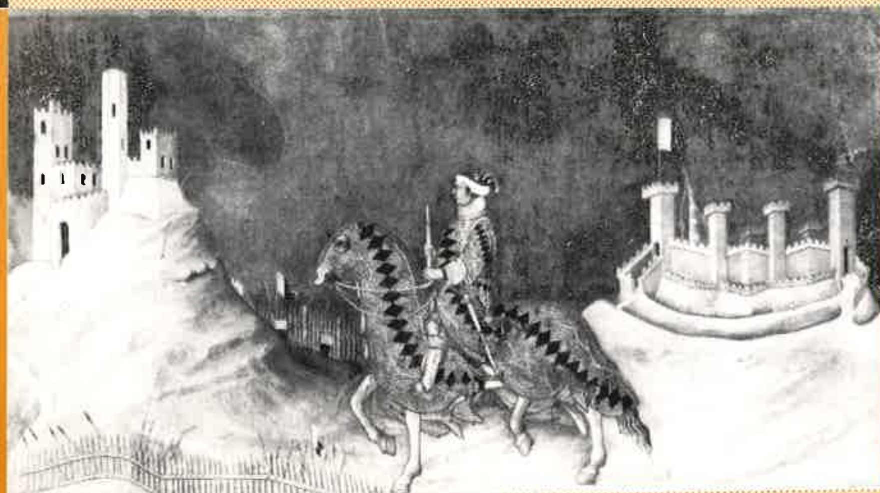
65

UNIVERSITA' DEGLI STUDI DI SIENA
Facoltà di Scienze Economiche e Bancarie

QUADERNI DELL'ISTITUTO DI ECONOMIA

Nicola Dimitri

" GENERALIZATIONS OF SOME CONTINUOUS TIME EPIDEMIC MODELS "



QUADERNI DELL'ISTITUTO DI ECONOMIA

Alla presente Collana "gialla" di quaderni l'Istituto di Economia affianca una Collana "celeste" di monografie. I numeri pubblicati a tutt'oggi sono i seguenti:

1) GIUSEPPE DELLA TORRE, Istituzioni Creditizie ed Accumulazione del Capitale in Italia (1948-81), 1984.
2) SANDRO GRONCHI, Tasso Interno di Rendimento e Valutazione dei Progetti: una Analisi Teorica, 1984.
3) ALDINO MONTI, La Proprietà Immobiliare a Bologna in Età Napoleonica (1797-1810), 1984.
4) GIULIO CIFARELLI, Equilibrium and Disequilibrium Interpretations of Inflation, 1985.

• Redazione: Istituto di Economia della Facoltà di Scienze Economiche e Bancarie - Piazza S. Francesco, 17 - 53100 Siena - tel. 0577/49059

• La Redazione ottempera agli obblighi previsti dall'Art. 1 del D.L.L. 31.8.45 n. 660

• Le richieste di copie della presente pubblicazione dovranno essere indirizzate alla Redazione

• I **Quaderni dell'Istituto di Economia dell'Università di Siena** vengono pubblicati dal 1979 come servizio atto a favorire la tempestiva divulgazione di ricerche scientifiche originali, siano esse in forma provvisoria o definitiva. I **Quaderni** vengono regolarmente inviati a tutti gli istituti e dipartimenti italiani, a carattere economico, nonché a numerosi docenti e ricercatori universitari. Vengono altresì inviati ad enti e personalità italiane ed estere. L'accesso ai **Quaderni** è approvato dal Comitato Scientifico, sentito il parere di un referee.

Nicola Dimitri

GENERALIZATIONS OF SOME CONTINUOS TIME EPIDEMIC MODELS

Siena, marzo 1987

Nicola Dimitri is a post graduate student in the Department of Economics of the University of Siena.

1. *Introduction**

As a branch of Social Statistics the epidemic theory of news, rumours and ideas is very much developed; indeed quite a few books and reviews have dedicated space and attention to it and in [2]can be found a full account of the results achieved so far. Being a topic involving dynamic realizations of certain processes (growth of the number of hearers in a given population) often the analytical side of this subject faces difficulties when encountering very complicated differential equations.

This mainly occurs in the stochastic version of the models (the more realistic one) because of the presence of differential-difference equations almost always impossible to solve; research then tends to deal with more treatable formalizations of the process such as the deterministic approximation.

This work generalizes some known models, mostly considering the deterministic side of them, since otherwise we would have to handle a stochastic birth and death process whose general solution does not exist. Furthermore we will not consider numerical results and only the qualitative behaviour of the systems will be explored.

(*) This work has been inspired by the project permormed for the MSc' Statistics at the London School of Economics; the first version of this paper was written in 1986. I would like to thank Prof. D.J. Bartholomew for the fruitful discussions I had with him.

Because of the standardized terminology and notation used in the topic, we will refer to [2] as far as symbols and their meanings are concerned.

2. A model with interactive forgetting

In this paragraph we consider a generalization of a model formerly proposed by Bartholomew [1] which, in turn, was a development of a work done by Kermack-McKendrick (K.M.) much earlier [2].

The two papers above both admitted the possibility of "forgetting" the item spread; the basic difference was that in the latter once a knower stopped the spreading he could not become a hearer any longer, which was allowed instead by the former.

The equation ruling the deterministic version of Bartholomew's model is

$$(1) \qquad \frac{dn(T)}{dT} = (\alpha + \beta n(T))\,(N-n(T)) - \mu n(T)$$

whereas for the K.M.'s we have

$$\frac{dn(T)}{dT} = (\alpha + \beta n(T))\,(N-n(T)) - \mu n(T)$$

$$(2) \qquad \frac{dm(T)}{dT} = (\alpha + \beta n(T))\,(N-n(T))$$

$$\frac{dl(T)}{dT} = \mu n(T)$$

In (2). μ is not a "forgetting" coefficient but expresses the propensity for a knower-spreader to become a knower who stopped spreading. Infact, once people have ceased communicating they don't become ignorants but "stiflers".

The characteristic bounding together (1) and (2) is the linearity of the "loss of interest" part; in other words people can only forget individually. What we do here is to generalize (1) by considering a quadratic expression in n(T) as far as the forgetting side of the model is concerned.

The justification to this is given by the observation that also interacting between themselves, and not only individually, people could make the spreading of the item "problematic".

In this framework the number of contacts, for the "loss of interest" section, is then going to be $n(T)^2$, and (1) becomes

$$(3) \qquad \frac{dn(T)}{dT} = (\alpha + \beta n(T))\,(N-n(T)) - \mu n(T)^2$$

The stochastic version is still a birth and death process as happens in (1). Transition probabilities are now the following: if we are at state n

$$P(n \rightarrow n+1 \text{ in } (T, T+\delta T)) = \lambda_n \delta T + o(\delta T) =$$

$$=(\alpha + \beta n(T))\ (N-n(T))\ \delta T + o(\delta T)$$

$$P(n \rightarrow n-1 \text{ in } (T,T+\delta T)) = \mu_n \delta T + o(\delta T) =$$
$$= \mu n(T)^2\ \delta T + o(\delta T)$$

and so

$$P'_n(T) = -[(\alpha+\beta n(T))(N-n(T)) + \mu n(T)^2]\ P_n(T) +$$
$$+ [(\alpha+\beta(n(T)-1))\ (N-n(T)+1)]\ P_{n-1}(T) +$$
$$+ \mu\ (n(T)+1)^2\ P_{n+1}(T)$$

is the differential-difference equation of the process.

Solution to (3) can easily be found and is given by

$$(3')\qquad n(T) = -\frac{\lambda_1 - \lambda_2\ [(\lambda_1+An(0))/(\lambda_2+An(0))]\ e^{(\lambda_2-\lambda_1)T}}{A\ [1-((\lambda_1+An(0))/(\lambda_2+An(0)))e^{(\lambda_2-\lambda_1)T}]}$$

where

$$\lambda_{1,2} = [(\beta N-\alpha) \pm ((\beta N-\alpha)^2+4(\beta+\mu)\alpha N)^{1/2}]/2$$

and

$$A = -(\beta+\mu)$$

Obviously

$$n(\infty) = [(N-\omega)+((N-\omega)^2 + 4(1+\varrho)\omega N)^{1/2}]\ /\ 2\ (1+\varrho)$$

where

$$\omega = \alpha/\beta, \quad \varrho = \mu/\beta, \quad d = N/\varrho$$

One important feature to check is if, when $\omega=0$ or $\alpha=0$, a threshold effect exists, that is whether there is a critical value for **d** which makes the epidemics impossible or not.

In (1) this value is $d=1$ or $N=\varrho$; in our model, if $t \rightarrow \infty$ and $\omega=0$ (clearly $n(0)>0$) we have

$$(4)\qquad n(\infty) = N\ /\ (1+\varrho)$$

which is always bigger than zero unless $\mu=\infty$ or $\beta=0$, of course having $N>0$. To have $u=\infty$ means that the epidemics will never take off because the "forgetting" is so high that the interaction is pointless; $\beta=0$ instead means that there are no chances for the individuals to communicate and so the transmission will not occur.

On the other hand rewriting (4) as

$$n(\infty) = Nd \,/\, (N+d)$$

we can see that if d=1, we get

$$n(\infty) = N/(N+1) < 1$$

Evidently, also in our model, d=1 represents a critical value (when $\omega=0$) for the development of the epidemics.

Considering d=2, a comparison with K.M. and Bartholomew's models shows that the ultimate proportion of people knowing the item is much lower: indeed while for the first two we have respectively $n(\infty)=0.8N$ and $n(\infty)=0.5N$ in our case $n(\infty)=N/(N/2+1) \leqslant 0.5N$ for any $N \geqslant 1$.

With $\varrho=1$, i.e. $\mu=\beta$, here we have $n(\infty)=0.5N$ whereas Bartholomew has, for instance, $n(\infty)=N-1$. The conclusion which can be drawn is that the number of hearers grows much slower for us than for the other two, and the asymptotic proportion of knowers is always lower than the others.

This is obviously due to the presence of the quadratic in the forgetting coefficient.

Although it is impossible to give a complete picture of the probability transition functions at each instant of time, we are in any case capable of writing down and solve the equilibrium equation for the steady-state density.

In a birth and death process, it is well known that as $t \to \infty$ we have

$$\pi_n \, \mu_n = \pi_{n-1} \, \lambda_{n-1}$$

which in our case is

$$(4') \qquad \pi_n \mu n^2 = (\alpha+\beta(n-1))\,(N-n+1)\,\pi_{n-1}$$

leading to the following distribution

$$(4'') \qquad \pi_n = \frac{(n!)^{-1}\binom{N}{n}\varrho^{N-n}\,\Gamma(n+\omega)}{\sum_{n=0}^{N}(n!)^{-1}\binom{N}{n}\varrho^{N-n}\,\Gamma(n+\omega)} \qquad n=0,1,\ldots,N$$

Equation (4) indicates that we have a situation where the deterministic analysis overestimates the stochastic mean, while it is reasonable to expect them to be the same. This occurs also in (1).

From the balance equation we have infact that

$$\pi_{n+1} \, \mu_{n+1} = \lambda_n \, \pi_n$$

and so

$$\mu(n+1)^2 \, \pi_{n+1} = (\omega+n)\,(N-n)\,\beta\,\pi_n$$

summing both sides we get

$$\mu \sum_{n=0}^{N-1} (n+1)^2 \pi_{n+1} = \beta \sum_{n=0}^{N} (\omega+n)\,(N-n)\pi_n$$

thus

$$\mu E(n^2) = \beta N E(n) + \beta N\omega - \beta E(n^2) - \omega \beta E(n)$$

which gives

$$E(n^2) = \frac{E(n)\,(N-\omega)}{(1+\varrho)} + \frac{\omega N}{(1+\varrho)}$$

since

$$Var(n) = E(n^2) - E(n)^2 =$$

$$= \frac{E(n)(N-\omega)}{(1+\varrho)} + \frac{\omega N}{(1+\varrho)} - E(n)^2 \geqslant 0$$

then

$$(5) \qquad E(n) \leqslant \frac{(N-\omega)}{2(1+\varrho)} + \frac{1}{2}\left(\left(\frac{N-\omega}{1+\varrho}\right)^2 + \frac{4N\omega}{1+\varrho}\right)^{1/2}$$

The right hand side of (5) corresponds to $n(\infty)$ of the deterministic

analysis, when $\alpha = 0$, as it can be seen from (4).

We approach now the discussion on the nature of the stationary solutions of our system [5] in terms of local stability.

The analysis considers only the case when $\omega = 0$ so that the pair of equations we concern with is

$$(6) \qquad \frac{dn(T)}{dT} = \beta n(T)\, y(T) - \mu\, n(T)^2 = N(T)$$

$$(6') \qquad \frac{dy(T)}{dT} = -\beta n(T)\, y(T) + \mu\, n(T)^2 = Y(T)$$

Linearizing (6) and (6') we obtain

$$(7) \qquad \frac{dn(T)}{dT} = a_{11}\, \bar{n}(T) + a_{12}\, \bar{y}(T)$$

$$(7') \qquad \frac{dy(T)}{dT} = a_{21}\, \bar{n}(T) + a_{22}\, \bar{y}(T)$$

where

$$\bar{n}(T) = (n(T) - n_0)\ ; \quad \bar{y}(T) = (y(T) - y_0)$$

and

$$a_{11} = \frac{dN(T)}{dn(T)}\Big|_{n_0, y_0}\ ; \quad a_{12} = \frac{dN(T)}{dy(T)}\Big|_{n_0, y_0}\ ;$$

$$a_{21} = \frac{dY(T)}{dn(T)}\Big|_{n_0,y_0} \quad \text{and} \quad a_{22} = \frac{dY(T)}{dy(T)}\Big|_{n_0,y_0}$$

being n_0 and y_0 the stationary solutions for (6). In this context we only have one pair of steady-state solutions

(i) $(n=0,\ y=N)$

and

(ii) $\left(n = \frac{N}{1+\varrho}\ ,\ y = \frac{N\varrho}{1+\varrho}\right)$

It is straightforward to obtain, for (i), $a_{11}=\beta N$, $a_{12}=0$, $a_{21}=-\beta N$, $a_{22}=0$ which leads to the characteristic equation $\lambda^2-\beta N\lambda=0$ whose roots are $\lambda_1=0$ and $\lambda_2=\beta N$, implying that (i) is unstable.

Considering now (ii) we have instead

$$a_{11} = -\frac{N\mu}{1+\varrho}\ ;\quad a_{12} = \frac{\beta N}{1+\varrho}\ ;$$

$$a_{21} = \frac{N\mu}{1+\varrho}\ ;\quad a_{22} = -\frac{\beta N}{1+\varrho}$$

giving the following characteristic equation

$\lambda^2 + [N(\beta+\mu)\ /\ (1+\varrho)]\lambda = 0$ that solved furnishes $\lambda_1=0, \lambda_2 = -N(\beta+\mu)\ /\ (1+\varrho)$ saying that (ii) is Ljapunov stable.

Summarizing we can assess how a small perturbation around (i) will be such that the system escapes from it whereas if (ii) is left the system will always gravitate around it.

3. Generalization of Rushton-Mautner model

We take into consideration now the case of a segmented population where only one of these segments can receive the item from the source. Our aim is the generalization of a system proposed by Rushton-Mautner who also gave the solution for the situation of homogenous mixing rates between segments.

They started from a deterministic equation given by

$$(8)\quad \frac{dn_i(T)}{dT} = (N_i-n_i(T))(\alpha+\beta_i n_i(T)) + (N_i-n_i(T))\sum_{j\neq i}\gamma_{ij}n_j(T)\quad i=1,2,\dots s$$

where α and β_i have the usual meaning while γ_{ij} expresses the contact rate between knowers of the j^{th} subpopulation and ignorants of the i^{th}.

A possible extension of (8) could be

$$(9)\quad \frac{dn_1(T)}{dT} = (\alpha+\beta_1 n_1(T))(N_1-n_1(T)) - \mu_1 n_1(T) + \sum_{j=2}^{s}\gamma^{|j-1|}n_j(T)(N_1-n_1(T))$$

$$(9')\quad \frac{dn_i(T)}{dT} = \beta_i n_i(T)(N_i-n_i(T)) - \mu_i n_i(T) +$$

$$+\sum_{j\neq i}^{s} \gamma^{|j-i|}\, n_j(T)(N_i-n_i(T))$$

$$i=2,3,\ldots,s$$

The rates γ, in this context, mean that the strenght of the interaction depends on the "distance" between the subpopulations.

Solution to (9) and (9') can be derived from the general one, provided by Rushton and Mautner [6], for (8).

What instead we are going to do is the discussion of a particular case.

Let S=2, $\beta_1=\beta_2=\beta$, $\mu_1=\mu_2=0$ and $\gamma^1=\beta$, (9) and (9') then become

$$(10) \quad \frac{dn_1(T)}{dT} = (\alpha+\beta n_1(T))\,(N_1-n_1(T))$$

$$(10') \quad \frac{dn_2(T)}{dT} = \beta n(T)\,(N_2-n_2(T)) \qquad n=n_1+n_2$$

$$(10'') \quad \frac{dn(T)}{dT} = (\alpha+\beta n_1(T))(N_1-n_1(T)) + \beta n(T)[(N-N_1)-(n(T)-n_1(T))]$$

Clearly, ultimately, all the elements of the population will possess the item.

(10) can be solved to give

$$n_1(T) = [\,N_1K_0 - \alpha e^{-(\alpha+\beta N_1)T}\,] \,/\, [\,K_0 + \beta e^{-(\alpha+\beta N_1)T}\,]$$

where

$$K_0 = [\beta n_1(0)+\alpha\,] \,/\, [\,N_1-n_1(0)\,]$$

Solution to (10") is thus the following

$$n(T) = -\frac{\lambda_1-\lambda_2\,[(\lambda_1+An(0))/(\lambda_2+An(0))]\; e^{(\lambda_2-\lambda_1)T}}{A\left\{1-[(\lambda_1+An(0))/(\lambda_2+An(0))]\; e^{(\lambda_2-\lambda_1)T}\right\}}$$

where

$$\lambda_{1,2} = [(N-N_1+n_1)\beta \pm (((N-N_1+n_1)\beta)^2+4\beta(\alpha+\beta n_1)(N_1-n_1))^{1/2}\,]\,/\,2$$

and

$$A = -\beta$$

As $t\rightarrow\infty$ we have $n(\infty) = -\frac{\lambda_1}{A}$ and since $n_1(\infty)=N_1$ we get

$$n(\infty) = \frac{N\beta + N\beta}{2\beta} = N$$

This shows that in the simple epidemics ultimately everybody will know. Let us consider now the case where those inside the closed population can forget: we consequently have

$$(12) \quad \frac{dn_1(T)}{dT} = (\alpha+\beta n_1(T))(N_1-n_1(T))-\mu n_1(T)$$

$$(12') \quad \frac{dn(T)}{dT} = (\alpha+\beta n_1(T))(N_1-n_1(T)) - \mu n_1(T)+\beta n(T)(N+n_1(T)-N_1-n(T))$$

In this particular case solution to (12) was given by Bartholomew [1] namely

$$(13) \quad n_1(T) = \frac{1}{2}(N_1-\varrho-\omega) + a \left\{\frac{be^{2a\beta T}-1}{be^{2a\beta T}+1}\right\}$$

where

$$a = (N_1\omega + \frac{1}{4}(N_1-\varrho-\omega)^2)^{1/2} \text{ and}$$

$$b = [a+n_1(0) - \frac{1}{2}(N_1-\varrho-\omega)] \;/\; [a-n_1(0) + \frac{1}{2}(N_1-\varrho-\omega)]$$

As $t \to \infty$ we have

$$(13') \quad n_1(\infty) = \frac{1}{2}(N_1-\varrho-\omega)+(N_1\omega + \frac{1}{4}(N_1-\varrho-\omega)^2)^{1/2}$$

Solution to (12') is still given in [2] where now the roots are

$$\lambda_{1,2} = [(N-N_1+n_1)\beta \pm (((N-N_1+n_1)\beta)^2 + 4\beta((\alpha+\beta n_1)(N_1-n_1)-\mu n_1))^{1/2}] \;/\; 2$$

and $A = -\beta$

In (13) we can observe the well known threshold-effect, i.e. if $\omega = 0$ (obviously $n(0) > 0$) to have $n_1(\infty) > 0$ we need $N_1 > \varrho$ implying $d > 1$.

As usual $n_1(\infty) = -\frac{\lambda_1}{A}$; now we can have either

i) $N_1 = \varrho$ so $n_1(\infty) = 0$

or

ii) $N_1 > \varrho$

In the first case nobody of the closed population ultimately know the new but a portion of the entire population will know in the end; in particular this will be formed by those who don't forget.

Indeed

$n(\infty) = N-N_1$

The threshold effect has only a partial influence since part of the population hears anyway.

In the second case we have $n_1(\infty) = N_1 - \varrho$ and

$$n(\infty) = [(N-\varrho)\beta + (((N-\varrho)\beta)^2 + 4\beta\mu(N_1-\varrho) - 4\beta\mu(N_1-\varrho))^{1/2}] \;/\; 2\beta = N-\varrho$$

which is the same value obtained by Bartholomew in his model. So if $N > \varrho$ the final number of knowers does not depend any longer crucially only upon the size of the closed population but also on ϱ .

Suppose now that only those outside the closed population can forget: we then obtain

$$(14)\quad \frac{dn_1(T)}{dT} = (\alpha+\beta n_1(T))(N_1-n_1(T))$$

$$(14')\quad \frac{dn(T)}{dT} = (\alpha+\beta n_1(T))(N_1-n_1(T))+$$

$$+\ \beta n(T)\,[(N-N_1)-(n(T)-n_1(T))] \quad -\mu(n(T)-n_1(T))$$

Now we get $n_1(\infty)=N_1$ and λ_1, λ_2 to be put into the general solution are

$$(15)\quad \lambda_{1,2} = [\beta(N-N_1+n_1)-\mu] \pm \{[(N-N_1+n_1)\beta-\mu\,]^2 +$$

$$+\ 4\beta((\alpha+\beta n_1)(N_1-n_1)+\mu n_1)\}^{1/2} \quad /\ 2$$

$$A = -\beta$$

so

$$n(\infty) = [(N-\varrho)/2] \ + [(N-\varrho)^2 / 4+\varrho N_1]^{\ 1/2}$$

which depends upon both N_1 and ϱ. In this particular instance, provided that

when $\alpha=0$ is $n(0)>0$, the number of final hearers (if the knowers belong to the closed population) will be exactly the same as above.

If the initial knowers belong to the second population then $n(\infty)=(N-N_1)-\varrho$.

Finally suppose that both the populations can forget, in other terms that we have

$$(15')\quad \frac{dn_1(T)}{dT} = (\alpha+\beta n_1(T))(N_1-n_1(T))-\mu n_1(T)$$

$$(15'')\quad \frac{dn(T)}{dT} = (\alpha+\beta n_1(T))(N_1-n_1(T))-\beta n(T)(N-N_1-n(T)+n_1(T))-\mu n(T)$$

Solution to (15') is still (13) with the same $n_1(\infty)$ (see (13')) but for (15'') $\lambda_{1,2}$ to be substituted inside the general expression, are now

$$\lambda_{1,2} = [(N-N_1+n_1-\mu)\beta \pm (((N-N_1+n_1-\mu)\beta)^2 +$$

$$+\ 4\beta(\alpha+\beta n_1)(N_1-n_1))^{1/2}]\ /\ 2$$

and

$$A = -\beta$$

As far as $n(\infty)$ is concerned two cases are possible (when $\alpha=0$)

i) $N_1 = \varrho$

ii) $N_1 > \varrho$

If $N_1 = \varrho$, $n_1(\infty)=0$ and we have, if the knowers belong to n_2,

$$n(\infty) = (N-N_1-\mu)\beta + [((N-N_1-\mu)\beta)^2]^{1/2} = (N-N_1)-\mu$$

When $N_1 > \varrho$ instead it is

$$n_1(\infty) = N_1 - \varrho$$

so

$$n(\infty) = [(N-\varrho-\mu)\beta + (((N-\varrho-\mu)\beta)^2 +$$

$$+ 4\beta^2(N_1-\varrho)\mu)^{1/2}] / 2\beta$$

4. Switching population and multiple sources model

We briefly propose here a model which could represent a possible real life situation: a population of size N is informed by more than one external source broadcasting items of different nature.

The elements don't forget but only change "their minds" so that they are always active whatever is the population they belong to; the number of ignorants will thus eventually be zero.

Analysis will merely concern the deterministic approximation of the system and the local stability properties of the stationary solutions.

The system is consequently described by

$$(16) \quad \frac{dn_i(T)}{dT} = (\alpha_i + \beta_i n_i(T))(N-n_i(T)) -$$

$$- \mu_i n_i(T) + \sum_{j \neq i} \mu_j n_j(T) \qquad i=1,2,\ldots S$$

where S is the number of subpopulations.

This is a general situation where the different suffices of the parameters mean possibly different values.

Let instead consider

$$S=2, \quad \beta_i=\beta, \ \mu_1=0, \ \mu_2=\mu, \ \alpha_i=0 \qquad i=1,2$$

so that (16) is now

$$(17) \quad \frac{dn_1(T)}{dT} = \beta n_1(T)(N-n_1(T)) + \mu n_2(T)$$

$$(17') \quad \frac{dn_2(T)}{dT} = \beta n_2(T)(N-n_2(T)) - \mu n_2(T)$$

Though (16) can be solved explicitly (being a system of first order non linear differential equations with variable coefficients) we are going to explore the nature of the stationary solutions of (17), that is the roots of this system

(18) $\beta n_1(N-n_1) + \mu n_2 = 0$

(18') $\beta n_2(N-n_2) - \mu n_2 = 0$

Evidently we must have either $n_1(0) > 0$ or $n_2(0)>0$ (or both) since no outside impulse is operating.

The "physically" acceptable stationary states are the pairs (i) ($n_1=0$, $n_2=0$) and (ii) ($n_1=N$, $n_2=0$). Infact, from equation (18') we would have another solution $N-\varrho$, but it is unacceptable.

Indeed if $N>\varrho$ the first relation, having as roots

(18'') $n_1^{1,2} = [N \pm (N^2+4\varrho N-4\varrho^2)^{1/2}]/2$

will furnish solutions either bigger than N or negative, which are obviously meaningless.

On the other hand, if $N<\varrho$, from (18'') it is easy to see that putting $f(N)=N^2+4\varrho N-4\varrho^2$, with $0\leqslant N<\varrho$, we have $f(N)<0$ which would imply a complex stationary solution. The only meaningful value for N, when considering (18''), is then $N=\varrho$, which leads to the two pairs already found.

Evaluating the coefficients of the linearized system we get, for (i)

$$a_{11} = \beta N,\ a_{12} = \mu,\ a_{21} = 0,\ a_{22} = \beta N-\mu$$

and for (ii)

$$a_{11} = -\beta N,\ a_{12} = \mu,\ a_{21} = 0,\ a_{22} = \beta N-\mu$$

The characteristic equation for the two states are

(a) $\lambda^2 - (2\beta N-\mu)\lambda + \beta N(\beta N-\mu) = 0$

(b) $\lambda^2 + \mu\lambda - \beta N(\beta N-\mu) = 0$

Solutions to (a) are

$$\lambda_{1,2}^{(a)} = \begin{cases} \beta N-\mu \\ \beta N \end{cases}$$

and to (b)

$$\lambda_{1,2}^{(b)} = \begin{cases} -\beta N \\ (\beta N-\mu) \end{cases}$$

If $d>1$ then (i) is unstable, as well as if the threshold effect were operating (i.e. $d\leqslant 1$). Whereas (ii) is asymptotically stable iff. $d<1$; when equality holds (ii) is stable only in the sense of Ljapunov, while if $d>1$ is unstable.

5. *Generalization of Kermack - Mc Kendrick and Bartholomew models*

It could be interesting to generalize the two extreme and main models, with only one population, including forgetting, that appeared in the literature so far: that is the Kermack-McKendrick and the Bartholomew one.

We already said that the former assumes how once an element has lost interest in the information it will not ever come back again to the state of potential spreader, causing the epidemics to stop when the last knower forgets. The latter allows the possibility that, after forgetting, elements can come back to the state of ignorants and potential spreaders.

What we propose is a "middle way" process and the only analytical effort will be to write down the deterministic model and the equations for the mean time of absorption relating the stochastic version.

The starting hypothesis is that once we have forgotten we can either become ignorants, with probability $\vartheta\delta T$, or stop spreading forever with probability $\mu\delta T$, all the other assumptions remaining the same.

Following Bartholomew's notation the deterministic system of equations is

$$\frac{dn(T)}{dT} = (\alpha + \beta n(T))(N-n(T)) - \mu n(T) - \vartheta n(T)$$

$$\frac{dm(T)}{dT} = -(\alpha + \beta n(T))(N-n(T)) + \vartheta n(T) \qquad (19)$$

$$\frac{dl(T)}{dT} = \mu n(T)$$

where $n(T)$ is the number of spreaders at time T, $m(T)$ the number of ignorants and $l(T)$ the number of those who stopped spreading but know.

As long as transition probability functions are concerned it is enough to consider two states (n,m) for describing all the possible movements.

Indeed we have in $(T, T+\delta T)$

$$P[\,(n,m) \rightarrow (n+1,m-1)\,] = (\alpha+\beta n)\; m\; \delta T + o(\delta T)$$

$$P[\,(n,m) \rightarrow (n-1,m+1)\,] = \vartheta n\, \delta T + o(\delta T)$$

$$P[\,(n,m) \rightarrow (n-1,m)\,] = \mu n\, \delta T + o(\delta T)$$

$$P[\,(n,m) \rightarrow (n,m)\,] = \{1 - [(\alpha+\beta n)\; m + \vartheta n + \mu n]\}\, \delta T + o(\delta T)$$

The differential-difference equation for the transition probability function of the process is then

$$P'_{nm}(t) = -[(\alpha+\beta n)\; m + \vartheta n + \mu n]\; P_{n,m}(t) +$$

$$+ (\alpha + \beta(n-1))(m+1)\; P_{n-1,m+1}(t) + \vartheta(n+1) P_{n+1,m+1}(t) +$$

$$+ \mu(n+1) P_{n+1,m}(t)$$

The expected time to absorption will be given by solving

$$\pi_{n,m} = [(\alpha+\beta n) m + \vartheta n + \mu n]^{-1} +$$

$$+ \pi_{n+1,m-1} [(\alpha+\beta n) m / ((\alpha+\beta n) m + \vartheta n + \mu n)] +$$

$$+ \pi_{n-1,m} \{n\mu / ((\alpha+\beta n) m + \vartheta n + \mu n)] +$$

$$+ \pi_{n-1,m+1} [n\vartheta / ((\alpha+\beta n) m + \vartheta n + \mu n)]$$

Summary

In this paper we saw a number of generalizations for models related to epidemic theory. The work was only analytical and the effort mostly regarded the deterministic approximations of the stochastic systems.

Riassunto

In questo articolo si sono considerate alcune generalizzazioni di modelli

epidemici. Il lavoro, di carattere esclusivamente analitico, riguarda essenzialmente l'approssimazione deterministica della versione stocastica di tali modelli.

Bibliography

[1] BARTHOLOMEW D.J. (1976), "Continous Time Model with Random Duration of Interest", J.M.S., 4, 187-199.

[2] BARTHOLOMEW D.J. (1981), "Stochastic Models for Social Processes", 3th Ed. Interscience (London).

[3] DE PALMA A. (1981), "Modelles Stochastiques des Comportament Collectif dans le Systemes Complexes", Unpublished Ph. D. Thesis in Physics at the Universitè Libre de Bruxelles.

[4] KARLIN S.-TAYLOR H. (1966), "A First Course in Stochastic Processes", New Academic Press (New-York).

[5] NICOLIS G.-PRIGOGINE I. (1977), "Self-Organization in non Equilibrium Systems", Interscience (London).

[6] RUSHTON S.-MAUTNER A.J. (1955), "The Deterministic Model of a Simple Epidemic for More than One Community", Biometrika, 42, 126-132.

Elenco dei Quaderni pubblicati

n. 1 (febbraio 1979) **MASSIMO DI MATTEO**, Alcune considerazioni sui concetti di lavoro produttivo e improduttivo in Marx

n. 2 (marzo 1979) **MARIA L.RUIZ**, Mercati oligopolistici e scambi internazionali di manufatti. Alcune ipotesi e un'applicazione all'Italia

n. 3 (maggio 1979) **DOMENICO MARIO NUTI**, Le contraddizioni delle economie socialiste: una interpretazione marxista

n. 4 (giugno 1979) **ALESSANDRO VERCELLI**, Equilibrio e dinamica del sistema economico-semantica dei linguaggi formalizzati e modello keynesiano

n. 5 (settembre 1979) **A. RONCAGLIA - M. TONVERONACHI**, Monetaristi e neokeynesiani: due scuole o una?

n. 6 (dicembre 1979) **NERI SALVADORI**, Mutamento dei metodi di produzione e produzione congiunta

n. 7 (gennaio 1980) **GIUSEPPE DELLA TORRE**, La struttura del sistema finanziario italiano: considerazioni in margine ad un'indagine sull'evoluzione quantitativa nel dopoguerra (1948-1978)

n. 8 (gennaio 1980) **AGOSTINO D'ERCOLE**, Ruolo della moneta ed impostazione antiquantitativa in Marx: una nota

n. 9 (novembre 1980) **GIULIO CIFARELLI**, The natural rate of unemployment with rational expectations hypothesis. Some problems of estimation

n. 10 (dicembre 1980) **SILVANO VICARELLI**, Note su ammortamenti, rimpiazzi e tasso di crescita

n. 10 bis (aprile 1981) **LIONELLO F. PUNZO**, Does the standard system exist?

n. 11 (marzo 1982) **SANDRO GRONCHI**, A meaningful sufficient condition for the uniqueness of the internal rate of return

n. 12 (giugno 1982) **FABIO PETRI**, Some implications of money creation in a growing economy

n. 13 (settembre 1982) **RUGGERO PALADINI**, Da Cournot all'oligopolio: aspetti dei processi concorrenziali

n. 14 (ottobre 1982) **SANDRO GRONCHI**, A Generalized internal rate of return depending on the cost of capital

n. 15 (novembre 1982) **FABIO PETRI**, The Patinkin controversy revisited

n. 16 (dicembre 1982) **MARINELLA TERRASI BALESTRIERI**, La dinamica della localizzazione industriale: aspetti teorici e analisi empirica

n. 17 (gennaio 1983) **FABIO PETRI**, The connection between Say's law and the theory of the rate of interest in Ricardo

n. 18 (gennaio 1983) **GIULIO CIFARELLI**, Inflation and output in Italy: a rational expectations interpretation

n. 19 (gennaio 1983) **MASSIMO DI MATTEO**, Monetary conditions in a classical growth cycle

n. 20 (marzo 1983) **M. DI MATTEO - M.L. RUIZ**, Effetti dell'interdipendenza tra paesi produttori di petrolio e paesi industrializzati: un'analisi macrodinamica

n. 21 (marzo 1983), **ANTONIO CRISTOFARO**, La base imponibile dell'IRPEF: un'analisi empirica

n. 22 (gennaio 1984) **FLAVIO CASPRINI**, L'efficienza del mercato dei cambi. Analisi teorica e verifica empirica

n. 23 (febbraio 1984) **PIETRO PUCCINELLI**, Imprese e mercato nelle economie socialiste: due approcci alternativi

n. 24 (febbraio 1984) **BRUNO MICONI**, Potere prezzi e distribuzione in economie mercantili caratterizzate da diverse relazioni sociali

n. 25 (aprile 1984) **SANDRO GRONCHI**, On investment criteria based on the internal rate of return

n. 26 (maggio 1984) **SANDRO GRONCHI**, On Karmel's criterion for optimal truncation

n. 27 (giugno 1984) **SANDRO GRONCHI**, On truncation "theorems"

n. 28 (ottobre 1984) **LIONELLO F. PUNZO**, La matematica di Sraffa

n. 29 (dicembre 1984) **ANTONELLA STIRATI**, Women's work in economic development process

n. 30 (gennaio 1985) **GIULIO CIFARELLI**, The natural rate of unemployment and rational expectation hypotheses: some empirical tests.

n. 31 (gennaio 1985) **SIMONETTA BOTARELLI**, Alcuni aspetti della concentrazione dei redditi nel Comune di Siena

n.32 (febbraio 1985) **FOSCO GIOVANNONI**, Alcune considerazioni metodologiche sulla riforma di un sistema tributario

n. 33 (febbraio 1985) **SIMONETTA BOTARELLI**, Ineguaglianza dei redditi personali a livello comunale

n. 34 (marzo 1985) **IAN STEEDMAN**, Produced inputs and tax incidence theory

n. 35 (aprile 1985) **RICHARD GOODWIN**, Prelude to a reconstruction of economic theory. A critique of Sraffa

n. 36 (aprile 1985) **MICHIO MORISHIMA**, Classical, neoclassical and Keynesian in the Leontief world

n. 37 (aprile 1985) **SECONDO TARDITI**, Analisi delle politiche settoriali: prezzi e redditi nel settore agroalimentare

n. 38 (maggio 1985) **PIETRO BOD**, Sui punti fissi di applicazioni isotoniche.

n. 39 (giugno 1985) **STEFANO VANNUCCI**, Schemi di gioco simmetrici e stabili e teoremi di possibilità per scelte collettive.

n. 40 (luglio 1985) **RICHARD GOODWIN**, The use of gradient dynamics in linear general disequilibrium theory.

n. 41 (agosto 1985) **M. MORISHIMA** and **T. SAWA**, Expectations and the life span of the regime.

n. 42 (settembre 1985) **ALESSANDRO VERCELLI**, Keynes, Schumpeter, Marx and the structural instability of capitalism.

n. 43 (ottobre 1985) **ALESSANDRO VERCELLI**, Money and production in Schumpeter and Keynes: two dichotomies

n. 44 (novembre 1985) **MARCO LONZI**, Aspetti matematici nella ricerca di condizioni di unicità per il Tasso Interno di Rendimento.

n. 45 (dicembre 1985) **NIKOLAUS K.A. LAUFER**, Theoretical foundations of a new money supply hypothesis for the FRG.

n. 46 (gennaio 1986) **ENRICO ZAGHINI**, Una dimostrazione alternativa dell'esistenza di equilibri in un modello di accumulazione pura

n. 47 (febbraio 1986) **ALESSANDRO VERCELLI**, Structural stability and the epistemology of change: a critical appraisal

n. 48 (marzo 1986) **JOHN HICKS**, Rational behaviour: observation or assumption?

n. 49 (aprile 1986) **DOMENICO MARIO NUTI**, Merger conditions and the measurement of disequilibrium in labour-managed economies

n. 50 (maggio 1986) **SUSAN SENIOR NELLO**, Un'applicazione della Public Choice Theory alla questione della riforma della Politica Agricola Comune della CEE.

n. 51 (maggio 1986) **SERENA SORDI**, Some notes on the second version of Kalecki's Business Cycle Theory

n. 52 (giugno 1986) **PIERANGELO GAREGNANI**, The classical theory of wage and the role of demand schedules in the determination of relative prices

n. 53 (luglio 1986) **FRANCO BIRARDI**, Crescita, sviluppo ed evoluzione: l'affermazione dei nessi strutturali sistematici e la combinazione di scienza, tecnologia ed economia nelle prime industrializzazioni

n. 54 (agosto 1986) **M. AMENDOLA - J.L. GAFFARD**, Innovation as creation of technology: a sequential model

n. 55 (settembre 1986) **GIULIO CIFARELLI**, Exchange rates, market efficiency and "news". Model specification and econometric identification

n. 56 (settembre 1986) **ALESSANDRO VERCELLI**, Technological flexibility, financial fragility and the recent revival of Schumpeterian entrepreneurship

n. 57 (ottobre 1986) **FABIO PETRI**, The meaning of the Cambridge controversies on capital theory: an attempt at communication

n. 58 (ottobre 1986) **DANIELE CHECCHI**, The international coordination of counter-cyclical policies

n. 59 (novembre 1986) **ALESSANDRO VERCELLI**, Probabilistic causality and economic models: Suppes, Keynes, Granger

n. 60 (dicembre 1986) **LIONELLO F. PUNZO**, J. Von Neumann and K. Menger's mathematical colloquium

n. 61 (dicembre 1986) **MICHAEL BACHARACH**, Three scenarios for fulfilled expectations

n. 62 (gennaio 1987) **J. ESTEBAN** and **T. MILLAN**, Competitive equilibria and the core of overlapping generations economies

n. 63 (gennaio 1987) **ROBERT W. CLOWER**, New directions for keynesian economics

n. 64 (febbraio 1987) **MARCO P. TUCCI**, A simple introduction to flexible functional forms and consumer behaviour theory