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Financial Contagion in Industrial Clusters: A Dynamical Analysis and Network Simulation

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Abstract - In this work we analyse the resilience of industrial districts to exogenous economic shocks. Firstly, we define a basic industrial district through a set of assumptions which prove to be critical for systemic risk in the event of a financial shock. In the course of the work we progressively relax the assumptions to make room for more complex representations. Consequently, depending on two dimensions of complexity (structure of economic interactions and the degree of heterogeneity of the industrial population), we develop three different models of industrial clusters, employing non-linear ordinary differential equations and percolation dynamics in graph theory. A mechanism of financial contagion is introduced and a threshold condition is derived in order to study each model's resilience. Eventually, we prove that it is the structure of economic interactions which produces a structural change in the threshold characterization.

Keywords – diffusion, networks, industrial clusters, crisis

JEL Classification – C02, D85, G01, H12, L14, 033

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I. Introduction

In recent years, dynamical processes related to economic agglomeration have been subject to intensive research [6]. Generally, scholars addressed their interest in understanding the patterns of agglomeration and consequently focused on the process of "market entry" in order to explain profitability of agglomeration. Specifically, positive returns engendered by agglomeration may be due either to the relation between the *intrinsic* geographical benefits of a market and the number of firms already active in it [1], [6] or to the diffusion of business information within a pre-existent economic cluster [7].

In this work we maintain the focus on diffusion processes which take place in pre-existent economic districts, yet with a different goal. Our purpose is to investigate the systemic effects of an exogenous economic shock which may affect a community of firms organized by means of an industrial district. Specifically, we aim to find the analytical conditions which determine the ultimate economic outcomes of an initial shock. However, as argued in chapter II, the industrial district is a complex and non-univocal form of economic organization subject to historical and geographical constraints, hence, we firstly reduce the multiplicity involved with actual districts to a tractable set of features which prove to be determining for the spread of a financial crisis in the event of an exogenous economic shock[1]. Subsequently, we implement in the remaining dimensions a characterization for the financial shock diffusion. In this regard, the dimensions we ended up with are the *structure* of economic interactions, which may be occasional or long-term specific, and the level of heterogeneity in specialization of the industrial population, which can include one homogeneous population or multiple groups. The dynamical features of the resulting models are then analytically studied in order to inspect the *systemic risk* of the economic clusters and determine their maximal tolerance to exogenous shocks. Hence, we derive a *threshold condition* which separates the two possible general outcomes of the initial shock, that is, whether it bursts forth in the district through a *default cascade* or it remains contained within the economic agents who directly suffered the losses of the initial shock.

Eventually, we demonstrate that different structures of relationships bring very diverse results in the resilience of the districts. Technically speaking, we ground the model of the

[1] Consider for instance the earthquakes which shook between the 20-29 of May 2012 the highly industrialized area of Emilia-Romagna (Italy)[28], where 37.9% of workers (Osservatorio Nazionale Distretti Italiani, 2012) are employed in tight networks of small and medium enterprises which are directly involved into the leading national export sectors (agroindustry, biomedical, precision machining and textile)

second chapter on the analysis of *S-I-R* (Susceptible, Infected, Removed) model studied by [20], which we carefully analyse with simulations and a Lyapunov function which we derive to ascertain its stability, explicating the advantages and costs of this modelling approach. The third chapter is all about diffusion in models with a graph structure. With regards to the latter, we implemented in the methodology of [13] a diffusion process both for a single and a composite layer of relationships. Our implementation allowed us to consider the bond percolation problem with the consequent critical findings of [22], which we adapted to an economic environment.

II. Financial shock and contagion in industrial districts relations

The original *S-I-R* (Susceptible, Infected, Removed) model is focused on deriving the dynamics of inter-relations within a generic and wide population composed by three homogeneous classes of individuals, and it is aimed to keep track of the overall composition of the aggregate population within the progression of a pandemic disease. Although the origin of the model traces back to the seminal work of [14], it is still considered a workhorse for its capability to provide a stylized yet realistic description of generalized contagions and diffusion processes. In fact, in the lapse of sixty years, the basic model characterized by a single homogeneous population had been widely expanded in order to account for various heterogeneities in the attempt to provide a realistic explanation of complex diffusion processes. However, no matter how intricate we construct a deterministic *SIR* model, it is still constrained to a set of onerous assumptions concerning the kind of relations agents will undertake. In fact, as it has been explicitly recognized by mathematic-epidemiologists and statistical physicists[2], the very possibility to construct a set of nonlinear differential equations in the traditional deterministic *SIR* models requires the model population(s) to be "fully mixed", that is, an individual in any group is equally likely to spread the infection to any other member of the population or the subpopulations [22].

This incorporates the two following strict assumptions:

1. Every member of each (sub)population may be in contact with any other individual belonging to the others (sub)populations with equal probability. In our case this means that in any instant each company has *approximately* the same number of business *contacts*.

[2] [22] concisely enucleates the issue. See[9] for a detailed exposition and a comprehensive literature review.

2. Each financial contact between an impaired company and a healthy one may endanger (i.e. infection ensues) the latter with a fixed probability which depends on the healthy firm group's resilience to infection.

Assumptions (1) and (2) are such that traditional models make sense in the context of large populations, where we expect to obtain some degree of homogeneity within the groups. Chapter III is entirely focused on relaxing both these demanding assumptions: there we implement in our models a graph structure in which the financial contagion may propagate only through the fixed layer of given *relationships* (and not just *contacts*) between *specific* agents[3]. However, conditional to these two assumptions**,** I am intended to point out the fruitful outcomes of the traditional modeling approach in respect to three general goals:

1. Characterize in analytical terms a process of diffusion and the ultimate outcomes caused by an exogenous shock which affects a part of the population of a given economic environment
2. Determine the possible existence of a *threshold* which engenders a *qualitative* transformation of the diffusion process
3. Provide a metric to test the *resilience* of different economic *sets* to a given exogenous shock [consequent to (1) and (2)]

2.1. Defining industrial districts within a set of assumptions

When Charles Perrow in its 1993 celebrated paper discussed the economic organization of Small Firm districts[4], he accurately wrote:

> Imagine breaking up the integrated firm into units whose average number of employees is ten each. For example, instead of 2,000 employees in one firm, there would be 200 firms of 10 employees each. […] The firms are usually very small – say ten people. They interact with each-other, sharing information, equipment, personnel, orders, even as they compete with one another. They are supplied by a smaller number of […] financial service firms. […] Three things help account for the success of Small Firm Networks: economies of scale through networks (still insufficiently theorized),

[3] In the sense that every agent is endowed with a determined number of *specific* business relations such that it may have diverse degree of exposure depending on its partners.

[4] He actually defines "networks" this specific form of productive organization. However, due to the technical implications this word has in our discussion, we leave it for chapter third, where we actually deal with mathematical graph structures

> trust and cooperation coexisting with competition and welfare effects that increase the efficiency of the region. (Perrow, 1993)

The description is coherent with the standard definition produced by Giacomo Becattini,

> I define the industrial district as a socioterritorial entity which is characterized by the active presence of both a community of people and a population of firms in one naturally and historically bounded area. In the district – and unlike in other environments, such as the manufacturing town – the community and the firms tend, as it were, to merge (Becattini, 2004)

We provide a first approximation of a homogeneous horizontal industrial district by means of a set of three main assumptions synthetized by [24] and [16] which characterizes the production system:

1) the social cohesion
2) the homogeneity of skills engendered by the easiness of various local spillovers (information, equipment, personnel and even orders) and complete transferability
3) Uniformity of interactions among producers

Although virtually no author questions assumption (1) in dealing with industrial districts different from the horizontal homogeneous one[5], assumptions (2) or (3) prove to be too restrictive for a broader framework. Especially, things get dramatically more complicated with relaxation of Assumption (3). In the one hand, uniform (i.e. deterministic) interaction is useful to approximate an important feature of several horizontal (or even vertical) small firm networks, and is partially coherent with "*Long term relationships, but possibly quite intermittent contacts*" [24], on the other hand it just tells a part of the story. In Chapter 3 we will drop point (3) allowing for an evident discrimination between *contacts* and *solid relationships* by means of a model based on graph theory. Concerning (2), it has been widely recognized [25] that *vertical* and mature industrial districts are characterized by a composite physiognomy of non-transferable skills. We will relax this assumption in the second part of the third chapter taking into account a first degree of specialization introducing in the district a well-developed financial system.

[5] Even within a critical standpoint with regards to the Marshallian "standard" model of district, [16] endogenizes Marshallian cooperation and internal cohesion as a central peculiarity for most of the district typologies she identifies.

In the current model, stringent social ties which engender uniformity are justified by several functional reasons. According to [4] one of the leading reasons is the capability to hedge producers against uncertainty[6]: agents who are affected by sudden individual shocks (e.g technical failures, bad investments) may get temporal relief from other firms, postponing or even deflecting the default by means of temporary liabilities easing or asking for favorable loans. However, considered the usual limited capitalization of business in industrial districts, this relief may constitute a significant cost to the enterprises which grant their help. In fact, healthy firms may get impaired by others' defaults, leading other firms or even the district at large to a systemic crisis or even extinction. It is then the objective of the current model to provide some insights about the dynamics which characterize a homogeneous industrial district in the event of an exogenous economic shock. Consequently, the identification of the critical factors which are responsible for the district's economic resilience may contribute to define an optimal size for the district, thus explaining why such a kind of industrial organization is subject to spatial bounds which prevent the diffusion of a single district even in the absence of geographical or political constraints.

2.2 Characterization of a Small-Firm homogeneous horizontal district

Assumptions (1), (2) and (3) derived in 1.1.2 imply that we take no constraining specificity for firms belonging to horizontal districts. This, along with the extreme fragmentation of production processes and the easiness of spillovers leads to perfect substitutability for economic collaborations. Consequently, a high degree of homogeneity rules in the district. It seems reasonable that this thick layer of horizontal relations operates also in the event of dramatic shocks: a firm which suffered *irreversible* losses may rely on any other firm which belongs to the district. Although this relief may temporarily help the impaired firm, in the absence of exogenous (e.g. institutional) help it will eventually default given our current definition of financial infection (irreversible losses). Moreover, the default may impair the firm's creditors leading to a secondary infection and so on. Hence, the number of competitive firms decreases proportionally to the thickness of the district industrial layer. Although the capability to tolerate a small chronic number of firms which suffer exogenous shocks defined by $I_0 \subseteq N/\{0\}$ is implicit to the very existence of the district, a more precise depiction of the dynamics is required to evaluate the drawbacks of a given shock in the set N of businesses. In fact, it may be the case that in given environments, particularly severe crashes could lead the

[6] As they point out, "Organizations participate in networks to reduce uncertainty, while the behavioural environment is changing rapidly. Networking can be seen as "relational contracting"." (p.172)

entire district to crisis or even rapid extinction. Hence, we define a parameter $\beta > 0$ which provides a metric for measuring the rate of defaulted (removed) firms.

Thus, we consider a horizontal industrial district with a large but finite population of $N = \{i, \dots, n\}$ small price taker firms which interact among each other concurring to the supply of a common *niche* good. We define a *district* $\mathbf{C}(\mathrm{t}) = (S(t), I(t), R(t), t)$ the following set of non-linear differential equations

$$\frac{dS}{dt} = \dot{S} = -\alpha S(t)I(t) \tag{1.1}$$

$$\frac{dI}{dt} = \dot{I} = \alpha S(t)I(t) - \beta I(t) \qquad \mathrm{t} \in R_{+} \tag{1.2}$$

$$\frac{dR}{dt} = \dot{R} = \beta I(t) \tag{1.3}$$

$S \subseteq N$ represents the number of healthy firms in the district which interact with $I \subseteq N$ firms that had been directly or indirectly impaired by the exogenous shock. With no defaults (i.e $\beta = 0$), the number of impaired firms would grow at $\dot{I} = -\dot{S}$. However, the default parameter acts in limiting the diffusion of the defaults at a rate proportional to the number of impaired firms. Summing all together we get that the aggregate dynamic reveals to be conservative with respect to time. The sum implies the upper constraint:

$$S(t) + I(t) + R(t) = constant = N \tag{1.5}$$

Hence, given the α and β parameters and the vector $\mathbf{V_0} \in R^3$ of the initial conditions:

$$R(0) = 0, \qquad S(0) = S_0 > 0, \qquad I(0) = I_0 > 0 \tag{1.6}$$

We define a *district environment* $E = (\alpha, \beta, \boldsymbol{V_0})$.

2.3 The Qualitative behaviour of homogeneous industrial districts

In order to study E, we extend the analysis focusing on the qualitative behaviour implied by (1.1) and (1.2) to sketch the first particularity of this non-linear dynamic system. Hence, we linearize[7] the system of equations and we obtain the Jacobian Matrix in which the two rows are the gradients $\nabla\dot{\mathrm{I}}(\boldsymbol{X}_0)$ and $\nabla\dot{\mathrm{S}}(\boldsymbol{X}_0)$.

$$\boldsymbol{J}|_{(S*,P*)} = \begin{Vmatrix} -\alpha I & -\alpha S \\ \alpha I & -\alpha S - \beta \end{Vmatrix} \tag{1.7}$$

It follows immediately that whatever s^* the solution $I^* = 0$ causes an interesting singularity in $\boldsymbol{J}$ (the first column is $\mathbf{0}$). Singularity implies that one eigenvalue is zero; hence the entire dynamics will be "stretched" around the eigendirection defined by the other one. Through standard procedures, we find that $\lambda_1 = 0$ and $\lambda_2 = -\alpha S - \beta = \tau(J)$, which, due to non-negative conditions of (1.6), it implies that: $\lambda_2 < 0$. Consequently, the solution of the *linearized* system is stable. The solution $I^* = 0$ is attractive independent of S. Hence, $I^* = 0$ is a line of singularities and $I(\infty) = 0$. However, the stability which we ascertained from the linearized system is a necessary but insufficient condition: in fact, in the presence of $\lambda_1 = 0$ (i.e. an eigenvalue with zero real part), the *Hartman–Grobman theorem* does not apply. That is to say that we should rely on other instruments to ascertain the stability of the actual non-linear system. I offer a proof of local asymptotic stability in the domain defined by the N firms by means of a *Lyapunov Function* $L: \mathbb{R}^2 \to \mathbb{R}$. The L function is such that

$$L(S, I) > 0 \quad \forall (I, S) \in N\backslash\{0\}$$
$$\frac{dL}{dt}(S, I) < 0 \; \forall (I, S) \in N\backslash\{0\}$$

The existence of such a function is a necessary and sufficient condition to ascertain the system *local* stability. Thus, we construct the following equation:

$$L = \varphi S^2 + I^2$$

Which is a simple quadratic equation (with a parameter φ) which satisfies the first requirement. We develop $\dot{L}$ in order to ascertain the feasibility of the second requirement

$$\frac{dL}{dt}(S, I) = 2\varphi \dot{S}S + 2I\dot{I} = 2\varphi S(-\alpha SI) + 2I(\alpha SI - \beta I)$$
$$= 2(\alpha SI(I - \varphi S) - \beta I^2)$$

Now, notice that both $S(t)$ and $I(t)$ must be non-negative and constrained by (1.5). Further, we defined $\beta > 0$. Hence, the second part of the equation is negative. This is also the case for the first part if $I - \varphi S < 0$. Thus, it suffices to impose $\varphi = N$ to satisfy both the requirements. Further, when $I = I^* = 0 \to \frac{dL}{dt}(S, I) = 0$ and the system is in equilibrium. ■

[7] For Linear stability Analysis see, for instance,[26] or [10]

2.4 The contagion cascade: definition and threshold derivation

Given C, we can shed some light on the diffusion process triggered by the initial shock which affected the I_0 firms. In order to do so, let's focus on the behavior of (1.2) at time $t = 0$

$$\left.\frac{dI}{dt}\right|_{t=0} = I_0(\alpha S_0 - \beta) \tag{1.8}$$

It is fair to notice that financial contagion may or not spread across the agents depending on ***E***. That is:

$$\begin{cases} \left.\dot{I}\right|_{t=0} > 0 \; if \; S_0 > \dfrac{\beta}{\alpha} \\ \left.\dot{I}\right|_{t=0} < 0 \; if \; S_0 < \dfrac{\beta}{\alpha} \end{cases} \tag{1.9}$$

Given our specification of (1.1), it is $\dot{I} \leq 0$. Hence, in the case of a shock that affect some agent (causing I_0 to be positive), $(t) < S_0 \, \forall \, t \in R_+$. This relation is relevant because it allows inferring from $\left.\dot{I}\right|_{t=0}$ the dynamics of $\dot{I} \, \forall \, t \in R_+$. If $S_0 < \frac{\beta}{\alpha}$:

$$\frac{dI}{dt} = I_t(\alpha S_t - \beta) \leq 0 \quad \forall \, t \in R^+ \tag{1.10}$$

Hence, $I_0 > I_t \to 0$ for t $\to \infty$. The converse applies for some time interval if $S_0 > \frac{\beta}{\alpha} \to I(t) > 0$ increases for some initial time span leading to contagion. Hence, we define

$$\frac{\beta}{\alpha} = \rho = S_c$$

the *contagion cascade threshold*. It offers a measure for the tolerance of **C**. Now it is relevant for the subsequent analysis to develop a metric[8] for the secondary contagions generated by the primary contagion. Consider the *transmission rate:*

$$R_0 = \frac{\alpha S_0}{\beta} = \frac{S_0}{\rho} \tag{1.11}$$

[8] See [20] which refers to Diekman et al. (1990) for thresholds in a heterogeneous population

It is easy to see that $R_0 > 1$ implies $\alpha S_0 > \beta$; that is, the first *indirect* wave of contagions overtakes the default rate. Hence, we are back to the conditions derived in (1.9)

2.5 Financial Contagion dynamics

Definition 1.1 I define the set of the all the possible firms in the district which will experience irreversible losses as $\boldsymbol{I} = \{(n, t) \colon \forall\, t\ \in R_+\ n \in [0, N]\ \}$

Definition 1.2 I define a *contagion outbreak* as the set $\boldsymbol{O} = \{\forall (t, n) \in R_+ x\ I :\ \ I(t) > I(0)\}$.

Apart from the threshold we found in section 1.1.2, the McKendrick model offers other interesting insights, which we can derive from the following analytical results. First, we construct a differential equation which defines the variation of the precarious firms relative to the changes in firms' population on which these businesses may rely for help. Dividing (1.1) by (1.2) we obtain:

$$\frac{dI}{dS}\frac{dt}{dt} = -\frac{I(\alpha S - \beta)}{\alpha S I} = \frac{\rho}{S} - 1 \tag{1.12}$$

The dynamics of which we can summarize in the following:

$$\frac{dI}{dS} = \begin{cases} > 0\ if\ S <\ \rho \\ < 0\ if\ S > \rho \end{cases} \tag{1.13}$$

From (1.13) I notice the existence of a maximum for the single argument real function $f(S) = \frac{dI}{dS}$ which implies non-trivial behaviour. Setting $\frac{dI}{dS} = 0$ we consequently obtain that:

$$\max_s\ f(s) = \max_s \left(\frac{\rho}{S} - 1\right) \text{ produces }\ S^* = \rho \tag{1.14}$$

Hence, I define:

$$\forall\, t, S_0 \in (R_+ x\ N) \colon\ \rho\ \ \leq S_0 \leq (N-1), \exists\, I(t) \in \boldsymbol{I}\ s.t.\ \{I(t) > I(0)\}\ \neq \{\emptyset\} \tag{1.15}$$

Which equals to say that the contagion outbreak set $\boldsymbol{O}(t)$ is non-empty. It is then worth to derive explicit solutions of (1.12) through separation of variables:

$$\int \frac{dI}{dS} dS = \int (\frac{\rho}{S} - 1)\, dS + K \text{ hence} \int dI = -\int dS + \rho \int \frac{1}{S} dS + K$$

The analytic result immediately follows:

$$I(t) + S(t) - \rho \ln S(t) = \text{constant} \tag{1.16}$$

Where we employ the initial conditions for the constant: $K = S_0 + I_0 - \rho \ln S_0$.

We gathered all the elements necessary to determine the financial contagion outbreak. In order to do so, we rearrange (1.15) plugging into (1.13). That gives

$$I(t) = N - \rho + \rho \ln\left(\frac{\rho}{S_0}\right) \tag{1.17}$$

Secondarily, differentiating S on R (i.e.: dividing (1.1) by (1.3)) we bring in place the analytical solution for the firms susceptible of financial contagion, that is:

$$\frac{dS}{dR} = -\frac{S}{\rho} \tag{1.18}$$

Again, separating the variables:

$$S(t) = S_0 e^{-\frac{R(t)}{\rho}} > S_0 e^{-\frac{N}{\rho}} > 0 \tag{1.19}$$

The asymptotic behaviour of the system is then bounded by (1.5):

$$0 < S(\infty) < N \; as \; \mathrm{t} \to \infty \tag{1.20}$$

Combining (1.20) with the resulting dynamics derived in (1.5) for the set of solutions $(\mathrm{I}^*, \mathrm{S})\ \forall\ \mathrm{S} \in (0, N]$, we obtain the number of total defaulted firms after the diffusion triggered by the initial chock: $R(\infty) = N - S(\infty)$

2.6 Simulation of two homogeneous District Environments

It may be useful to have a graphical perspective of the behaviour of a composite class of district environments $E_i \epsilon (\alpha, \beta, \boldsymbol{V_0})$. To understand the mechanics behind the different scenarios and observe the diffusion of financial contagion and inspect the possibility for default-cascades I compiled with Mathematica^{TM} a basic SIR model. Both the districts C_1, C_2 are defined on N = 418 small. They both undergo an initial shock which directly affects a fixed number of $I_0 = 24$ factories. For the sake of realism, I will consider a period of three years observed in months (i.e. 36 periods). The parameters are $\beta = 1$ and $\alpha = 0.000574$. Imposing $\beta = 1$ we are saying that with no diffusion or default cascades, $I(t) \to 0$ as $t \to 6$ months. As it emerges in Figure 2, the initial shock does not produce a default-cascade. This follows from the fact that the tolerance offered by the contagion-cascade threshold in the actual context is equal to $\rho_1 = 1754$. Now consider a second example of district environment. Everything equal, let us consider a district which much lower resilience than C_1. That is, district C_2 is calibrated with $\alpha_2 = 4{,}5x10^{-3}$ Hence, the tolerance is lower than S_0, that is, $\rho_2 = 218$ and the infection ensues with dramatic effects, as notable in Figure 3.

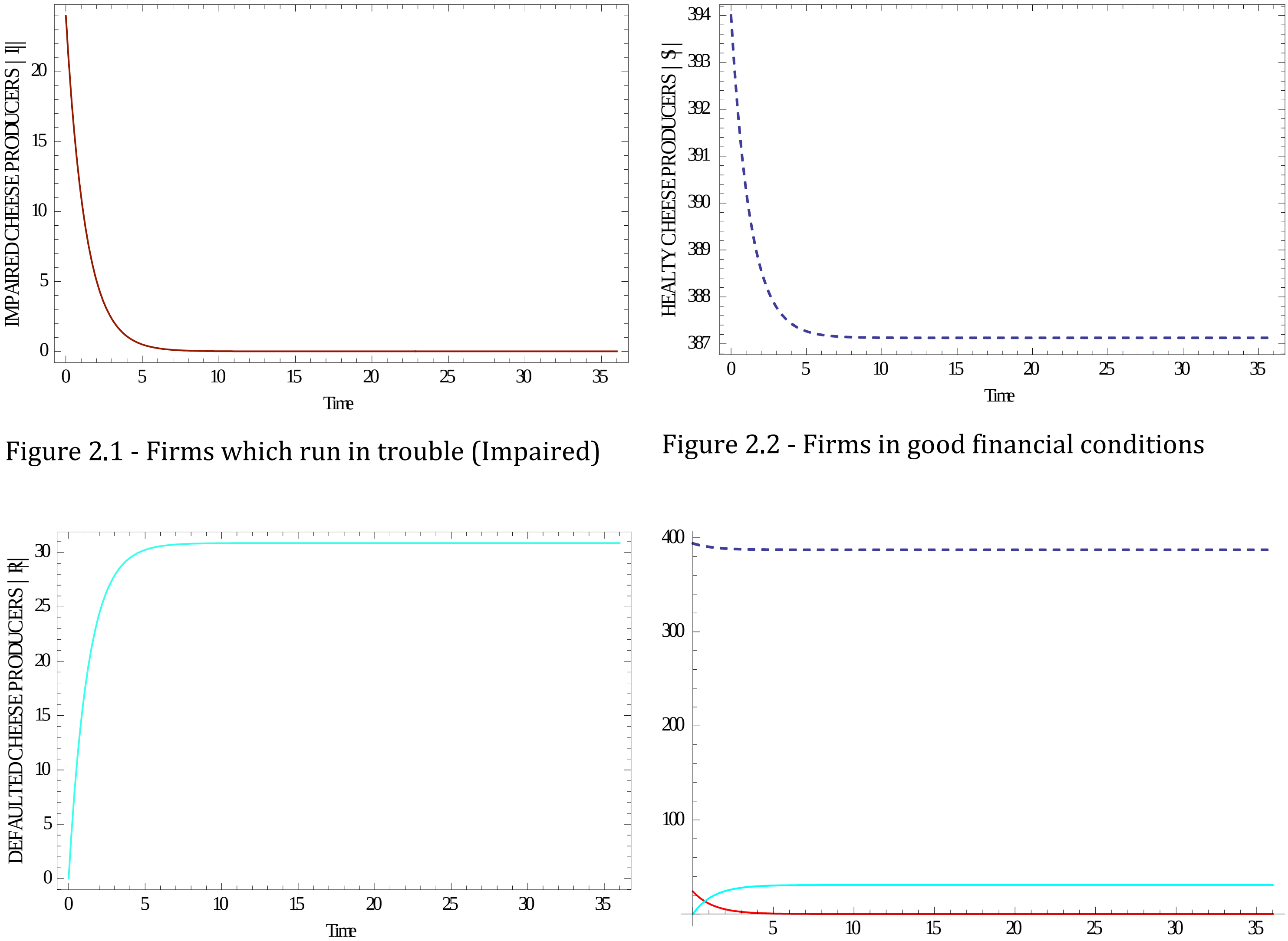

Figure 2.1 - Firms which run in trouble (Impaired)

Figure 2.2 - Firms in good financial conditions

Figure 2.3 - Firms which incurred in Default

Figure 2.4 – The General Picture

Figure 2 – The District Environment $E_1 = \{\alpha_1 = 5{,}74x10^{-4}, \beta = 1,\ S_0 = 394,\ I_0 = 24\}$

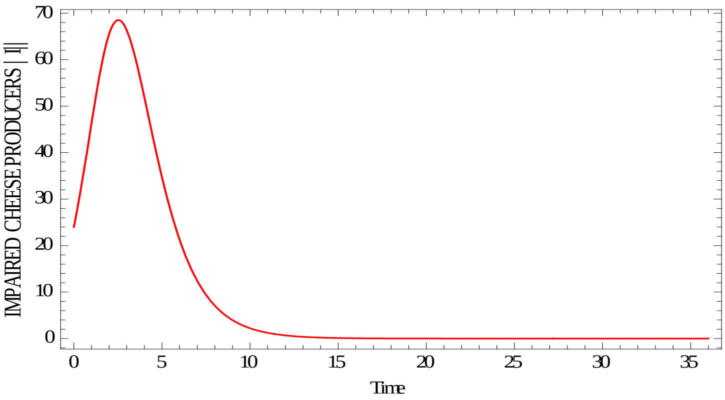

Figure 3.1 - Firms which run in trouble (Infected)

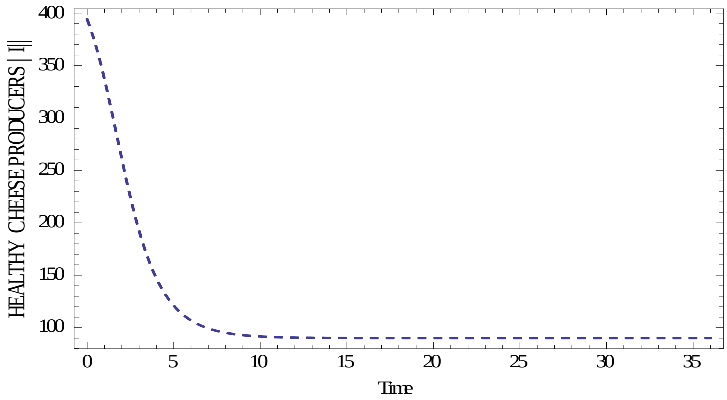

Figure 3.2 - Firms in good financial conditions

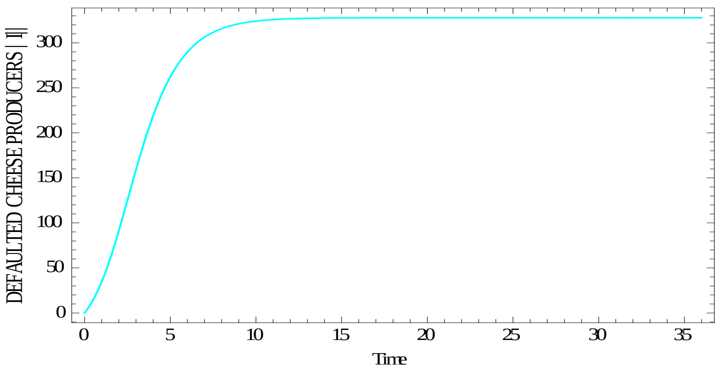

Figure 3.3 - Firms which incurred in Default

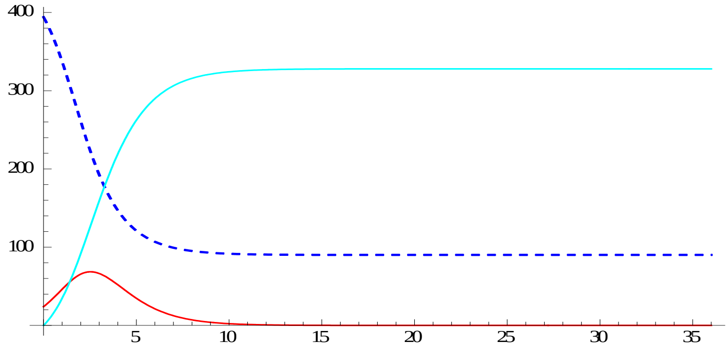

Figure 3.4 – The General Picture

Figure 3 – The District Environment $E_2 = \{\alpha_2 = 4{,}5x10^{-3}, \beta = 1,\ S_0 = 394,\ I_0 = 24\}$

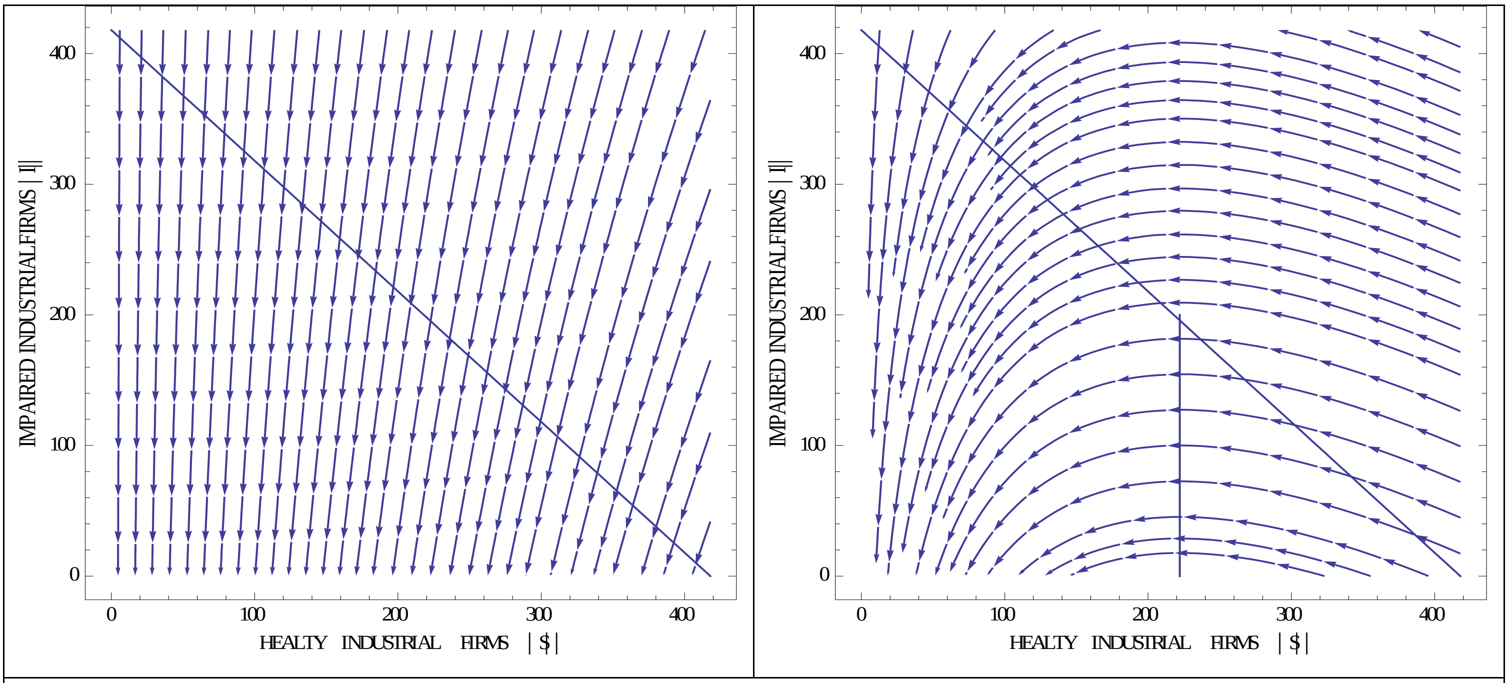

Figure 4 - Phase Portraits for C_1 and C_2

We can summarize the critical elements of the basic industrial district model through the comparison of the two environments' phase portraits. The condition (1.5) engenders the initial condition $S_0 + I_0 = N$. In fact, whatever point $x_0 = (S_0, I_0)$ we will start from, it will be bounded to the N line. Now notice the difference between environments C_1 and C_2. With respect to the former, the tolerance ρ_1 is far from being put under stress by S_0. Hence, from (1.13) and (1.19) it follows that the shock impact is contained into the district and no peak is observable in the default dynamics. The maximum number of firms in trouble in a given moment is I(0). That is to say, every trajectory which origins in the line N of Figure 4.1 is monotonically attracted by the $I = 0$ solution. Conversely, in Figure 4.2 the cascade threshold ρ_2 is exactly equal to 218. Hence, given the set of the all the possible firms which may experience irreversible losses $\boldsymbol{I}$ and N$-I_0 > \rho_2$, condition (1.15) applies and the set $\boldsymbol{O}$ *(*Definition 1.2) is non-empty.

III. Financial shocks and contagion in industrial network relationships – a stochastic approach

So far, one of the leading assumptions which we relied on in dealing with financial infection in industrial clusters was *homogeneity* of *contacts* among the economic actors of our cluster. While this may fairly proxy many features of reality in a stylized fashion, it fails in carving the underlining *structure* of fundamental interactions for economic agents. In fact, the "fully-mixed" assumption which rules in deterministic equations assumes that economic agents are roughly exposed *at random* to an equal number of business *contacts* in any instant of time ([22], [20]). Further, the business district is assumed to be thickly interconnected in a way such that any agent is able to potentially cooperate with any other *directly*. While this may be the case in horizontal districts or districts where specificity of investments is low and spillovers are eased by a relatively limited degree of specialization[9], the definition of an underling structure becomes a priority in composite districts in which the externalization of functions[10] produces a layer of complex *solid* business relationships wherein technical spillovers funnel through [6]. Consequently, as [2] stances:

> The Marshallian 'industrial district' (see Bellandi 1989b) is a localized 'thickening' (and its strengths and weaknesses both lie in its spatial limitation) of inter-industrial

[9] [12] discuss and provide empirical findings related to the evolution of industrial districts in Italy and Taiwan. [6] identify the category of Horizontally Diversified Agglomerations

[10] By means of a detailed empirical investigation, [25] proves that technical development and specialization in industrial districts result in a process of "*vertical disintegration*". Subcontracting is one of the possible routes to expansion, but definitively not the unique one [6].

relationships which is reasonably stable over time. Its composite nature, tending toward the multisectorial, gives it, even in the midst of intense change, a stability which a unit such as a single industry, in the technological sense of the term, lacks (p. 16)

And, in this precise framework, [24] concisely points out that small firm industrial networks firstly (but not exclusively) represent an *organizational* answer to the production process, hence the need to study the solid layer which underlies under any apparently casual transaction. These remarks lead us to a simple but crucial observation: the implementation of an underling graph *structure* in the next models, far from being a mere sophistication, is consequent to the very concept of industrial district. In this very light we eventually introduce the last degree of complexity, building a network model with two kinds of economic actors.

3.1. The Industrial Cluster through the graph theory

As anticipated, the models which we are going to present in this section are developed within the bounds of *graph theory*. Following [15], [21] and [13] I define a network $(N, \boldsymbol{G})$ as a finite set of $N = \{1, \dots, i, \dots, n\}$ nodes (agents) and $G \subset N\text{x}N$ edges (links) described by means of *Adjacency* matrix $\boldsymbol{G}_{\boldsymbol{nxn}} \in \{0,1\}^n$, such that

$$if\ \exists(i,j) \in N\text{x}N\ s.t\ (i,j) \in G\ \rightarrow g(i,j) = g_{ij} \neq 0$$

Hence, the Adjacency matrix describes the set of relations of our network. Two agents are said to be neighbours if a relationship between them exists ($g_{ij} + g_{ij} \neq 0$). For the sake of the goal of our models (i.e. explain the contagion dynamics), we apply two important but non-strictly necessary restrictions to the **G** matrix:

1. $\boldsymbol{G}_{\boldsymbol{nxn}}$ is symmetric, and consequently, the graph is *undirected*
2. $g_{ij} = \begin{cases} 1\ if\ (i,j) \neq 0 \\ 0\ otherwise \end{cases}$ that means that the relations are *non-weighted*

While these restrictions appear to be compelling in an economic context where *relative* liabilities may exactly be encapsulated by means of the adjacency matrix, they are useful to provide a straightforward illustration of the contagion dynamics. We define a *component* the set of vertices that can be reached from any element of the component by paths running along edges of the graph [21]. Hence, a component is a non-empty sub-network (N', $\boldsymbol{g}'$) such that $N' \subset N/\{\emptyset\}, \boldsymbol{g}' \subset \boldsymbol{g}$. As we will demonstrate, understanding the size of a component relative to the overall dimension of a network is crucial in our context. Finally, the *degree* d_i of a node *i* is such that $d_i = \sum_{j=1}^{\infty} g_{ij}$

3.2. Stochastic Graphs modelling with generating functions

Firstly, we define a random graph as a network in which the agents' connections are described by a probabilistic rule given by a probability distribution $P(d)$ of the degrees *d,* that is, of the number of connections each agent have. After having defined the basic concepts on which I develop the model, it is fundamental to spend some words justifying the sense of random graph employment. As random graphs are defined by means of processes which are known *ex-ante,* it is possible to contrast existent forms of networks (such actual industrial districts) with the proprieties distilled from different degree distributions. However, as noticed in [13], stochastic graph modelling plays a major role especially when dealing with complex diffusion processes which heavily depend on the structure of the graph (i.e. the actual distribution of edges among agents). These models identify general rules (boundary conditions) and approximate the edge structure of actual networks within a set of known proprieties. In the following models we implemented *fixed* graphs, in which the structure of relations does not change in time during the infection. In order to study the resilience of an industrial cluster in a generalized network context (i.e. without *a priori* definition of the degree distribution), we will rely on the analysis of the components structure. Hence, we follow [8] and [22] that develop an elegant and exact methodology rooted in the employment of (ordinary) generating functions. These formal power series are thus defined:

$$G_P(x) = \sum_{d=0}^{\infty} P(d)x^d \tag{2.1}$$

The functions prove to be extremely useful to characterize and in defining the properties of the degree distributions *P(d).* For instance, setting $x = 0$, after the repeated derivative we obtain:

$$\left.\frac{d^d G_P(x)}{dx^d}\right|_{x=0} = P(d)[d(d-1)(d-2)\dots 1] = P(d)d!$$

$$P(d) = \left.\frac{d^d G_P(x)}{dx^d}\right|_{x=0} \frac{1}{d!} \tag{2.2}$$

That is, the generative function G_P generates the probability distribution *P(d)*. In order to elucidate the infection process, we make use of the following array of properties derived in [13]:

1. $G_P(1) = \sum_{d=0}^{\infty} P(d)1^d = \sum_{d=0}^{\infty} P(d) = 1$ provided that *P(d)* is a normalized probability distribution

2. $\frac{d\, G_P(1)}{dx} = G_P'(1) = \sum_{d=0}^{\infty} P(d)dx^{d-1} = < d >$ is the network mean degree

However, the fundamental contribution of generating functions to the percolation problem is given by tracing the two following properties

3. Consider two independent draws of a random variable D. The probability that their sum is d is given by:$\sum_{i=0}^{d} P(i)P(d-i) = P_2(d)$. Hence, given that $P_2(d)$ is a probability function itself, it has an associated generating function:

$$G_{P_2}(x) = \sum_{d=0}^{\infty} P_2(d)x^d = \sum_{d=0}^{\infty}\sum_{i=0}^{d} P(i)P(d-i)x^d$$

Conversely,

$$[G_P(x)]^2 = \left[\sum_{d=0}^{\infty} P(d)x^d\right]^2 = \sum_{i,j}^{\infty} P(i)P(j)x^{i+j} = G_{P_2}(x)$$

The result is generalizable to *m* independent draws by means of $[G_P(x)]^m$.

4. Consider a distribution P generated by picking a distribution P_i from a series of distributions $(P_1, P_2, \dots)$ with probability π_i and consequently drawing from it. Then, it ensues that $G_P = \sum \pi_i G_{P_i}$

3.3. The Defaulting process in a graph structure: a network in the network

Suppose that we generate our industrial cluster by means of a *configuration model*[11]. In this framework, agents are endowed with a degree distribution *P(d)*. That is, the production structure is such that the N firms have a probability *P(d)* to be in partnership with *d* agents. Further, the production structure is such that the firms are vertically distributed along the production chain in a way that we do not experience short-loops (figure 7.2). Now suppose that every given economic *relationship* settled between two firms $i, j \in N$ in the graph has a

[11] See [13] for details

probability T_{ij} to transform into insolvency in the case one of the two firms defaults. Assuming that defaults are random and uncorrelated, a second configuration model is formed inside the district graph through the sub-network of impaired *relationships* and it is characterized itself by a new degree distribution. Testing the resilience of our industrial network is then equal to study this second layer of *potential insolvencies*, whose size is defined by means of a given transmissibility rate which averages all the T_{ij}. Hence, we are questioning whether an initial random default may generate a *default stream* trough the network of relationships which may eventually reach *any* firm in the component to which the first impaired firm belongs. Therefore, the dimensions involved in the actual transmission problem are thus summarized:

- The original network of firms and relationships generated by means of a configuration model with distribution *P(d)*
- A second-level layer of *potential insolvencies* (i.e. impaired relationships) whose *size* is dependent on T_{ij} $\forall\ i, j \in N$.
- A randomly chosen firm belonging to a component of the original network which get impaired by an exogenous shock

In fact, this exact question corresponds to the classical bond percolation problem of Physics [22]. As before, we want to find a synthetic metric for the resilience of the industrial district (i.e. a threshold) given the initial conditions of the environment. In a graph theory perspective, the capability of the district to undertake an initial shock is conditional on whether a component with *maximum* size (i.e. a *giant* component) of *potential insolvencies* emerges or not in the long run *inside* the original network structure[12]. In the case of emergence, the ultimate size of the default stream is equal to the size of the component to which the first impaired firm belongs. I follow [13] for the methodology to inspect the component size of an actor taken at random[13] and I subsequently develop on this method the crucial results of [22] both for a homogeneous population of industrial firms and for a mixed population of firms and saving banks. On the one hand, the two-step modeling approach is necessary to boil down the complexity of the second model by presenting the central issues with the homogeneous case and then implementing the heterogeneity. On the other hand, as stated before, homogeneous districts with a layer of solid relationships are a fact of reality.

[12] As [21] points out, a further characterization of the outbreak evolution in time would require a mean-field approximation.
[13] Jackson methodology slightly departs from the one developed in [23] and enriched in [22] for percolation

Hence, I first derive the methodology and then implement the percolation problem. The examined method is consistent with our initial assumption concerning the industrial district; in fact, as pointed both by [22] and [13], the method presumes a tree local structure. That is, no short-loops (i.e. loops between close agents) exist.

3.4. Generating Functions and agents reachability

As anticipated in the previous section, the study of the resilience characterized by means of a graph structure deals with the eventual length of a *default stream*. Hence, we need to understand the size of the population which may be eventually affected via indirect connections by a randomly impaired firm. We define Q as the distribution of the *number* of firms which are reachable by randomly choosing a business relation in the network, selecting one of the two agents which are therethrough connected and summing every agent which is connected to this agent by all the paths except the one we originally used to reach him. Hence, $G_Q(x)$ is the generating function associated with the distribution Q. Now define:

$$\tilde{P}(d) = \frac{dP(d)}{\sum_d dP(d)} = \frac{dP(d)}{< d >} \tag{2.3}$$

That is, $\tilde{P}(d)$ represents the number of connections we expect the neighbor of an agent to have. Put in other terms, $\tilde{P}(d)$ is the expected degree of an agent found after randomly picking a *link* and considering one of the two agents connected by it. This metric is fundamental since it is able to account for the different relevance of agents (in terms of connections) belonging to a network. In fact, while we expect an agent chosen at random in the network to be economically involved with *<d>* partners, we understand that randomly picking a *link* and choosing one of the two agents connected yields a higher probability to reach an agent with a relatively higher connectivity. Coming back to the construction of Q, there is a probability $\tilde{P}(d)$ to reach an agent involved in d economic partnerships. Consequently, the original link generates $d - 1$ partnerships and each of these $d - 1$ links may be followed to find Q additional agents. That means that with a large number of agents involved, Q is a random variable. Consequently, the generating function of the *sum* of $d - 1$ draws from Q is encapsulated by the generating function $G_Q(x)$ by means of **Property 3**: $[G_Q(x)]^{d-1}$. The generating function of the distribution of the agents found after the first agent is then given by **Property 4**: $\sum_d \tilde{P}(d)[G_Q(x)]^{d-1}$. However, we should account also for the agent we initially found, hence updating the generating function. The relation between the distribution P of d

and the one of $d + 1$ is such that the distribution of d is $\bar{P}(d) = P(d-1)$. Consequently, the generating function is:

$$G_{\bar{P}}(x) = \sum_{d=1}^{\infty} P(d-1)x^d = xG_P(x)$$

Eventually, the generating function for the component size is thus defined by

$$G_Q(x) = x\sum_{d=1}^{\infty} \tilde{P}(d)[G_Q(x)]^{d-1} \tag{2.4}$$

Here we have a departure of Jackson from [23]. In spite of $\tilde{P}(d)$, they define a generating function G_1 which accounts for the number of links departing from a node excluding the one used to reach it. However, it is possible to bridge the two methods noticing that:

$$[G_Q(x)]^2 = x\sum_{d=1}^{\infty} \tilde{P}(d)[G_Q(x)]^{d} = xG_{\tilde{P}}\left(G_Q(x)\right) \tag{2.5}$$

$$\frac{G_{\tilde{P}}\left(G_Q(x)\right)}{G_Q(x)} = \frac{G_Q(x)}{x} = \sum_{d=1}^{\infty} \tilde{P}(d)[G_Q(x)]^{d-1}$$

Given (2.3):

$$\frac{G_{\tilde{P}}\left(G_Q(x)\right)}{G_Q(x)} = \frac{\sum_{d=1}^{\infty} P(d)d[G_Q(x)]^{d-1}}{< d >} = \frac{G'_P(G_Q(x))}{G'_P(1)} = G_1$$

3.5. The Infection Dynamics

We gathered everything we need to introduce the financial infection dynamics in our economic network by means of a percolation problem. The passage from the deterministic model in which fully mixed classes (compartments) of homogeneous agents interact to a network structure where actors are characterized by a heterogeneous layer of relations has a cost in terms of model solvability. In fact, the N firms of our stochastic network are individually characterized by a composite set of features:

1. The number of agents which a single agent is in partnership with, represented by the distribution *P(d)*

2. The average financial exposure of agent i with respect to agent *j,* defined by the random variable β_{ij} which is distributed by $P_i(\beta)$

3. The time τ_i during which an impaired firm *i* continues its activity with its partners before declaring bankruptcy, which follows a distribution *P(τ)*

As pointed in [21], the stringent connection between percolation in a network and traditional deterministic diffusion was identified by [11], which reduced the complexity of the problem by means of a crucial assumption on the transmission mechanism which I briefly depict. If we assume that β, τ are independent and identically distributed (*IID*), it is possible to boil down the characterization of 2 and 3. Let us define:

$$1 - T_{ij} = (1 - \beta_{ij}\Delta t)^{\frac{\tau_i}{\Delta t}}$$

As the probability that the financial shock is *not* going to be *transmitted* from the impaired firm *i* to firm *j*. Setting $\Delta t = 1$ we obtain [22] the probability of transmission in discrete time-steps:

$$T_{ij} = 1 - (1 - \beta_{ij})^{\tau_i} \qquad (2.6)$$

Obviously, β_{ij} may vary among individuals and it is not realistic to assume it to be symmetric (firms in a production chain usually do not develop reciprocal liabilities). Also, τ_i typically differs according to a series of peculiar characteristics of agents. However, being both τ_i and β_{ij} IID, it follows that T_{ij} is independently and identically distributed too. Hence, we can define *a-priori* the probability of transmission of the default T_{ij} by averaging over the two distributions $P_i(\beta)$ and *P(τ):*

$$T = 1 - \int d\beta \sum_{\tau=0}^{\infty} P(\beta)P(\tau)(1-\beta)^{\tau} \qquad (2.7)$$

Hence, although in the absence of a mean field approximation we are not able to define "laws of motion" for the diffusion of the infection such as the ones of the previous chapter, the T parameter provides a powerful yet simple characterization for the default process. Further, under the assumption of I.I.D[14] for β_{ij} and τ_i it leaves room for a heterogeneous characterization of the economic agents. The percolation problem is then applied to the study of the resilience of our economic set by noticing that the distribution of the impaired components which contain a single firm impaired by the initial exogenous shock corresponds to the distribution of occupied clusters of the classical bond percolation problem. Firstly, we

[14] Which may also be realistically tackled, as proved in [22]

develop the infection spreading among a single population and then we will generalize to a bipartite economic set in which industrial and financial firms are strictly interconnected.

3.6. Generating Functions for Impaired Firms

Let me define two new generating functions which account for the number of *potential insolvencies as* a function of the probability of transmission T defined in (2.7). We make use of the binomial distribution to express the probability that a firm faces exactly k defaults among the d agents with which it is in partnership. Hence, the probability is given by

$$\binom{d}{k} T^k (1-T)^{d-k} \tag{2.8}$$

We implement the distribution *P(d)* with (2.8) to design the probability distribution of the *potential insolvencies*. We generate it via:

$$G_P(x,T) = \sum_{d=0}^{\infty} P(d) \sum_{k=0}^{d} \binom{d}{k} (1-T)^{d-k} (Tx)^k \tag{2.9}$$

By the virtues of the Binomial Theorem we know that

$$\sum_{a=0}^{n} \binom{n}{a} x^{n-a} y^a = (x+y)^n$$

Consequently (2.9) becomes:

$$G_P(x,T) = \sum_{d=0}^{\infty} P(d)\big(1+T(x-1)\big)^d = G_P(1+T(x-1)) \tag{2.10}$$

Notice that when *T = 1* the generating function for the distribution of the impaired relationships boils down to the generating function of the degree distribution P(d). That is, $G_P(x,1) = G_P(x)$. Hence, when the average financial exposure β and the average time of activity of an impaired firm τ are great enough, the transmissibility T is such that the layer of *potential insolvencies* corresponds to the *entire* network of relationships, and the size of the component of the defaulted firms will equal the size of the component to which the initially impaired firm does belong. Hence,

Now consider:

$$G'_P(x,T) = T\sum_{d=0}^{\infty} P(d)d\big(1+T(x-1)\big)^{d-1} \tag{2.11}$$

And notice that

$$G'_P(1,T) = T < d >$$

From which I construct $G_{\tilde{P}}(x,T)$. The probability to reach an impaired firm starting randomly from a bond is

$$G_{\tilde{P}}(x,T) = x\frac{G'_P(x,T)}{G'_P(1,T)} \tag{2.12}$$

Similarly, from (2.5) we may define the $G_{\tilde{P}}\big(G_Q(x),T\big)$ for impaired firms:

$$G_{\tilde{P}}\big(G_Q(x),T\big) = x\frac{G'_P(G_Q(x),T)}{G'_P(1,T)} \tag{2.13}$$

Now, we go back to (2.5) and update and reformulate (2.5)

$$G_Q(x,T) = x\frac{G_{\tilde{P}}\big(G_Q(x,T),T\big)}{G_Q(x,T)}$$

$$G_Q(x,T) = \sqrt{xG_{\tilde{P}}\big(G_Q(x,T),T\big)} \tag{2.14}$$

Even if we don't know how to solve (2.14), it is possible to use **Property 2** differentiating by *x* and substituting $x = 1$ in order to obtain the *average* number of impaired firms found from one randomly chosen link.

$$G_Q{}'(x,T) = \tfrac{1}{2}\Big(xG_{\tilde{P}}\big(G_Q(x,T),T\big)\Big)^{-\frac{1}{2}}\Big(G_{\tilde{P}}\big(G_Q(x,T),T\big) + xG'_{\tilde{P}}\big(G_Q(x,T),T\big)G_Q{}'(x,T)\Big) \tag{2.15}$$

Hence, given that

$$G_Q(1,T) = G_Q(1) = 1 = G_{\tilde{P}}\big(G_Q(1,T),T\big)$$

by means of **Property 1** (2.15) boils down to

$$G_Q{}'(1,T) = \frac{1}{2}\Big(1 + G'_{\tilde{P}}\big(G_Q(1,T),T\big)G_Q{}'(1,T)\Big) \tag{2.16}$$

The derivative of $G'_{\tilde{P}}\big(G_Q(x,T),T\big)$ equals:

$$G'_{\tilde{P}}\big(G_Q(x,T),T\big) = G'_{\tilde{P}}(1,T) = \frac{G'_P(G_Q(x,T),T)}{G'_P(1,T)} + x\frac{G''_P(x,T)}{G'_P(1,T)}$$

Hence, plugging $G_Q(1,T) = 1$ and simplifying T

$$G'_{\tilde{P}}\big(G_Q(1,T),T\big) \;=\; \left[T\,\frac{\sum_{d=0}^{\infty} P(d)d(d-1)1^{d-2}}{< d >} + 1\right] \tag{2.17}$$

$$= \frac{T}{< d >}(< d^2 > - < d >) + 1$$

$$= \; T\left(\frac{< d^2 >}{< d >} - 1\right) + 1$$

Where $< d^2 > = \sum_{d=0}^{\infty} P(d)d^2$ is a part of the variance of the degrees[15]. Hence, solving for $G_Q{}'(1,T)$ equation (2.16) gives:

$$G_Q{}'(1,T) \;= \frac{1}{1 - T\left[\frac{< d^2 > - < d >}{< d >}\right]} \tag{2.18}$$

Now, $G_Q{}'(1,T)$ represents the average number of *impaired* firms which we may find starting from a randomly chosen *partnership*. But we may also find the average *outbreak* of the initial shock, that is, the average size of the component of the impaired firms. In order to do so, I consider a single randomly chosen impaired firm with a number of partners which we know from (2.10) is given by *P(x,T)*. The generating function of the extended neighborhood size is $\left[G_Q\,(x,T)\,\right]^d$ plus the initial impaired firm. Thus, in the very fashion of (2.4), we define $G_D(x,T)$ the generating function for the distribution of the components size of randomly chosen impaired firm[16]

$$G_D(x,T) = x\sum_{d=0}^{\infty} P(d)\left[G_Q\,(x,T)\,\right]^d = xG_p(\,G_Q\,(x,T),T) \tag{2.19}$$

We easily find the *average* size of this component by means of **Property 2**

$$G'_D(1,T) = G_P(\,(1,T),T) + G'_P(\,(1,T),T)G'_Q\,((1,T),T) \tag{2.20}$$

Plugging (2.10) and (2.14) and applying **Property 1**

$$G'_D(1,T) = 1 + \frac{T < d >}{1 - T\left[\frac{< d^2 > - < d >}{< d >}\right]} \tag{2.21}$$

[15] Which is given by $E(d^2) - E(d)^2 = < d^2 > - < d >^2$

[16][21], [22] define this very generating function such as $H_0(x) = x(H_1(x,T);T)$. However, as I demonstrated, their method is easily implemented into [13] methodology by means of the referred proprieties of generating functions which we further developed in order to account for the defaulting process.

3.7. The Threshold for Homogeneous districts

Therefore, I offer the crucial result of the analysis. As stated, we cannot track down for the expansion of the crisis in our business district without the employment of mean-field approximation. However, we may use (2.16) to settle the threshold for the *general* diffusion of the crisis engendered by the initial shock. In fact, notice that when the transmission parameter defined in (2.7) is such that the denominator of (2.14) reaches the critical value of *0*, both $G_Q{}'(1,T)$ (the number of impaired firms) and $G'_D(1,T)$ (the relative size of the sub-networks they belong to) skyrocket. That is, in the limit we expect a non-zero fraction of the firms of the district to default (given by the size of the component which the initially impaired firm belongs to). The critical value is then obtained through:

$$1 - T_D \left[\frac{< d^2 > - < d >}{< d >} \right] = 0$$

$$T_D = \frac{< d >}{< d^2 > - < d >} \qquad (2.22)$$

We had obtained the threshold condition $T_D(E[d]; < d^2 >)$, *independently* of the chosen distribution for the business relations of the clusters' firms. As noticed by various authors ([18], [13], [22]) the variance of the agents' connections is the critical element which determines resilience to contagion. Essentially, the threshold for the system to be affected by an exogenous shock is negatively influenced by the connectivity fluctuations: high variance implies that our network is populated by a number of agents heavily connected with a majority of low connected ones. Once the more connected agents are impaired, the shock is easily diffused to other firms, which are plausibly connected with the higher connected ones. In that sense it is easy to understand the positive effect produced by the average number of connections < *d* >*.* An economic district endowed with a regular structure of relations is less prone to financial infection diffusion than a district characterized by unbalanced relationships, as it could be with the presence of a dominant company which coordinates a number of small sub-suppliers. Therefore, the economic insight of this outcome is straightforward. Notice that this result differs from the *contagion cascade threshold* (1.9) in the sense that it derives an optimal size with respect to the overall distribution and number of *relations* among agents, instead of the number of economic *actors.* However, as it is proved in chapter 4, it is possible to bridge the two results by means of a Poisson process for the relationships generation, if we further impose that each agent has $(n-1)$ relationships in the graph.

3.8. Heterogeneity of population in the Industrial District

The tools we developed in section 2.3 were addressed at deriving the crucial condition (2.22), which does not depend on any pre-determined distribution we may decide to use to describe the business relations within an industrial district. However, given that we want to study the financial resilience of a developed district with specialized agents (i.e. not all the skills are transferable), in which financial intermediates act to mediate the business risk, collect savings and ease the economic transitions, we should bring one step further the characterization of the district we modeled in Section 2.1. Consequently, we make a major assumption concerning the *structure* of the economic relations apparently similar to the one we used in Section 1.2.1 for our complete deterministic model. That is, we assume that *any* financial relation between the firms is mediated by the saving banks and these latter have only indirect connections among them. In Figure 8 I produced two examples of *bipartite* networks, in which agents of one group are connected with the other type. At one (unrealistic) extreme, every industrial firm is connected with each bank and vice versa. Conversely, if we assume for the sake of realism that the district is populated by a scarce relative number of saving banks which connect a multiplicity of firms which do not usually have direct relation with many banks, we are approaching the right part of Figure 8.

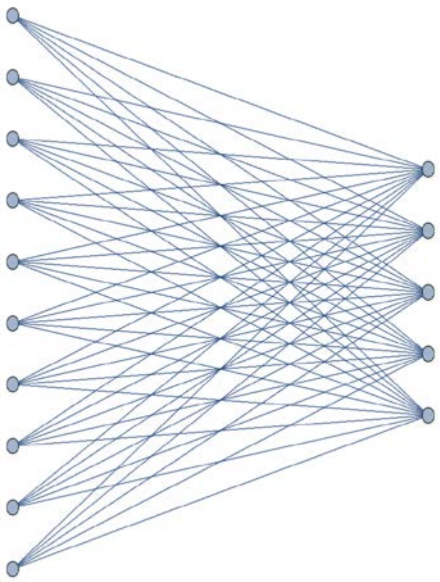

Figure 8.1 - Complete Bipartite Network for 10 firms and 5 saving banks

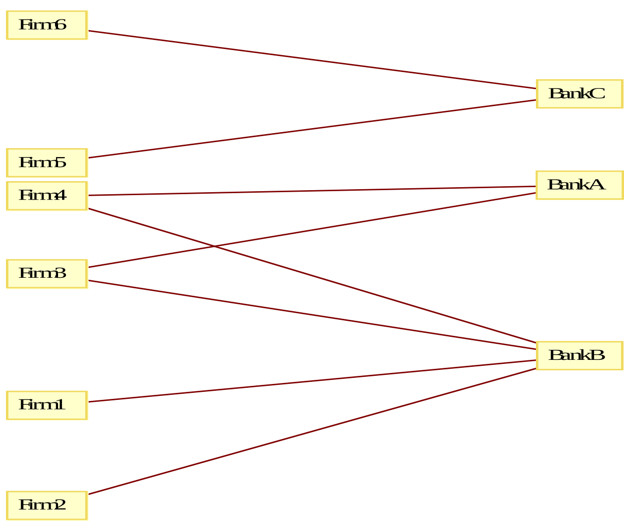

Figure 8.2 - Bipartite Network for 4 firms and 2 saving banks

The bipartite characterization of our business district is also useful in a technical sense. In fact, a negligible number of cycles are present in the network layer and consequently we are able to adopt the methods we derived in the previous section taking into account the heterogeneity which we discussed.

3.9. Implementing two classes of agents in the network

Consider the industrial cluster (N, **g**) populated by $N^I = \{n_1^I, n_2^I, \dots, n_{N^I}^I\}$ small and medium industrial enterprises and $N^F = \{n_1^F, n_2^F, \dots, n_{N^F}^F\}$ saving banks, such that:

$$N = \bigcup_{j \in \{I,F\}} \cup_i n_i^j.$$

The heterogeneity of economic agents is accounted by means of the following specifications:

1. Two different probability distributions for the degrees of economic connections, expressed by $P_I(d)$ for the industrial firms and $P_F(d)$ for the saving banks
2. Two *average* integrated probabilities of financial infection represented by T^{IF} for the contagion which stems from industrial firms to banks and T^{FI} for the backward mechanism. These two parameters are obtained in the very same fashion with which we derived (2.7)
3. $\forall\, k, j \in i, i \in \{I, F\} \nexists \left(n_k^i n_j^i\right) s.t\; P\left[n_k^i n_j^i\right] \neq 0$. That is to say, two agents of the same type may be only indirect neighbours, as Firm 4 and Firm 3 (but not Firms 1-4 with Firms 5-6) in Figure 8.2

Specifications 1-3 require us to build up a new set of generating functions to explain the diffusion process *among the actors of a class*. In fact, the bipartite layer we decided to adopt for our network prevents us to offer a direct translation of the results we derived in the last section. This notwithstanding, it is still possible to find a sufficient amount of regularity in our graph. For instance, consider a random actor i : we know for sure that its *neighbours's neighbours* belong to its very class and are characterized by the same generating function that it is. Starting from the generating function for our two populations, we define, coherently with (2.1) and (2.4)

$$G_{I_P}(x) = \sum_{d=0}^{\infty} P_I(d)x^d \tag{2.23a}$$

$$G_{F_P}(x) = \sum_{d=0}^{\infty} P_F\,(d)x^d \tag{2.23b}$$

$$G_{I_Q}(x) = x\sum_{d=1}^{\infty} \tilde{P}_F(d)[G_{I_Q}(x)]^{d-1} \tag{2.23c}$$

$$G_{F_Q}(x) = x\sum_{d=1}^{\infty} \tilde{P}_I(d)[G_{F_Q}(x)]^{d-1} \tag{2.23d}$$

Now we have to define the distribution of the *neighbours' neighbours* taking into account the aforementioned heterogeneity of the layer. In order to do so, we go back to the generating function

$$G_j^1 = \sum_{d^j=1}^{\infty} \tilde{P}_j(d)[G_{j_Q}(x)]^{d_j-1}$$

Which we derived from (2.5). This function accounts for the number of links departing from a node excluding the one used to reach it. Now consider - without loss of generality- a random industrial firm which is in financial relations with d_I banks. The probability that this firm is *indirectly* connected with *j* firms (the neighbours' neighbours) equals the probability that the *sum* of all the $(d_F - 1)$ relations of the d_I banks is *j.* Hence, by means of **Property 3** we account for all the d_I relations of our random firm and obtain

$$G_{I_P}[G_F^1(x)] = \sum_{d=1}^{\infty} P_I(d)\left[\sum_{d^F=1}^{\infty} \tilde{P}_F(d)\left[G_{F_Q}(x)\right]^{d_F-1}\right]^d = \sum_{d=1}^{\infty} P_I(d)\,[G_F^1(x)]^d \tag{2.24a}$$

In the same fashion, a generating function $G_{F_P}[(G_F^1)]$ for the neighbours' neighbours of a saving bank may be derived:

$$G_{F_P}[G_I^1(x)] = \sum_{d=1}^{\infty} P_F(d)\left[\sum_{d^I=1}^{\infty} \tilde{P}_I(d)\left[G_{I_Q}(x)\right]^{d_I-1}\right]^d = \sum_{d=1}^{\infty} P_F(d)\,[G_I^1]^d \tag{2.24b}$$

And in a similar manner we can define $G_{I_{\tilde{P}}}[(G_F^1(x))]$ and $G_{F_{\tilde{P}}}[(G_I^1(x))]$.

Generating functions (2.24a) and (2.24b) are our workhorses. We are going to implement in them the two average integrated transmissions through which we will obtain the generating functions of the potential insolvencies for firms and saving banks of the bipartite network. For the sake of brevity we develop the results just for the industrial firms, given that the derivation for saving banks is symmetric. Again, we employ the binomial distribution to express the diffusion process of the initial shock among the industrial firms. Hence, the probability is given by

$$G_{I_P}[(G_F^1(x,T^{FI}),T^{IF}] = \sum_{d=0}^{\infty} P_I(d) \sum_{k=0}^{d} \binom{d}{k} (1-T^{IF})^{d-k} \, (T^{IF}(G_F^1(x,T^{FI}))^k \qquad (2.25)$$

And, applying the Binomial Identity,

$$G_{I_P}[(G_F^1(x,T^{FI}),T^{IF}] = G_{I_P}(1+T^{IF}(G_F^1(x,T^{FI})-1) \qquad (2.26)$$

How it can be easily inferred, explicit equations become cumbersome. However, I use the form derived in (2.12) and proprieties of generating functions in order to develop a clean expression for the updated version of (2.16). From (2.12) we know that

$$G_{\tilde{P}}(x,T) = x\frac{G'_P(x,T)}{G'_P(1,T)}$$

Consequently,

$$G'_{\tilde{P}}(1,T) = \frac{G'_P(1,T)}{G'_P(1,T)} + \frac{G''_P(x,T)}{G'_P(1,T)} = 1 + \frac{G''_P(x,T)}{G'_P(1,T)} \qquad (2.27)$$

Hence, plugging (2.27) into the updated expression for (2.16)

$$G'_{I_Q}(1,T^{FI},T^{IF}) = \frac{1}{2}\left(1 + G'_{I_{\tilde{P}}}(G_{I_Q}(1,T^{FI},T^{IF})G'^1_F(1,T^{FI},T^{IF})G'_{I_Q}(1,T^{FI},T^{IF})\right)$$

We get the new expression for (2.18)

$$G'_{I_Q}(1,T^{FI},T^{IF}) = \qquad (2.28)$$

$$= \frac{1}{2 - \left[1 + \frac{G''_{I_P}\left[\left(G_F^1(G_{I_Q}(1,T^{FI})\right),T^{IF}\right]G'^1_F(1,T^{FI},T^{IF})}{G'_{I_P}\left[\left(G_F^1(G_{F_Q}(1,T^{FI})\right),T^{IF}\right]}\right]}$$

Similarly to the homogeneous network case, $G'_{I_Q}(1,T^{FI},T^{IF})$ represents the average number of impaired industrial firms which we may find starting from a randomly chosen saving bank. However, we are still interested in the size of the average component of the defaulted firms generated by the initial exogenous shock which impaired one industrial firm. The impaired firms which constitute this random firm's second order neighbours are described through the generalization of the generating function defined in (2.19), such that

$$G_{I_D}(x, T^{FI}, T^{IF}) = \tag{2.29}$$

$$= x \sum_{d=0}^{\infty} P_I(d) \left[G_{I_Q}\big((G_F^1(x, T^{FI}), T^{FI}, T^{IF}\big)\right]^d$$

$$= xG_{I_p}(G_{I_Q}\big((G_F^1(x, T^{FI}), T^{IF}\big))$$

Then, the *average* size of the component of impaired industrial firms is again obtained by means of Property 2

$$G'_{I_D}(1, T^{FI}, T^{IF}) = 1G_{I_P}(G_{I_Q}(1, T^{FI}, T^{IF})) + G'_{I_P}(G_{I_Q}(\cdot))G'^1_F(1, T^{FI}, T^{IF})G'_{I_Q}(\cdot)$$

Hence,

$$G'_{I_D}(1, T^{FI}, T^{IF}) = 1 + G'_{I_P}(1, T^{FI}, T^{IF}))G'^1_F(1, T^{FI}, T^{IF})G'_{I_Q}(1, T^{FI}, T^{IF}) \tag{2.30}$$

Now, let's derive $G'_{I_P}(1, T^{FI}, T^{IF})G'^1_F(1, T^{FI}, T^{IF}) =$

$$= T^{IF}T^{FI} \left[\frac{< d^2{}_F > - < d_F >}{< d_F >}\right] < d_I >$$

Now notice that in the long run the effects of the initial shock which impaired one firm are no more locally bounded within a limited group of firms when $G'_{I_D}(1, T^{FI}, T^{IF}) \to \infty$. This implies that either $G'_{I_P}(1, T^{FI}, T^{IF}) \to \infty$ or $G'_{I_Q}(1, T^{FI}, T^{IF}) \to \infty$. However, we are not concerned with the first possibility, since we are considering a network with a non-trivial stratification and consequently$< d_F > \neq 0$. Hence, the latter is the crucial one. Let's go back to (2.28) and notice that

$$G''_{IP}\left[\left(G_F^1(G_{F_Q}(1, T^{FI})\right), T^{IF}\right] G'^1_F(1, T^{FI}, T^{IF}) = (T^{IF})^2 T^{FI} \left[\frac{< d^2{}_F > - < d_F >}{< d_F >}\right] [< d^2{}_I > - < d_I >]$$

We know that $G'_{I_Q}(1, T^{FI}, T^{IF}) \to \infty$ when its denominator goes to 0, and that is the case when

$$1-\frac{G''_{I_P}\left[\left(G_F^1(G_{F_Q}(1,T^{FI})\right),T^{IF}\right]G'^1_F(1,T^{FI},T^{IF})}{G'_{I_P}\left[\left(G_F^1(G_{F_Q}(1,T^{FI})\right),T^{IF}\right]}=0$$

Hence, this leads us to our final result:

$$(T^{IF})^2T^{FI}\left[\frac{<d^2{}_F>-<d_F>}{<d_F>}\right][<d^2{}_I>-<d_I>]=T^{IF}<d_I>$$

$$(T^{IF}T^{FI})=\frac{1}{\left[\frac{<d^2{}_F>-<d_F>}{<d_F>}\right]\left[\frac{<d^2{}_I>-<d_I>}{<d_I>}\right]}$$

$$(T^{IF}T^{FI})^*=\frac{<d_I><d_F>}{[<d^2{}_F>-<d_F>][<d^2{}_I>-<d_I>]} \tag{2.31}$$

Thus, the composite threshold is determined as the product of the two coefficients for financial infection defined in Specification 2. This is coherent both with the results of the bipartite percolation model developed by [22] and the dynamical mean-field approximation of [27], where the first and second moments of *both* the degree distributions determine the threshold. Specifically, the two variances $Var[d_F], Var[d_I]$ have a negative effect in lowering the threshold, while the expected degrees $E[d_F], E[d_I]$ act contrariwise. Hence, the result in (2.31) is coherent with (2.22).

3.10. Some considerations

This leads us toward some economic consideration concerning the layer of economic relations which bounds the agents of a heterogeneous industrial district of the kind such the one we illustrated. It is easy to understand that districts populated by firms and saving banks with homogeneous connections are less prone to financial contagion than industrial agglomerations in which at least one of the two sectors is characterized with a "fat-tailed" distribution, in which agents with many and few business relations (i.e. the bank sector) dominate the ones with the average degree. In fact, this is the crucial outcome offered in [18], which pointed out that "fat-tailed" distributions show a *vanishing* threshold to infections. Although their result is obtained in the context of a power-law distributed network with homogeneous agents described by

$$P(d) = cd^{-\gamma} = \frac{2m^2}{k^3}$$

the compatibility between (2.22) and (2.31) assures that their result is generalizable as well to the case of heterogeneous relations such the ones we built in our latter model.

IV. Simulating Financial Contagion in Homogeneous Networks

In this section I simulate the defaulting process which we implemented in the graph structure of the industrial district generated by means of a Poisson process. The model is a modification of a SIS model with immunization and random outbreak of infection created by Phillip Bonacich. I characterize the graph by means of a simple Poisson random process for homogeneous relationships formation. This is useful to trace some parallelism with the models derived under assumption 3.

4.1. Threshold derivation and a bridge toward Assumption 3

Applying the results we obtained in section 2.3.5 we may directly calculate the threshold for the infection to spread among the whole cluster. Given that we employ a simple Poisson random process for generating the network relationships, we can extract both $< d >$ and $< d^2 >$ from the probability mass function:

$$P(d) = \frac{k^d}{d!} e^{-k} \tag{3.1}$$

By means of $\sum_{d=1}^{\infty} dP(d)$ and $\sum_{d=1}^{\infty} d^2 P(d)$. Starting from the first moment $< d >$

$$\sum_{d=1}^{\infty} d \frac{k^d}{d!} e^{-k} = \sum_{d=1}^{\infty} k \frac{k^{d-1}}{(d-1)!} e^{-k} = k \sum_{d=0}^{\infty} \frac{k^d}{d!} e^{-k} = k e^k e^{-k} = k = < d >$$

Then, the second moment $< d^2 >$ is

$$\sum_{d=1}^{\infty} d^2 \frac{k^d}{d!} e^{-k} = \sum_{d=1}^{\infty} d \frac{k^d}{(d-1)!} e^{-k}$$
$$= \sum_{d=1}^{\infty} kd \frac{k^{d-1}}{(d-1)!} e^{-k} = \sum_{d=0}^{\infty} k(d+1) \frac{k^d}{d!} e^{-k}$$
$$= e^{-k} k \left[\sum_{d=0}^{\infty} d \frac{k^d}{d!} + \sum_{d=0}^{\infty} \frac{k^d}{d!} \right] = e^{-k} k \left[k \sum_{d=0}^{\infty} \frac{k^{d-1}}{(d-1)!} + \sum_{d=0}^{\infty} \frac{k^d}{d!} \right]$$
$$= e^{-k} k \left[k e^k + e^k \right] = k(k+1) \tag{3.2}$$

To summarize, with a Bernoulli distribution, $< d > = k$ and $< d^2 > = k^2 + k$

The threshold derived in section 2.3.5 is then equal to

$$T_D = \frac{< d >}{< d^2 > - < d >} = \frac{k}{k(k+1-1)} = \frac{1}{k} \tag{3.3}$$

Now, notice that the simple Poisson random models narrow the distance with models restricted by assumption 3 (homogeneous relations). In fact, we can account for assumption 3 in graph theory imposing every firm to be connected with any other company, by generating a *complete graph*. In this case, $k = n - 1$. Consequently, exactly as in the cases of second chapter, the wider the population, the lower the threshold. Or, to state it in other terms,

$$T_D = \frac{1}{n-1} \rightarrow (n-1) = \frac{1}{T_D}$$

Where $n - 1 = S_0$ is the network population after the initial shock which impaired one firm. Hence, in the case of *complete* relationships, we are back within the boundaries of assumption 2, and consequently the threshold identifies the district optimal size in terms of the *number* of agents. Apart from the mathematical insights, the outcome I just derived is interesting to remark the increase in realism we obtained relaxing assumption 3 through the graph structure. In fact, in order to get back within assumption 3 we had to impose to our graph not only the distribution (the Poisson process) but also the number of connections among the agents.

4.2. Financial contagion with Mathematica: simulation outcomes

Now, assume that we are considering a generic District Environment similar to the one depicted in **Section 1.1.4.** In this case, the industrial district is populated with 300 firms, one of which is randomly affected by a severe loss. Before defaulting (at a fixed rate per unit of time), the impaired firm continues its activity for $\tau_i = 1$ receiving financial assistance (thus contracting debts) from its business partners. Hence, the transmission of the shock is defined by means of the T parameter (section 2.3.3) which we examine for the critical value T^D derived in (3.3). This allows us to boil down the characterization of the financial shock transmission to the average *financial exposure* in the district.

$$T = 1 - e^{-\beta} \rightarrow \ln e^{-\beta} = \ln[1 - T]$$

$$-\beta^* = ln\left[\frac{k-1}{k}\right] \rightarrow \ \beta^* = ln\left[\frac{k}{k-1}\right]$$

Hence, the critical value which determines the emergence of a giant component of *insolvent relationships* is derived by means of average financial exposure. Finally, we characterize the relationships layer assuming that the probability for a link coming into existence is $p = \frac{1}{n+1}$. So, in the limit of big populations the probability for two clusters to be attached through the same edges goes to zero of loops appear[17]. We execute the simulation for T = 100 periods.

Now, we calculate the threshold imposing the aforementioned tree structure by means of $p = \frac{1}{n+1}$ and applying the conditions derived in section 3.1. With 300 firms the average degree is nearly 2.3. Hence, substituting the critical value T_D which we derived in the contagion process and plugging $\tau_i = 1$ we obtain the *critical* level of *financial exposure* of our district:

$$\beta^* \cong \ \ln\left[\frac{2{,}4}{2{,}4-1}\right] = 0.53$$

It is then possible to run a simulation for a given District Environment $\bar{E}_1 = \{\, \bar{\beta} = 4{,}5x10^{-3}, \tau_i = 1\,,\ S_{\mathbf{0}} = 299,\ I_{\mathbf{0}} = 1\}$ with an infection parameter equal to the one we derived in section 1.2.3 and obtain that the diffusion does not spread. The result is reported in Figure 10. Clearly, given the initial conditions and the degree distribution, no infection had followed the initial shock and the impaired firm does not transmit to its cluster the contagion. Now we replicate the simulation for a slightly different environment, in which the diffusion parameter $\bar{\beta}$ is such that $\overline{\beta} > \beta^*$. Considering $\bar{\beta}$ = 0.6 we define a second Industrial District Environment $\bar{E}_1 = \ \{\bar{\beta} \ = \ 0.6\,, \tau_i = 1\,,\ S_{\mathbf{0}} = 299,\ I_{\mathbf{0}} = 1\}$. the result is reported in Figure 11**.** As we can see, the entire component has been infected as the outcome of the epidemic. However, notice two fundamental differences from the results of section 1.1.4. First, the threshold derived in the graph structure is realistically higher. In fact, it

[17] See [13], p. 92 for further details on this and other thresholds for Poisson random graphs.

stances that a given firm may have outstanding credits for 60% of its capital with respect to one its impaired relationships. Secondly, even in the case of transmissions higher than the threshold, it can be the case that a *default cascade* does *not* follow [22] because the initial shock affected an isolated *small* component.

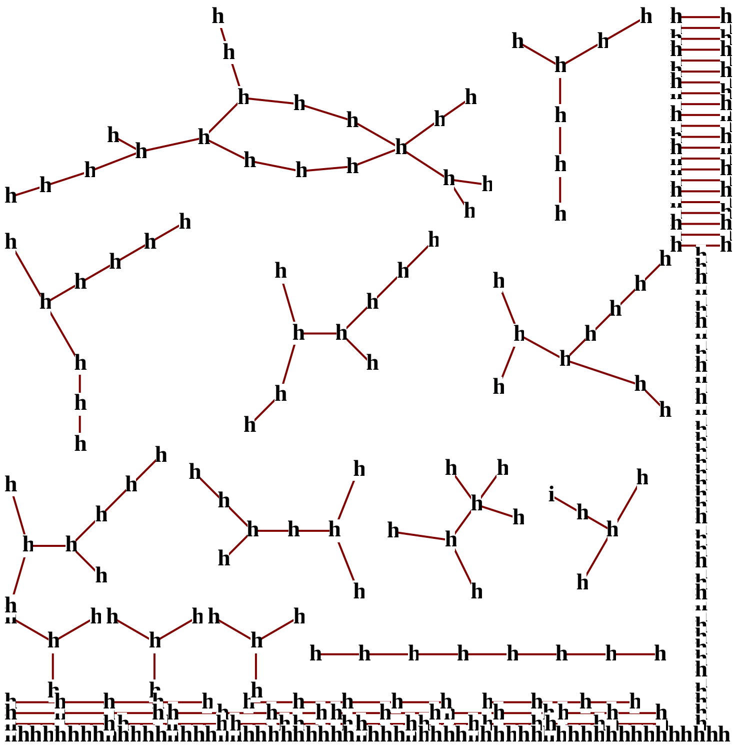

Figure 10.1 The network structure for $\bar{\bar{E}}_1$

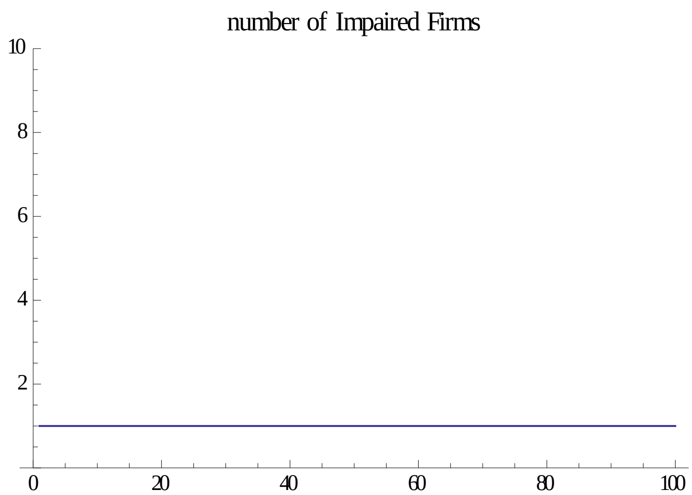

Figure 10.1 Cumulate number of impaired firms for $\overline{E}_1$

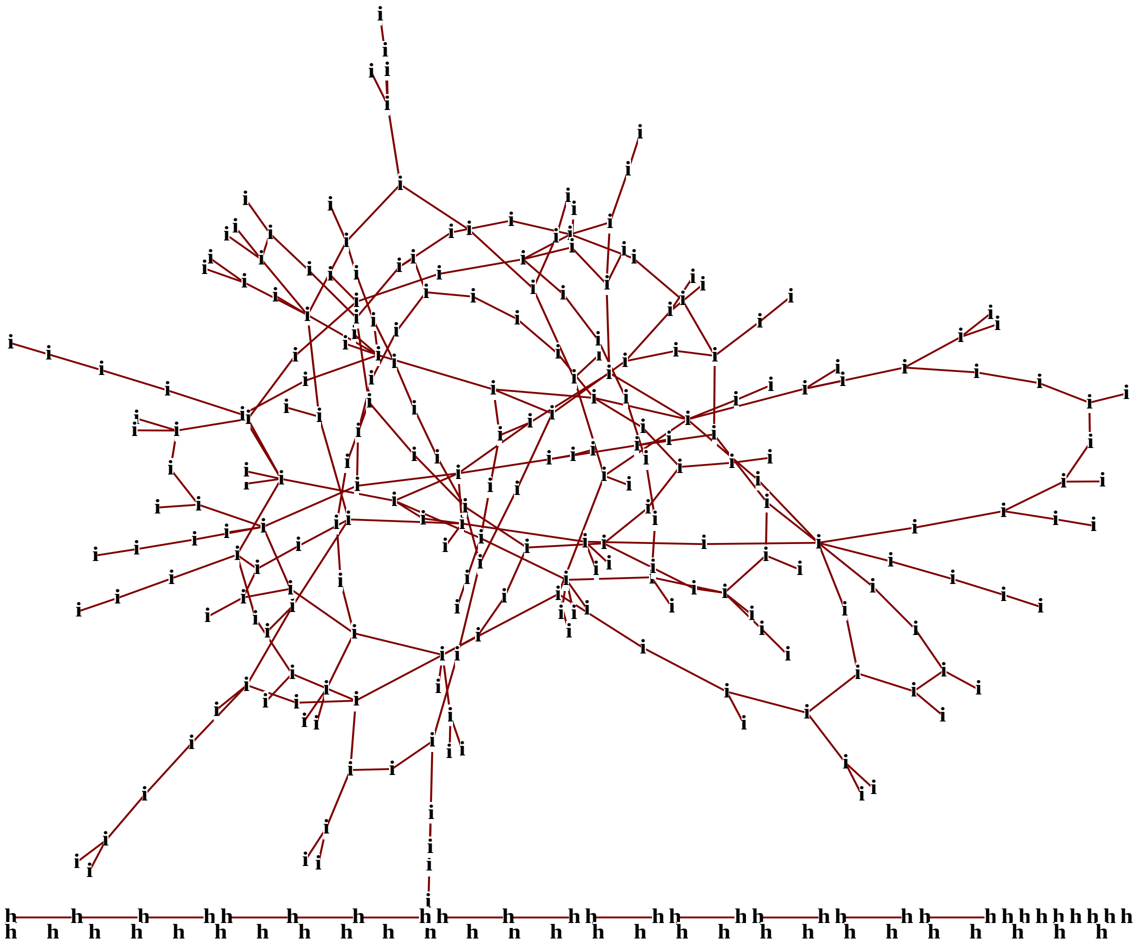

Figure 11.1 The network structure for $\overline{E}_2$

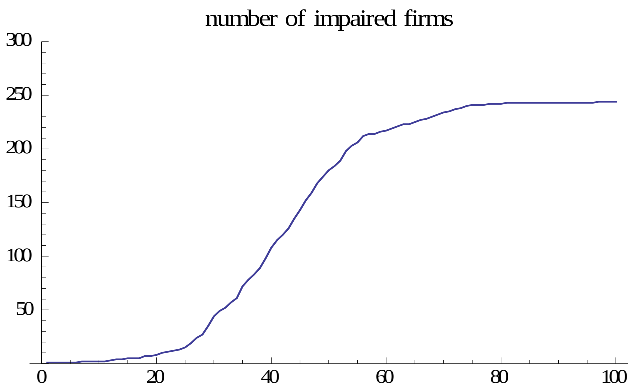

Figure 11.2 Cumulate number of impaired firms for $\overline{E}_2$

V. Conclusions

In this work, we attempted to frame some relevant features of the economic organization of industrial districts within an analytic structure with the purpose to study the systemic effects of an exogenous economic shock. However, the coexistence of a variety of industrial districts endowed with a wide array of features impedes a univocal characterization based on a universal model. In fact, at least two levels of complexity act simultaneously in shaping an industrial district: the *structure* of economic interactions and the degree of *heterogeneity* of the industrial population. Consequently, grounded on the definitions of the reviewed literature, we provided an identification of the "simplest" economic organization of an industrial district – the homogeneous horizontal district. Subsequently, in the course of the analysis, we proceeded by a progressive relaxation of assumption 3 and 2, introducing a higher level of complexity in order to build a more general framework. Eventually, two dimensions of characterization emerge. These are defined by the possible presence of a composite population of specialized agents (saving banks and industrial firms) which interact by means of casual relations or through a specific layer of relationships, which I modelled by implementing an underlying graph structure. When financial contagion dynamics are introduced in every scenario, our findings demonstrate that it is in the interactions dimension (relations versus relationships) that things change manifestly. With this regard, we implemented a diffusion system in the sets in order to study the *systemic* capability of each district typology to withstand the possible outcomes of an exogenous financial shock which impairs some agents belonging to the set. In each district, the possible diffusion of the crisis and the extent of the consequent *default cascade* ensued by the first burst of defaults is dependent on a set of conditions which we were able to derive in analytical terms. In every scenario, this leaded us to the definition of a *threshold* t*hat* establishes the ultimate outcomes of an initial default burst. That is, whether a *default cascade* follows to the initial shock or not and how ruinous the cascade could be. We tested the analytic results by means of a series of simulations in chapter 4. The thresholds encapsulate the very peculiarities of the complexity that had been considered. As stated before, when we change the structure of interaction from simple (casual) *relations* to a network of solid *relationships,* all things equal (exposure to impaired agents and default rates) we also observe a qualitative change in the structure of the threshold. With casual relations, the threshold identifies the optimal size of the industrial district in terms of the *number* of agents, conversely, when relationships are taken into account, the district tolerance is determined in terms of first and second moments of the degree distribution; that is, by the average number of relationships (positive effect which increases the tolerance) and their dispersion among the agents (negative effect). However, in chapter 4 we showed that it is possible to bring back the graph model under the assumption 3 by means of a Poisson process: if this is the case, the connection between the thresholds derived in section one and two follows straightforwardly. Even if the characterization offered by means of bond percolation provides realistic insights, further work should be addressed to relax the assumptions stated in section (3.1). It is in fact a priority for an economic description of an industrial district to develop a model with weighted and directed relationships, in order to account for different degrees and typologies of economic relationships among the agents.

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