) = \mu(f(\bot, x_{N\smallsetminus\{1\}}), \pi_1(x_1), f(\top, x_{N\smallsetminus\{1\}}))$,

i.e. $f_{x_{N\smallsetminus\{1\}}} = \mu(f(\bot, x_{N\smallsetminus\{1\}}), \pi_1, f(\top, x_{N\smallsetminus\{1\}}))$.

Thus, for all $x_1 \in X$, $f_{x_{N\smallsetminus\{1\}}}(x_1)$ is the first term of the canonical μ-sequence with a basis $(\{f(x^*)\}_{x^*\in\{\bot,\top\}^N}, \sigma)$ such that $\sigma(1) = 1$. Next, consider $f_{x_{N\smallsetminus\{1,2\}}} : X^2 \to X$ as defined by the following rule: for all $x_1, x_2 \in X$, $f_{x_{N\smallsetminus\{1,2\}}}(x_1, x_2) = f(x_1, x_2, x_{N\smallsetminus\{1,2\}}) = \mu(f(\bot, x_{N\smallsetminus\{1\}}), \pi_1(x_1), f(\top, x_{N\smallsetminus\{1\}})) =$

$\mu(f(\bot, x_2, x_{N\smallsetminus\{1,2\}}), \pi_1(x_1), f(\top, x_2, x_{N\smallsetminus\{1,2\}}))$. By repeating the previous argument of this proof as applied to both $f(\bot, x_2, x_{N\smallsetminus\{1,2\}})$ and $f(\top, x_2, x_{N\smallsetminus\{1,2\}})$, it follows that $f_{x_{N\smallsetminus\{1,2\}}}(x_1, x_2) =$

$$= \mu(\mu(f(\bot, \bot, x_{N\smallsetminus\{1,2\}}), \pi_2(x_2), f(\bot, \top, x_{N\smallsetminus\{1,2\}})),$$
$$\pi_1(x_1), \mu(f(\top, \bot, x_{N\smallsetminus\{1,2\}}), \pi_2(x_2), f(\top, \top, x_{N\smallsetminus\{1,2\}})))$$

i.e. $f_{x_{N\smallsetminus\{1,2\}}}$ is the *fourth* term of a canonical μ-sequence with a basis $(\{f(x^*)\}_{x^*\in\{\bot,\top\}^N},$, $\sigma)$ such that $\sigma(1) = 1$ and $\sigma(2) = 2$. Repeated iteration of the very same argument establishes that, for all $x_N \in X^N$, $f(x_N) = \mu_{\Sigma_{i=1}^n 2^{n-i}}(x_N)$ i.e. $f(x_N)$ is the last term of the canonical μ-sequence $\boldsymbol{\mu}(\{f(x^*)\}_{x^*\in\{\bot,\top\}^N}, \sigma)(x_N) = \langle \mu_i(x_N) \rangle_{i\in\{1,\ldots,\Sigma_{i=1}^n 2^{n-i}\}}$ with basis $(\{x^h\}_{h\in\{1,..,2^n\}},$ $\sigma)$ such that $x^1 = f(\bot^N), \ldots\ldots, x^{2^n} = f(\top^N)$, and $\sigma(i) = i$ for all $i \in N$ whence the thesis follows. $\square$

The main implications of the foregoing lemmas are indeed summarized by the following

Theorem 1. *Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice. Then, the following statements on a social choice rule $f : X^N \to X$ are equivalent:*

(i) f is canonically median-representable;

(ii) f is monotonic with respect to $\mathcal{X}$;

(iii) f is strategy-proof on $U_{\mathcal{X}}^N$.

Proof. (i)$\Longrightarrow$(ii) It follows immediately from the definition of canonical median-representability, from the observation that projections and constants induce monotonic social choice rules, and from Lemma 3.

(ii) $\Longrightarrow$(iii) It follows from Lemma 1.

(iii)$\Longrightarrow$(i) Immediate from Lemma 1 and Lemma 4. □

Remark 2. *Notice that Theorem 5 implies in particular strategy-proofness of the simple majority social choice rule on unimodal domains (with an odd population of voters), since it can be quite easily shown that the former is monotonic (see e.g. Monjardet (1990) for a formal definition and study of the simple majority rule in a latticial framework). Therefore, in an arbitrary bounded distributive lattice there exist social choice rules -such as e.g. the simple majority rule- that jointly satisfy anonymity (i.e. symmetric treatment of voters), neutrality (i.e. symmetric treatment of outcomes), unanimity (i.e. faithful respect of unanimity of votes) and strategy-proofness on the full unimodal domain. It turns out, however, that in an arbitrary bounded distributive lattice-as further discussed below and in Section 3- strategy-proof social choice rules that satisfy unanimity (including in fact the simple majority rule) may fail to satisfy coalitional strategy-proofness or even efficiency on the full unimodal domain.*

Indeed, it can also be established that strategy-proofness and coalitional strategy-proofness of a social choice rule are *not* equivalent on unimodal domains in bounded distributive lattices. This is made precise by the following

Proposition 1. *Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice. Then the following holds:*

(i) if $f : X^N \to X$ is strategy-proof on $U_{\mathcal{X}}^N$ and $|X| \leq 3$ then f is also coalitionally strategy-proof on $U_{\mathcal{X}}^N$;

(ii) if $|X| \geq 4$ then there exists a sublattice $\mathcal{Y} = (Y, \leqslant_Y)$ of $\mathcal{X}$ (with $|Y| \geq 4$), a subdomain $D \subseteq U_{\mathcal{X}}$ and a restricted social choice rule $f' : Y^2 \times X^{N \smallsetminus \{1,2\}} \rightarrow X$ that is strategy-proof but not coalitionally strategy-proof on $D^2 \times U_{\mathcal{X}}^{N \smallsetminus \{1,2\}}$;

(iii) if $|X| \geq 4$ and $\mathcal{X}$ is not a linear order then there exists a sublattice $\mathcal{Y} = (Y, \leqslant_Y)$ of $\mathcal{X}$ (with $|Y| \geq 4$) and a social choice rule $f' : Y^N \rightarrow Y$ that is strategy-proof but not coalitionally strategy-proof on $U_{\mathcal{Y}}^N$.

Proof. (i) It follows from a straightforward adaptation of the proof of Theorem 1 of Barberà, Berga and Moreno (2010) to social choice rules as combined with Proposition 1 of the same paper;

(ii) Take restricted social choice rule f' as introduced in Remark 2 above, where it was also shown that f' is strategy-proof.

Now, consider any preference profile $(\succcurlyeq_i)_{i \in N}$ such that $\succcurlyeq_1 = \succcurlyeq'$ and $\succcurlyeq_2 = \succcurlyeq$ hence $top(\succcurlyeq_1) = d$, $top(\succcurlyeq_2) = a$. Then, for any $x_{N \smallsetminus \{1,2\}} \in X^{N \smallsetminus \{1,2\}}$, both $f'(a, d, x_{N \smallsetminus \{1,2\}}) \succ_1 f'(top(\succcurlyeq_1), top(\succcurlyeq_2), x_{N \smallsetminus \{1,2\}})$ and $f'(a, d, x_{N \smallsetminus \{1,2\}}) \succ_2 f'(top(\succcurlyeq_1), top(\succcurlyeq_2), x_{N \smallsetminus \{1,2\}})$: it follows that coalition $\{1, 2\}$ can manipulate the outcome at $(\succcurlyeq_i)_{i \in N}$ namely f' is *not* coalitionally strategy-proof.

(iii) Let us assume without loss of generality that $|X| = 4$ and let $X = \{a, b, c, d\}$ and $\Delta_X = \{(x, x) : x \in X\}$. Next, define

$\leqslant^{**} = \{(a, b), (a, c), (a, d), (b, d), (c, d)\} \cup \Delta_X$.

It is easily checked that $\mathcal{X}^{**} = (X, \leqslant^{**})$ is the Boolean lattice 2^2 with $a = \top$, $d = \perp$.

Now, define the family $\{f(x^*)\}_{x^* \in \{\perp, \top\}^N}$ as follows: for all $x_{N \smallsetminus \{1,2\}} \in \{\perp, \top\}^{N \smallsetminus \{1,2\}}$

$f(a, a, x) = a$, $f(d, d, x) = d$, $f(a, d, x) = b$, $f(d, a, x) = c$.

Then, consider the *canonical μ-sequence with basis* $(\{f(x^*)\}_{x^* \in \{\perp, \top\}^N}, \sigma^{id})$ induced by social choice rule f at $x_N \in X^N$ (where $\sigma^{id}(i) = i$ for all $i \in N$), namely the sequence

$\boldsymbol{\mu}(\{f(x^*)\}_{x^* \in \{\perp, \top\}^N}, \sigma^{id})(x_N) = \langle \mu_i(x_N) \rangle_{i \in \{1, \ldots, \Sigma_{i=1}^n 2^{n-i}\}}$ as defined above (notice that f is by construction an extension to the full unimodal domain of f' as mentioned above under part (ii) of the present proof).

A few simple if tedious calculations immediately establish that for all $x_{N \smallsetminus \{1,2\}} \in X^{N \smallsetminus \{1,2\}}$

$f(a, c, x) = f(b, a, x) = f(b, c, x) = a$,

$f(b, b, x) = f(a, b, x) = f(b, d, x) = b$,

$f(c, c, x) = f(c, a, x) = f(d, c, x) = c$,

$f(c, d, x) = f(d, c, x) = f(c, b, x) = d$.

By construction, and in view of Lemma 3 above, f is monotonic with respect to $\mathcal{X}^{**}$. Therefore, by Lemma 1, f is also strategy-proof on $U_{\mathcal{X}^{**}}^N$.

Now, take $\succcurlyeq = \{(a,b),(a,c),(a,d),(b,c),(b,d),(c,d),(d,c)\} \cup \Delta_X$,

$\succcurlyeq' = \{(d,b),(d,c),(d,a),(b,c),(b,a),(c,a),(a,c)\} \cup \Delta_X$, as defined in Remark 2 above.

First, observe that both $\succcurlyeq$ and $\succcurlyeq'$are in $U^N_{\mathcal{X}^{**}}$ i.e. are unimodal with respect to $\mathcal{X}^{**}$: indeed, $top(\succcurlyeq) = a$, $top(\succcurlyeq') = d$ and it is immediately seen that

$$B_\mu(X, \quad \leqslant \quad ^{**}) = \left\{ \begin{array}{c} (a,b,d),(a,c,d),(b,a,c),(b,d,c),(d,b,a), \\ (d,c,a),(c,a,b),(c,d,b) \end{array} \right\} \cup$$
$$\cup \left\{(x,y,z) \in X^3 : x = y \text{ or } z = y\right\}.$$

But then, since $\{(b,d),(c,d),(a,b),(d,c)\} \cup \Delta_X$ is a subrelation of $\succcurlyeq$ and $\{(b,a),(c,a),(a,c),(d,c)\} \cup \Delta_X$ is a subrelation of $\succcurlyeq'$, it follows that $\succcurlyeq$ and $\succcurlyeq'$are also unimodal with respect to $\mathcal{X}^{**}$. Now, take any preference profile $(\succcurlyeq_i)_{i \in N}$ such that $\succcurlyeq_1 = \succcurlyeq'$ and $\succcurlyeq_2 = \succcurlyeq$ hence $top(\succcurlyeq_1) = d$, $top(\succcurlyeq_2) = a$. Then, for any $x_{N \smallsetminus \{1,2\}} \in X^{N \smallsetminus \{1,2\}}$, both $f(a,d,x_{N \smallsetminus \{1,2\}}) \succ_1 f(top(\succcurlyeq_1), top(\succcurlyeq_2), x_{N \smallsetminus \{1,2\}})$ and $f(a,d,x_{N \smallsetminus \{1,2\}}) \succ_2 f(top(\succcurlyeq_1), top(\succcurlyeq_2), x_{N \smallsetminus \{1,2\}})$: it follows that, again, coalition $\{1,2\}$ can manipulate the outcome at $(\succcurlyeq_i)_{i \in N}$ namely f is *not* coalitionally strategy-proof. □

In fact, as a further straightforward consequence of Proposition 7 (and of a few previously known results), we have the following

Corollary 1. *Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice. Then the following statements are equivalent:*

(i) for each sublattice $\mathcal{Y} = (Y, \leqslant_{|Y})$ of $\mathcal{X}$ and each social choice rule $f : Y^N \to Y$, f is strategy-proof on $U^N_{\mathcal{Y}}$ if and only if it is also coalitionally strategy-proof on $U^N_{\mathcal{Y}}$;

(ii) $\mathcal{X} = (X, \leqslant)$ is a linear order.

Proof. (i)$\Longrightarrow$(ii) It follows immediately from Proposition 7 (iii) above;

(ii)$\Longrightarrow$(i) It follows from a straightforward extension and adaptation of the proof of Proposition 4 of Danilov (1994) concerning social choice rules on unimodal domains of *linear orders* in *undirected trees* (details available upon request), and is indeed already stated without explicit proof in Moulin (1980). □

Thus, we have here a remarkable characterization of *bounded linear orders* as *the only bounded distributive lattices where equivalence of individual and coalitional strategy-proofness of social choice rules on full unimodal domains holds.*

3. Related literature and concluding remarks

The main results of the present paper may be summarized as follows:

(i) Theorem 5 provides an explicit characterization in terms of iterated medians of projections and constants of the class of strategy-proof social choice rules on full unimodal domains in bounded distributive lattices [that result extends in a new direction previous results due to Moulin (1980) for unimodal domains in bounded linear orders, and to Danilov (1994) for unimodal domains of linear orders in undirected trees];

(ii) Proposition 7 establishes that *equivalence between (individual) strategy-proofness and coalitional strategy-proofness on full unimodal domains* holds precisely in bounded linear orders, and *fails in bounded distributive lattices that are not linear orders.*

In order to properly appreciate the significance of the foregoing results a detailed discussion of a few strictly related previous contributions is to be entered here.

The seminal paper by Moulin (see Moulin (1980)) provides an explicit characterization in terms of 'extended medians' of the class of all strategy-proof social choice rules on the domain of *all* profiles of total preorders that are unimodal with respect to a fixed *bounded linear order.* Furthermore, Moulin (1980) establishes the equivalence of strategy-proofness and coalitional strategy-proofness for *all* social choice rules on such full unimodal domains.

In fact, Moulin's proof relies heavily on the following property of medians in bounded linear orders that *does not hold for medians in general bounded distributive lattices*: given an odd population of $n = 2k+1$ voters, for any $(x_i)_{i=1,..,n} \in X^N$ the extended median $\mu^*(x_1, ..., x_n)$ i.e. the (iterated) median $\mu(x_{2k}, \mu(x_{2(k-1)}, \mu(..(\mu(x_1, x_2, x_3))..), x_{2k-1}), x_{2k+1})$ is such that

($^\circ$) $\min(\#\{i \in N : x_i \leq \mu^*(x_1, ..., x_n)\}, \#\{i \in N : \mu^*(x_1, ..., x_n) \leq x_i\}) \geq k+1.$

Indeed, in bounded linear orders (extended) medians are sometimes *defined* by the foregoing property [notice that the extended median on a linear order is invariant with respect to arbitrary permutations of $(x_i)_{i=1,..,n}$].

However, take $n = 3$ (hence $k+1 = 2$) and consider again the Boolean lattice $\mathcal{X}^{**} = (X, \leqslant^{**})$ with $X = \{a, b, c, d\}$, $\#X = 4$, $\Delta_X = \{(x, x) : x \in X\}$,

$\leqslant^{**} = \{(a, b), (a, c), (a, d), (b, d), (c, d)\} \cup \Delta_X$,

i.e. $\mathcal{X}^{**} = (X, \leqslant^{**})$ is the Boolean lattice $\mathbf{2}^2$ with $a = \top$, $d = \perp$.

Clearly $\mu^*(a, b, c) = \mu(a, b, c) = a$, hence at $(x_1, x_2, x_3) = (a, b, c)$,

$\#\{i \in N : x_i \leq \mu^*(a, b, c)\} = 3$

but $\#\{i \in N : \mu^*(a, b, c) \leq x_i\} = 1$, and ($^\circ$) fails.

In a similar vein, Danilov (1994) provides a characterization in terms of (iterated) medians of the class of strategy-proof social choice rules on the domain of all unimodal *linear orders* (i.e. *antisymmetric* total preorders) when X is the vertex set of an undirected (finite) *tree* (see also Danilov and Sotskov (2002) for further discussion of this topic, and Demange (1982) for an early study of majority-like voting rules on domains of unimodal linear orders in undirected trees). Moreover, Danilov (1994) also shows that strategy-proofness and coalitional strategy-proofness of social choice rules on *unimodal profiles of linear preference orders in undirected trees* are *equivalent* properties. But in fact, it can be shown that Danilov's proofs can be readily extended to the wider full domain of *unimodal total preference preorders* (arguing along the lines of the first part of the proof of Lemma 1 above), and to the case of an underlying *bounded linearly ordered set* of alternatives.

The key step of Danilov's proof relies on the following property shared by *intervals* of linear orders and of undirected trees, namely :

(*) for all $x, y, v, z \in X$, if $x \in [y, v]$ and $y \in [x, z]$ then $x \in [v, z]$.

Notice however that (*) does *not* hold for intervals of arbitrary bounded distributive lattices: to see this, consider again the four-element Boolean distributive lattice $\mathcal{X}^{**} = (X, \leqslant^{**})$ with $X = \{a = \top, b, c, d = \bot\}$ introduced above in the text, and notice that e.g. $b \in [a, d]$, $a \in [b, c]$ but $b \notin [c, d]$ (see e.g. Sholander (1952, 1954(a)), Bandelt and Hedlikova (1983) for a thorough study of intervals in general median algebras, and Isbell (1980) for an even more general approach that also considers intervals in a larger class of ternary algebras).

Building upon some remarkable earlier contributions such as Barberà, Gul and Stacchetti (1993) and Barberà, Massò and Neme (1997), and relying on a *strict* notion of unimodality, Nehring and Puppe (2007(a)) offer a comprehensive study and an 'issue-by-issue voting-by-committees'-based characterization of souvereign (i.e. surjective) strategy-proof social choice functions on rich domains of *strictly unimodal linear orders* in certain *finite* 'median spaces' as induced by suitably defined 'property spaces' [it can be shown that such finite 'median spaces' do essentially correspond to *finite median algebras*: see e.g. Sholander (1954(a), 1954(b)) and Bandelt and Hedlikova (1983)]. In an interesting related paper- Nehring and Puppe (2007(b))- it is also shown that (under a suitably defined notion of 'dimension') even the median of projections is *not* efficient in 'median spaces' of dimension $k \geq 3$, including the Boolean lattice $\mathbf{2}^3$, and it is proved that efficiency *and* strategy-proofness of a social choice function f on a *rich* strictly unimodal domain of linear orders in a 'median space' jointly imply that either f is weakly dictatorial or that the dimension of the 'median space' is *two* at most. While such a paper is not concerned with the equivalence issue of individual and coalitional strategy-proofness, it should be noticed that the foregoing result has a clear *inequivalence* implication for souvereign strategy-proof social choice rules on rich strictly unimodal domains in higher-dimensional finite 'median spaces', since coalitional strategy-proofness and souvereignty of a social choice function or rule jointly imply efficiency of the latter.

However, Nehring and Puppe's results are entirely silent on the equivalence issue in two-dimensional 'median spaces'. Moreover, and perhaps less obviously, their general results are strictly speaking irrelevant to the same equivalence issue *even for any finite distributive lattice that is not a linear order.* That is so because the strict notion of unimodality they use is unsuitable for linear preference orders in general distributive lattices, typically resulting in vacuous -hence a fortiori *not rich*- domains (to check this, observe that the relevant notion of 'betweenness' used by Nehring and Puppe essentially amounts to saying that y is 'between' x and z precisely when $y \in [x, z]$; then, just take the four-element Boolean lattice $\mathcal{X}^{**}$ as defined above in the proof of Proposition 7 (iii) and observe that it admits no strictly unimodal linear preference orders). Therefore, the main theorems of Nehring and Puppe (2007 (a), (b)) concern finite lines and trees and also certain products of lines and trees but not e.g. Boolean distributive lattices with at least four elements (either finite or not).

More recently, Barberà, Berga and Moreno (2010) focused on a strict version of unimodality for total preorders (it should be noticed, however, that strict unimodality in a bounded linear order reduces to unimodality when total preorders are in fact antisymmetric i.e. linear orders: details are available from the authors upon request). Relying on a property they newly introduce and label 'sequential inclusion', Barberà et al.(2010) establish a general sufficient condition ensuring equivalence of individual and coalitional strategy-proofness, and show that *strictly unimodal domains of total preorders* as defined on a *linear order* $(X, \leqslant)$ do satisfy it. Thus, *prima facie* such a result seems to support the widely shared presumption of equivalence between individual and coalitional strategy-proofness on unimodal domains (see also Le Breton and Zaporozhets (2009) in that connection).

The argument of Barberà et al.(2010) for such an equivalence result however *cannot be extended to domains of unimodal total preorders* since it can be easily checked that profiles of unimodal -as opposed to strictly unimodal- total preorders need *not* satisfy 'sequential inclusion' (indeed, take a four-element *linear order* $(\{x, y, w, z\}, \leqslant)$ such that $x < y < w < z$, consider total preorders $\succcurlyeq_1, \succcurlyeq_2$ on X such that $y \succ_1 w \succ_1 x \sim_1 z$ and $y \succ_2 x \succ_1 w \sim_1 z$, and observe that $\succcurlyeq_1$and $\succcurlyeq_2$ are unimodal -though of course not strictly unimodal- *and* violate the 'sequential inclusion' property: details are available from the authors upon request).

Thus, it is worth contrasting the *lack of equivalence* between individual and coalitional strategy-proofness on unimodal domains established by Proposition 7 above with the earlier *equivalence* results concerning related unimodal domains. Interestingly enough, and perhaps somewhat surprisingly, Proposition 7 implies that *such equivalence cannot be extended to an arbitrary subdomain of unimodal total preorders* if $(X, \leqslant)$ is a bounded distributive lattice having at least *four* elements, *nor* to social choice rules on the *full domain* of total preorders that are unimodal with respect to a bounded distributive lattice that is *not* a linear order. As a consequence, the 'sincere' outcomes of strategy-proof social choice rules on unimodal domains in bounded distributive lattices are typically *not renegotiation-proof*: some supplementary ex-post bargaining may be required to ensure efficiency or indeed 'stability' of outcomes. But then, whenever voters do take that fact into account, new incentives to manipulate may be brought about.

Finally, it should also be noticed that some of the results of the present paper -notably, Lemma 1- can be easily reproduced in a more general setting e.g. in any median algebra (see Isbell (1980), Bandelt and Hedlikova (1983)). It remains to be seen which of the other results, if any, can also be lifted to the latter environment. This is however best left as a topic for future research.

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