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Orderings of Opportunity Sets

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Abstract. We consider an extension of the class of multi-utility hyper-relations, the class of semi-decent hyper-relations. A semi-decent hyper-relation satisfies monotonicity, stability with respect to contraction, and the union property. We analyze the class of semi-decent hyper-relations both associating them to an appropriate class of choice functions and considering decomposition of a decent relations via 'elementary' ones. Doing so, we consider images in the set of choice functions of three subclasses of semi-decent hyper-relations: the decent hyper-relations, the transitive decent hyper-relations , and transitive decent hyper-relations which satisfy the condition LE of lattice equivalence. We prove that the image of the set of decent hyper-relations coincides with of the set of heritage choice functions; the image of the set of transitive decent hyper-relations coincides with the set of closed choice functions; the image of the set of transitive decent hyper-relations which satisfy the LE coincides with the set of Plott functions. We consider, for each of the above subclasses of hyper-relations, the problem of the decomposition of a given hyper-relation into `elementary' ones, namely the representation of a given hyper-relation as the intersection of `elementary'ones.

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1. Introduction

Let X be a set of alternatives whose subsets are to be interpreted as opportunity sets (menus, or sets of offers) for a consequent (future) decision maker's choice. For a wide review of the different approaches to ordering opportunity sets see, for example, a survey paper [3]. We are interested in the so-called flexibility-based orderings of opportunity sets, i.e. a menu B is less preferred than a menu A (or A provide more flexibility of choices than B) whenever A has more options for a subsequent choice than B.

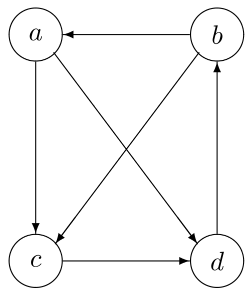
The simplest case is as follows. Suppose a decision maker compares alternatives of X according to a bounded utility function $u : X \to \mathbb{R}$. Then the 'indirect utility' of a subset $A \subseteq X$ is set by the rule $u(A) = \sup_{a \in A} u(a)$ ($u(\emptyset) = -\infty$), and the decision maker compares menus according to their indirect utility. However, this is not the most interesting case.

Kreps ([9]) enriched this framework by assuming that the decision maker's preference on the set of alternatives depends on the 'states of nature', i.e., to each state of nature $s \in S$ is associated some utility function $u_s : X \to R$. Then, for a menu A, we get the associated vector of utilities $(u_s(A))_{s \in S} \in \mathbb{R}^S$, and an ordering on 2^X is defined as a coordinate-wise ordering of utility's vectors in $\mathbb{R}^S$, that is B is preferred by A ($B \preceq A$), if for any state of nature $s \in S$, there holds $u_s(A) \geq u_s(B)$. Kreps in [9] axiomatically characterize such a *multi-utility* ordering on 2^X.

Observe that the ordering of menus in the simplest situation and the multi-utility ordering share the same properties – (i) monotonicity with respect to set-inclusion (a set is preferred to any part of it), (ii) transitivity and (iii) union (the union of worse sets remains a worse set). The only difference between the two cases is that in the simplest case the corresponding ordering is complete. This phenomenon takes place in a more general situation (see Examples 3 and 5 in the next section).

Namely, the multi-utility framework can be extended to a case with non-transitive binary relations. In order to distinguish binary relations on X and on 2^X, we say a *relation* for a binary relation on X and a *hyper-relation* for a binary relation on 2^X.

Let us start with an example. Suppose $X = \{a, b, c, d\}$ be a collection of four genres (say, science fiction (a), crime (b), drama (c) , and love story (d)). Typically, a book is made up of different genres. Thus, we understand a subset A of X as a book. A decision maker is a family (with more than two members). The family compares genres due to the majority voting as follows $a > d$, $a > c$, $b > c$, $b > a$, $c > d$, $d > b$ (see Picture 1). This defines a non-transitive relation on X.



Picture 1.

Then in order to have a hyper-relation on the set of books, the family could follow the rule: a book A is preferred by a book B, $A \preceq B$ if, for every genre x which occurs in A but not in B, $x \in A \setminus B$, there exists an genre y in B such that $x < y$. Such a hyper-relation is not transitive, but it satisfies the monotonicity, union and two properties which weaken transitivity: stability with respect to contraction and stability with respect to extension. These latter axioms reflect the following relationships between a hyper-relation and the set-inclusion: if a set provides - let us say- more freedom of choice than another, then this *a fortiori* holds for any subset of the latter, and that if a set offers - let us say- more 'suitable alternatives' than another set, then the set containing the former as its subset will certainly provide more 'suitable alternatives' than the latter. Let us illustrate the extension and contraction axioms on this example: the menu $\{a, c\}$ is preferred by b, b is a subset of the menu $\{b, d\}$, and $\{a, c\}$ is preferred by $\{b, d\}$; The menu $\{a, d\}$ is preferred by $\{b, c\}$ and any subset of $\{a, d\}$, either a, or d or the empty set is preferred by $\{b, c\}$. Such a hyper-relation is complete.

In this example we demonstrated an extension of the simplest situation to a case with a non-transitive relation on X.

Thus, for arbitrary relations, we get an extension of the class of multi-utility hyper-relations. A hyper-relation of this class we call *decent* and it satisfies monotonicity, stability with respect to contraction, stability with respect to extension, and the union property. We analyze the class of decent hyper-relations both associating them to an appropriate class of choice functions and considering decomposition of a decent relations via 'elementary' ones.

Namely, we consider a mapping from the set **HR**(X) of hyper-relations on X to the set **CF**(X) of choice functions on X, and a mapping from the set of choice functions to the set of hyper-relations, which provide a bijection between the decent hyper-relations and heritage choice functions. An idea to relate hyper-relations and choice functions was proposed by Puppe in [15]. Let us notice that he studied another mapping from the set of choice functions to the set of hyper-relations. Ryan in [18] studies one more different mapping from the class of Plott functions to hyper-relations.

Let us consider a mapping $\phi : \mathbf{HR} \to \mathbf{CF}$ which sends a hyper-relation $\preceq$ on 2^X to a choice function $f := \phi(\preceq)$ by the rule:

$$f(A) = \{a \in A \text{ such that does not hold } a \preceq A - a\}.$$

In other words if an element $a \in A$ is preferred by the complement $A - a$, then such an element a will be not chosen from A. For a motivation, suppose to be in the simplest case considered above, then a is not chosen from A if $u(a) \leq u(A - a)$.

In the example above, the corresponding choice function is specified as follows: $f(a) = a$, $f(b) = b$, $f(c) = c$, $f(d) = d$, $f(a,b) = b$, $f(a,c) = a$, $f(a,d) = a$, $f(b,c) = b$, $f(b,d) = d$, $f(c,d) = c$, $f(a,b,c) = b$, $f(a,c,d) = a$, $f(a,b,d) = f(b,c,d) = f(a,b,c,d) = \emptyset$. This choice functions is a heritage choice function, that is f satisfies *heredity* property **H** or, equivalently, the Chernoff axiom, or, equivalently, the Sen α-axiom. In Section 4 we will show that this is not a happenstance, but a general rule.

The reverse mapping $\kappa : \mathbf{CF} \to \mathbf{HR}$ is defined by the rule: let $\preceq_f := \kappa(f)$, then

$$A \preceq_f B \text{ if for every } a \in A \setminus B, \text{ there holds } a \notin f(a \cup B).$$

A motivation for such a definition goes as follows. Suppose a decision maker's choice function is rationalizable by a binary relation $<$ on X, $x < y$ means that y is strictly better than x. Then, we define a hyper-relation on 2^X by the rule $A \preceq B$ if for any alternative a which is in A but not in B either there is $b \in B$ such that $a < b$, or a is a *dummy*, that is $a < a$.

We show that the mapping κ is an injection, and its image coincides with the subclass **SDHR** of semi-decent hyper-relations, $\mathbf{SDHR} \subset \mathbf{HR}$ (we give precise definitions of all these notions in the next Section).

We consider images of three subclasses of semi-decent hyper-relations: the decent hyper-relations, the transitive decent hyper-relations (transitive hyper-relations were studied in different set-ups, see, for example, [9, 5, 7]), and transitive decent hyper-relations which satisfy the condition **LE** of lattice equivalence (in [5] these hyper-relations are called *framed dependency relation*) and we get the following results

- the image of the set of decent hyper-relations coincides with of the set of heritage choice functions;
- the image of the set of transitive decent hyper-relations coincides with the set of closed choice functions;
- the image of the set of transitive decent hyper-relations which satisfy the condition of lattice equivalence coincides with the set of Plott functions.

Remark. In the literature on stable matchings [17], the Heritage axiom is interpreted in terms of substitutability. Thus we may argue that the class of decent hyper-relations corresponds to ordering of opportunities for substitutable alternatives. For a case of complementary alternatives, the union axiom can be violated, and this case needs another analysis.

We consider, for each of the above subclasses of hyper-relations, the problem of the decomposition of a given hyper-relation into ‘elementary’ ones, namely the representation of a given hyper-relation as the intersection of ‘elementary’ones.[1] So doing, we get that in the corresponding subclasses of choice functions, ’elementary’ choice functions are specified by adding the concordance axiom. These choice functions are rationalizable by binary relations for the class of heritage choice functions, by para-transitive binary relations for closed choice functions, and by linear orders for Plott choice functions. Then, a given hyper-relation takes the form of intersection of the corresponding binary hyper-relations.[2]

2. Hyper-relations

We denote by X a universal set of alternatives, which is allowed to be infinite in Section 2–5 and finite in Section 6. The set of all subsets of X is 2^X, its elements denoted by $A, B, C, \ldots$, and are referred to as opportunity sets or menus, the empty set $\emptyset$ is also a menu. Recall that a binary relation on X we call a relation and denote usually by $<$. We assume that properties (ir)reflexivity, acyclicity and transitivity of relations are known to a reader. A binary relation on 2^X we call a hyper-relation and denote by $\preceq$. We denote by **HR** the set of all hyper-relations on X.

The set of all subsets of X is a partially ordered set (poset) with respect to the set inclusion. This poset is a Boolean lattice with respect to the union and the intersection. We are interested in hyper-relations which take into account this partial ordered structure.

First, the fact that every opportunity set is at least as good as every subset of it is without any doubt one of the less controversial requirement in the theoretical literature on ranking sets of opportunities in terms of the freedom. This obvious requirement is discussed in the works of several scholars (see, for example, [3]), Kreps attributes this axiom to Koopmans (axiom (1.3) in [9]).

- **Monotonicity with respect to set inclusion (Mon).** For all $A, B \in 2^X$, $A \subseteq B$ implies $A \preceq B$.

We introduce now two axioms that we call stability with respect to contraction and extension. They reflect the following relationships between a hyper-relation and the set-inclusion: if a set provides - let us say- more freedom of choice than another, then this *a fortiori* holds for any subset of the latter, and that if a set offers - let us say- more ‘suitable alternatives’ than another set, then the set containing the former as its subset will certainly provide more ‘suitable alternatives’ than the latter.

- **Stability with respect to contraction (Cont)**. For any A, A', $B \in 2^X$, $A' \subseteq A \preceq B$ implies $A' \preceq B$.
- **Stability with respect to extension (Ext)**. For any A, B, B'^X, $A \preceq B \subseteq B'$ implies $A \preceq B'$.

[1]Notice that the hyper-relation in the example of the family ordering books is elementary.

[2]We notice here that for transitive decent hyper-relations such a decomposition has been obtained in [9], and the ’elementary’ binary hyper-relations were associated to the states of nature.

These axioms might be consider as weakening of transitivity. Namely, if a transitive hyper-relation $\preceq$ satisfies (**Mon**) then $\preceq$ is stable with respect to contraction (**Cont**) and to extension (**Ext**). This was already noticed in Lemma 1 in [9].

We say that a hyper-relation satisfies the *union property* if it satisfies the following axiom.

- **Union (U).** For any family of subsets A_i, $i \in I$, and any subset $B \subseteq X$, if, for every $i \in I$, $A_i \preceq B$, then $\cup_{i\in I} A_i \preceq B$.

For finite families, the Union axiom coincides with the Robustness axiom in [3].

Definition. A hyper-relation $\preceq$ is *decent* (*semi-decent*) if it satisfies axioms (**Mon**), (**Cont**), (**Ext**), and (**U**) ((**Mon**), (**Cont**), and (**U**), respectively).

Remark 1. For a semi-decent hyper-relation $\preceq$, and two subsets A and B, $A \preceq B$ if and only if holds $a \preceq B$ for every $a \in A$.

Let us discuss some variants of the union axiom.

Kreps ([9], axiom (2.1)) and Ryan ([18], axiom **K**) consider the following axiom

- **K** for any A, B, $A \preceq B$ implies $B \preceq A \cup B \preceq B$.

For a transitive hyper-relation which satisfies monotonicity **Mon**, the axioms **K** and **U** are equivalent. In fact, due to **K**, $A \preceq C$ and $B \preceq C$ are nothing but $C \approx A \cup C$ and $C \approx B \cup C$. By transitivity, we get $A \cup C \approx B \cup C$. The latter due to **K** reads as $A \cup C \approx (A \cup C) \cup (A \cup B) = A \cup B \cup C$, and due to transitivity, we get $A \cup B \cup C \approx C$. Then due to monotonicity $A \cup B \preceq A \cup B \cup C$, that is (due to transitivity)$A \cup B \preceq C$.

A more general form of the union axiom is the following axiom of *additivity*:

- **A** If, for some collections of menus A_i, B_i, $i \in I$, for each i there holds $A_i \preceq B_i$, then $\cup_{i\in I} A_i \preceq \cup_{i\in I} B_i$.

From the following lemma follows that any decent hyper-relation satisfies the additivity **A**.

Lemma 1. *Let $\preceq$ be a decent hyper-relation. Then $\preceq$ satisfies* ***A***.

Proof. For each $i \in I$, we have $A_i \preceq B_i \subset \cup_{i\in I} B_i$. Then, from (**Ext**), we get $A_i \preceq B$, and, due to (**U**), there holds $\cup_{i\in I} A_i \preceq \cup_{i\in I} B_i$. □

Remark 2. The class of decent (semi-decent) hyper-relations is stable under the intersection. That means that for any decent hyper-relations $\preceq_1$ and $\preceq_2$, a hyper-relation defined by $A \preceq B$ if $A \preceq_1 B$ and $A \preceq_2 B$, is decent. The stability under intersection allows us to construct new dependency relations from the existing ones. In the next section we present a more powerful construction of decent (semi-decent) hyper-relation based on the use of choice functions.

We end this section with some examples.

Examples.

(1) The set-theoretical inclusion, i.e. $A \preceq B \Leftrightarrow A \subseteq B$, is a transitive decent relation.

(2) Consider a weak order $\leq$ on X. Let us say that a subset B is better than a subset A, $A \preceq B$, if for any $a \in A$ there exists $b \in B$ such that there holds $a \leq b$. Obviously such a defined hyper-relation $\preceq$ is decent, moreover, $\preceq$ is transitive and complete. For a finite X, the reverse statement is also true: any complete and transitive decent hyper-relation is obtained by the above rule for some weak order on X. In fact, for x, $y \in X$, define $x \leq y$ if $\{x\} \preceq \{y\}$. Because of the Remark 1, we have to check that if, for some $a \in X$, $B \subseteq X$, $a \preceq B$, then there exists $b \in B$ such that $a \leq b$ holds true. Since B is a finite set, we consider a maximal element $b \in B$ with respect to $\leq$. Then, because of the union axiom **U**, there holds $B \preceq b$. Hence $a \preceq B \preceq b$, and because of transitivity we get $a \leq b$. (Kreps in [9] gave another characterization of hyper-relations of such a form.)

(3) Let $<$ be a relation on X. Then, for subsets A and B, we set $A \preceq B$ if, for every $a \in A \setminus B$, there exists an element b in $a \cup B$ such that $a < b$. (We consider here the set $a \cup B$ instead of B in order to do not exclude the possibility $a < a$.) Such a hyper-relation is said to be *binary hyper-relation*. A binary hyper-relation is a decent hyper-relation, and is a transitive decent hyper-relation if the relation $<$ is transitive.

Let us remark, that if we consider a relation $<$ which is the strict part of some weak order $\leq$ on X, then the hyper-relation defined above differs from a hyper-relation of Example 2.

(4) A transitive decent hyper-relation is nothing but a dependency relation in [5] or a complete implication system (see, for example, [7].

(5) A generalization of binary hyper-relations in the framework with uncertainty is as follows. Suppose there is a set S of states of nature. A decision maker has a list of relations $<_s$ on X which depend on $s \in S$. Then, for subsets A and B, we set $A \preceq B$ if, for every $a \in A \setminus B$, and every $s \in S$ there exists an element b in $a \cup B$ such that $a <_s b$. Such a hyper-relation is decent. Later on we show that any decent hyper-relation takes such a form.

(6) For a finite set X, Pattanaik and Xu defined the cardinality-base ordering $\preceq_c$, $A \preceq_c B$ if $|A| \leq |B|$. This hyper-relation is transitive, satisfies **Mon**, but fails to satisfy the union axiom **U** and therefore is not semi-decent.

3. Fundamental mappings

We present here constructions of a mapping from hyper-relations to choice functions and a mapping from choice functions to hyper-relations.

Let us recall that a *choice function* on X is a mapping $f : 2^X \to 2^X$, such that $f(A) \subseteq A$ for any subset $A \subseteq X$. One can consider the set $f(A)$ as a set of 'chosen' alternatives from A. We allow empty choice for some sets, that means that it might occur $f(A) = \emptyset$ for some A's. If, for any non-empty menu $A \subseteq X$, $f(A) \neq \emptyset$, then f is called *non-empty-valued*. An alternative is said to be *dummy* if $f(x) = \emptyset$. If there are no dummies, we say that a choice function is *no-dummy*. The set of all choice functions on X is denoted by **CF**.

We associate to any relation $<$ on X a choice function $f_<$ on X by the rule

$$f_<(A) = \{a \in A \mid \nexists\, a' \in A \text{ such that } a < a'\}. \tag{3.1}$$

In other words the choice of a menu A is constituted from the undominated (within A) alternatives. Such a choice function is called *rationalizable* by the relation $<$. An alternative x is dummy if and only if $x < x$.

Let us construct a mapping κ from **CF** to **HR**. For a choice function f, we define a hyper-relation $\preceq_f = \kappa(f)$ by the rule

$$A \preceq_f B \text{ if for every } a \in A \setminus B, \text{ there holds } a \notin f(a \cup B). \tag{3.2}$$

Examples 7. For the choice function $\mathbf{1}$ ($\mathbf{1}(\mathbf{A}) = \mathbf{A}$ for every menu A), the corresponding hyper-relation $\preceq_{\mathbf{1}}$ is nothing but the set-theoretical inclusion $\subseteq$.

8. Let $<$ be a relation, and let f be the choice function rationalizable by $<$. Then $\preceq_f$ is exactly the hyper-relation from Example 3.

9. Let an element $a \in X$ be a dummy for a choice function f, that is $f(a) = \emptyset$. Then $a \preceq_f \emptyset$ that is a dummy for the corresponding hyper-relation $\preceq_f$.

One can easily check that, for a choice function f, the hyper-relation $\preceq_f$ defined in (3.2) is semi-decent. Moreover, any semi-decent hyper-relation takes the form $\preceq_f$ for some (uniquely determined) choice function. Namely, we have the following

Proposition 1. *The mapping $\kappa : \mathbf{CF} \to \mathbf{HR}$, $f \mapsto \preceq_f$, defines an antimonotone bijection between the set of choice functions **CF** and the subset* **SDHR** $\subset$ **HR** *of semi-decent hyper-relations.*

To prove this proposition, we construct a 'reverse' mapping $\varphi : \mathbf{HR} \to \mathbf{CF}$, which sends a hyper-relation $\preceq$ to a choice function $f := \phi(\preceq)$ by the rule:

$$f(A) = \{a \in A \text{ such that does not hold } a \preceq A - a\}. \tag{3.3}$$

Proposition 1 follows from two claims.

Claim 1. *For every choice function f there holds $f = \varphi(\kappa(f))$.*

In fact, for a set A and an element $a \in A$ we get that a is not chosen from A due to the choice function $\varphi(\preceq_f) \Leftrightarrow a \preceq_f A - a \Leftrightarrow a \notin f(a \cup (A - a)) = f(A)$.

In particular, this establishes injectivity of the mapping κ.

Claim 2. *For a semi-decent hyper-relation $\preceq$ there holds $\preceq = \kappa(\varphi(\preceq))$.*

Let f be a choice function defined by (3.3), $f = \varphi(\preceq)$. We have to show that $A \preceq B$ holds true if and only if $A \preceq_f B$ holds true. Since hyper-relations $\preceq$ and $\preceq_f$ satisfy the axioms **Cont** and **U**, one can assume that A is a singleton a. If $a \in B$, then $a \preceq B$ and $a \preceq_f B$ hold true. Suppose now that $a \notin B$. Then we have the following sequence of equivalences

$$a \preceq_f B \Leftrightarrow a \notin f(a \cup B) = \varphi(\preceq)(a \cup B) \Leftrightarrow a \preceq (a \cup B - a) = B.$$

This proves the claim 2 and the proposition. □

Remarks. 1. In [15] the notion of essential elements was introduced, namely, for a hyper-relation $\preceq$, an alternative $a \in A$ such that $A \succ A - a$ is said to be *essential* in A. Puppe in [15] considers a mapping $\mathcal{E} : \mathbf{HR} \to \mathbf{CF}$, $\mathcal{E}(\preceq)(A) = \{\text{essential elements in } A\}$. (We allow $\mathcal{E}(\preceq)(A)$ to be the empty set, but in [15] it was postulated non-emptyness of such sets.) We claim that the mappings $\mathcal{E}$ and φ coincide on the set of semi-decent hyper-relations **SDHR**.

In fact, $a \notin f_{\preceq}(A)$ if $a \preceq A - a$. By the union axiom **U** and monotonicity **Mon** ($A - a \preceq A - a$), $a \preceq A - a$ implies $A \preceq A - a$, that means that $a \notin \mathcal{E}(\preceq)(A)$. This shows the inclusion $\mathcal{E}(\preceq)(A) \subseteq f_{\preceq}(A)$. For showing the reverse inclusion, let us consider an element $a \notin \mathcal{E}(\preceq)(A)$, that is $A \preceq A - a$. We have $a \subset A \preceq A - a$, and hence due to **Cont**, there holds $a \preceq A - a$. This establish the reverse inclusion and the claim is proven.

2. There are at least three more possible mappings from the set of choice functions to the set of hyper-relations.

The first mapping is defined by associating to a choice function f a hyper-relation $\preceq'_f$ defined by

$$A \preceq'_f B \Longleftrightarrow a \notin f(a \cup B) \quad \forall\, a \in f(A) - B.$$

That means that there holds $a \notin f(a \cup B)$ only for elements $a \in f(A) - B$. Obviously, $A \preceq_f B$ implies $A \preceq'_f B$. However, such a defined hyper-relation satisfies only the Monotonicity axiom **Mon**.

The second mapping was considered in [15]: for a choice function f, a hyper-relation is defined by the rule

$$A \preceq''_f B \Longleftrightarrow f(A \cup B) \subseteq B.$$

This hyper-relation also satisfies only the Monotonicity axiom **Mon**. Let us notice, that if f is a heritage choice function (see next section for a formal definition of what is a heritage choice function) then the following inclusion holds true $\preceq_f \subseteq \preceq''_f$, that is $A \preceq_f B \Rightarrow A \preceq''_f B$. (For a reader being familiar with the heritage choice functions we give a proof of the latter claim. Suppose $A \not\preceq''_f B$, that is there exists $a \in f(A \cup B)$ such that $a \notin B$. Then since f is heritage choice function and $a \in A - B$, we get $a \in f(a \cup B)$ ($a \cup B \subset A \cup B$ and $a \in f(A \cup B)$). Hence $A \not\preceq_f B$, since $A \preceq_f B$ implies $a \notin f(a \cup B)$).

One more mapping was considered in [16, 18]

$$A \preceq^*_f B \Longleftrightarrow f(A \cup B) \cap B \neq \emptyset.$$

Such a defined hyper-relation satisfies only **Mon**. Thus, these three mapping do not have so many properties as the mapping defined in (3.2).

The set **CF** of all choice function is stable with respect to the union, and any choice function can be decomposed into the union of concordant choice functions. Recall that a choice function is *concordant* if it satisfies the following axiom

- **C** Let A_i, $i \in I$ be a non-empty collection of menus, and let a be such that $a \in f(A_i)$ for all $i \in I$. Then $a \in f(\cup A_i)$.

A semi-decent hyper-relation is said to be *concordant* if the mirror of the above condition is satisfied:

- If $x \preceq \cup_{i\in I} A_i$, then $x \preceq A_i$ for some $i \in I$.

For example, for a concordant hyper-relation $\preceq$, there holds if $x \preceq A$ for non-empty A, then $x \preceq a$ for some $a \in A$.

Thus, we get that any semi-decent hyper-relation can be represented as the intersection of concordant hyper-relations.

4. Decent hyper-relations and Heritage choice functions

In this section we prove that the mapping (3.3) establishes a bijection between the class **HCF** of the heritage choice functions and the class **DHR** of the decent hyper-relations.

Recall, that a choice function f is *heritage* function if it satisfies the *heredity* property

- **H** For any A, B, if $A \subseteq B$ then there holds $f(B) \cap A \subset f(A)$.

In other words, if an alternative $a \in A$ is not chosen in a smaller set A, then it is still the case in a bigger set B. For other equivalent formulation of this axiom[3] see, for example, [11, 12]

Proposition 2. *Let f be a heritage choice function. Then the hyper-relation $\preceq$, defined by (3.2), satisfies **Ext**, and hence (see Proposition 1) the hyper-relation $\preceq_f$ is decent.*

Proof. Consider three menus A, B, B' such that there holds $A \preceq B' \subseteq B$. Then, we have to show that $A \preceq B$ holds true. Let $a \in A - B$. Hence $a \in A - B'$ and, due to $A \preceq B'$, we have $a \notin f(a \cup B')$. Because of the heritage axiom **H**, a is also not chosen from a larger sets $a \cup B$. This implies $A \preceq B$. □

Proposition 3. *Let a hyper-relation $\preceq$ satisfy the axiom **Ext**. Then the choice function $c = c_{\preceq}$, defined by (3.3), is a heritage choice function.*

Proof. Let $A \subseteq B$, and let $a \in A \cap B$ be such that $a \notin c(A)$. Then, due to the rule (3.3), the latter means that $a \preceq A - a$. Since $A - a \subseteq B - a$ and $a \preceq A - a$, due to **Ext**, we get $a \preceq B - a$, that is $a \notin c(B)$. □

Propositions 1–3 together establish the following

Theorem 1. *The mapping φ being restricted to the set **DHR** of the decent hyper-relation is a bijection with the set **HCF** of the heritage choice functions.*

For decent hyper-relations (and heritage choice functions), the decomposition procedure is more interesting than for semi-decent ones. Namely, any heritage choice function can be represented as the union of choice functions, which satisfy axioms **H** and **C**. Any concordant heritage choice functions is rationalizable by a relation on X (see (3.1)). Because of this, we get the following

Proposition 4. *Any decent hyper-relation takes the form of a hyper-relation from Example 5.*

[3] In the literature on stable matchings [17], the Heritage axiom is interpreted in terms of *substitutability*: if a worker a is hired by a firm from the list A, she will be also hired in any shorter list $A \setminus a'$, $a' \neq a$.

Let a heritage choice function $f \in \mathbf{HCF}$ be decomposed as the union of a collection $\{f_i\}_{i\in I}$ of choice functions rationalizable by relations $\{<_i\}_{i\in I}$. Assume X is finite. If among the relations $\{<_i\}_{i\in I}$ there is at least one that is acyclic, then the corresponding choice function f_i is non-empty-valued. Hence f is also non-empty-valued. The reverse is also true. Namely, if a heritage choice function f is non-empty-valued, then there exists a decomposition $f = \cup f_{<_i}$, such that at least one of the relations $<_i$ is linear (hence such $<_i$ is acyclic).

Proposition 5. *Let $f \in \mathbf{HCF}$ be a heritage choice function on a finite set X, and let $\preceq = \preceq_f$ be the corresponding decent hyper-relation. Then the following are equivalent:*

(1)
(2) *f is non-empty-valued;*
(3) *There exists a linear order $<$ on X, such that the rationalizable choice function $f_<$ is inferior than f (that is, for any $A \in 2^X$, $f_<(A) \subseteq f(A)$);*
(4) *$\preceq$ is majorated by a binary hyper-relation associated to some linear order on X;*
(5) *The Plottization of f (a maximal Plott function dominated by f) is non-empty-valued.*

Proof. It is obvious that items 2 and 3 are equivalent. 2) $\Rightarrow$ 4) since Plottization of f contains the choice function $f_<$. It is also obvious that 4) $\Rightarrow$ 1). So, it remains to check the implication 1) $\Rightarrow$ 2).

Let x_1 belong to a non-empty set $f(X)$. Set $X_2 = X - x_1$. If X_2 is non-empty, then pick an element $x_2 \in f(X_2)$. Set $X_3 = X_2 - x_2$, and pick $x_3 \in f(X_3)$, and so on until we exhaust the whole X. Take a linear order $x_1 > x_2 > x_3 > \ldots$. We assert that the choice function $f_<$ rationalizable by $<$ is inferior than f, that is, for any menu A, the inclusion $f_<(A) \subseteq f(A)$ holds true. Let a be maximal element in A with respect to the linear order $<$ being restricted to A. Then $f_<(A) = \{a\}$. It suffices to show that $a \in f(A)$. Let $a = x_k$. This means that $A \subseteq X_k$. By the construction of the linear order $<$, we have $a = x_k \in f(X_k)$. Since $a \in A$, from the heritage axiom $\mathbf{H}$ follows $a \in f(A)$. □

Remark. Let $\preceq$ be a decent hyper-relation, such that the corresponding choice function $f = \varphi(\preceq)$ is non-empty valued. Then consider the following chain of sets $X_0 := X$, $X_1 := X_0 \setminus f(X_0)$, $X_2 := X_1 \setminus f(X_1)$, ...

For a set A define the *rank* of A, $rk(A)$, as a number k such that $A \subseteq X_k$ and $A \not\subseteq X_{k+1}$. Then we define a complete ordering by the rule

$$A \ll B \text{ if } rk(A) \geq rk(B).$$

This is a complete transitive decent hyper-relation. We claim that $A \preceq B$ implies $A \ll B$.

From the contrary. Suppose $A \preceq B$, but $rk(A) < rk(B)$. Let $rk(A) = k$ and $rk(B) > k$. Then there exists $a \in A$ such that $a \in f(X_k)$. Since $B \subseteq X_{k+1} = X_k - f(X_k)$, $a \notin B$. Moreover, $a \cup B \subseteq X_k$. Due to $\mathbf{H}$, we get that such a belongs to $f(a \cup B)$, that is $a \not\preceq B$, that contradicts to $A \preceq B$. □

Let us remark, that this rank function generalizes the *peeling rank* from Statistics. Namely, for a finite set of points X in an Euclidean space, let us consider the following chain of sets $X_0 := X$, $X_1 := X_0 \setminus ex(X_0)$, where $ex(Y)$ denotes the set of points of Y which belong to the boundary of its convex hull, $X_2 := X_1 \setminus ex(X_1)$, ... Then, for a subset $A \subset X$ its rank is defined as above for this chain. This peeling rank is used in non-parametric rank tests [8].

5. Transitive decent hyper-relations

Kreps in [9] studies the transitive decent hyper-relations. More precisely, in a setting with no dummy alternatives, a decomposition-type result was obtained in [9] based on a relationship between the transitive decent hyper-relations and the closure operators[4]. Since Kreps was not aware of the relation between the closure operators and the closed choice functions (see [6]), he did not establish any relations between the transitive decent hyper-relations and the closed choice functions. In this section, we fill this gap and also obtain a decomposition of a closed choice function as the union of choice functions rationalizable by para-transitive relations.

For a decent hyper-relation $\preceq$, denote by $D := \{d \in X \mid d \preceq \emptyset\}$ the set of of all dummy elements of X. Because of the following lemma, we can consider no-dummy transitive decent hyper-relations. (It is worth noticing here that for a no-dummy transitive decent hyper-relation the corresponding choice function can attain empty values for the sets whose cardinality is more than one.)

Lemma 2. *Let $\preceq$ be a transitive decent hyper-relation. Then the following are equivalent:*

(1) $A \preceq B$;

(2) $A - D \preceq B - D$.

Proof. We claim that, for any menu B, there holds $B \preceq B - D$. In fact, if an element b from B is not dummy, it lies in $B - D$ and $b \preceq B - D$. If the element b is dummy then we have $b \preceq \emptyset \subseteq B - D$, and by **Ext** there holds $b \preceq B - D$. Therefore $B \preceq B - D$.

Let now $A \preceq B$. Then, since $A - D \subseteq A \preceq B$, we get $A - D \preceq B$. Because $B \preceq B - D$, transitivity of $\preceq$ implies $A - D \preceq B - D$.

Vice versa. Let $A - D \preceq B - D$. Then, since $A \preceq A - D$, we get, due to transitivity, $A \preceq B - D$. Since $B - D \subseteq B$, by the axiom **Ext**, we get $A \preceq B$. □

Thus, without loss of generality, we consider no-dummy transitive hyper-relations.

Example 10. Let $\leq$ be a pre-order on X, that is a reflexive and transitive binary relation on X. Define a hyper-relation $\preceq := \preceq_{\leq}$ by the rule (compare with Example 2):

$$A \preceq B, \text{ if for any } a \in A, \text{ there exists } b \in B \text{ such that } a \leq b.$$

Such a defined hyper-relation is a no-dummy transitive decent hyper-relation. We call such a hyper-relation as a *hyper-relation associated with a pre-order*. In this section we show that any no-dummy

[4]Notice that the transitive decent hyper-relations have been studied in the literature on implication systems with the name of *complete implication system*, and the key Lemma 2 in [9] was proven by Anderson in 1974 see, for example, [7].

transitive decent hyper-relation takes the form of the intersection of a collection of hyper-relations associated with pre-orders.

Let us consider the following two prominent preorders: (1) $\leq$ is a weak order (see Example 2 above), (2) $\leq$ is a dichotomous weak order (namely, there exist at most two sets of equivalence). Then there holds

Proposition 6. *The following are equivalent:*

(1)
(2) *a hyper-relation $\preceq$ is no-dummy, transitive, and decent;*
(3) *$\preceq$ is the intersection of hyper-relations associated with pre-orders hyper-relations;*
(4) *$\preceq$ is the intersection of hyper-relations associated with weak orders;*
(5) *$\preceq$ is the intersection of hyper-relations associated with dichotomous weak orders.*

Proof. The implications 4) $\Rightarrow$ 3) $\Rightarrow$ 2) $\Rightarrow$ 1) are obvious. Thus, we prove the implication 1) $\Rightarrow$ 4).

Let $\preceq$ be a no-dummy transitive decent hyper-relation. A set F is called *closed*, if $x \preceq F$ implies $x \in F$. For example X is a closed menu. Since $\preceq$ is no-dummy, the empty set $\emptyset$ is closed. For a set A, we consider the set $\sigma(A) := \{x \in X \mid x \preceq A\}$. There holds $A \subseteq \sigma(A) \preceq A$.

Claim A. *The set $\sigma(A)$ is the minimal closed set which contains A.*

Firstly, $\sigma(A)$ is closed. In fact, let $x \preceq \sigma(A)$. Then, since $\sigma(A) \preceq A$, due to transitivity, we get $x \preceq A$, and hence $x \in \sigma(A)$. Secondly, let F closed and let $A \subseteq F$. Since, for any $x \in \sigma(A)$, we get $x \preceq A$, due to the axiom **Ext**, there holds $x \preceq F$, and, since F is closed, $x \in F$ holds true. Thus $\sigma(A) \subseteq F$ and the claim holds true. □

As a consequence of this claim, we get that F is closed iff $F = \sigma(F)$.

Claim B. *$A \preceq B$ if and only if $A \subseteq \sigma(B)$.*

The implication $A \preceq B \Rightarrow A \subseteq \sigma(B)$ is obvious. The reverse implication follows from $A \subseteq \sigma(B) \preceq B$ and the axiom **Cont** . □

To prove the implication 1) $\Rightarrow$ 4), let us define for each closed menu F a dichotomous weak order $\leq_F$ on X by the rule:

$$x \leq_F y, \text{ if either } x \in F, \text{ or } y \notin F.$$

The hyper-relation $\preceq_F$ associated with $\leq_F$ is as follows:

$$A \preceq_F B \text{ if either } A \subseteq F, \text{ or } B \not\subseteq F.$$

Claim C. *The hyper-relation $\preceq$ is equal to the intersection of hyper-relations $\preceq_F$, while F runs over the set of closed menus.*

Suppose $A \preceq B$, that is equivalent to $A \subseteq \sigma(B)$ (Claim B). We have to check that, for any closed menu F, $A \preceq_F B$ holds true. If $A \subseteq F$, then we are done. Assume $A \not\subseteq F$, then we show that there holds $B \not\subseteq F$ (that implies $A \preceq_F B$). From the contrary, suppose $B \subseteq F$,then due to Claim A, we have $\sigma(B) \subseteq F$. But due to the Claim B, we have $A \subseteq \sigma(B)$, that implies $A \subseteq F$, a contradiction. $\square$

Remark. The equivalence 1) $\Leftrightarrow$ 3) has been established in [9] (later this equivalence was established by Malishevski in [10]. The meet-presentation of no-dummy transitive decent hyper-relations due to Kreps [9] also follows from this. Let $\preceq = \cap_{s\in S} \leq_s$ be the decomposition from the item 3 of Proposition 6, and let u_s be a utility function representing $\leq_s$, for any state of the nature $s \in S$. Then, for a menu $A \subseteq X$ define the vector $u(A) \in \mathbb{R}^S$ as $u(A) = (\max_{a\in A} u_s(a), s \in S)$. Hence, $A \preceq B$ if and only if $u(A) \leq u(B)$ coordinate-wise.

Now we present a characterization of transitive decent hyper-relations in terms of choice functions. For that we need to recall the notion of a closed choice function.

Definition. A choice function f is said to be *closed* if f is heritage and satisfies the following axiom

- **W** Let $x \notin A$. If $x \in f(x \cup A)$ and $y \notin f(y \cup A)$ for every y from some set Y, then $x \in f(x \cup Y \cup A)$.

Remark. In [6] a particular form of this axiom with singleton Y was considered. For a finite X, this axiom is equivalent to its particular form with singleton Y.

Theorem 2. *For a decent hyper-relation $\preceq$ the following are equivalent*

(1) *The hyper-relation $\preceq$ is transitive;*
(2) *The corresponding choice function $f = \varphi(\preceq)$ is closed.*

Proof. Without loss of generality, we assume that $\preceq$ and f are no-dummy.

The implication 1) $\Rightarrow$ 2). Let x, y and A be as in the formulation of **W**, and suppose that $x \in f(x \cup A)$, $y \notin f(y \cup A)$ for every $y \in Y$, but $x \notin f(x \cup Y \cup A)$. Due to definition of the mapping φ, $y \notin f(y \cup A)$ means that $y \preceq A$ for every $y \in Y$. Hence $Y \preceq A$ and (due to **U**) $Y \cup A \preceq A$. The relation $x \notin f(x \cup Y \cup A)$ means $x \preceq A \cup Y$. Now, due to transitivity of $\preceq$, we get $x \preceq A$, that is $x \notin f(x \cup A)$. A contradiction.

The reverse implication 2) $\Rightarrow$ 1). We have to check that, for a closed choice function f, the hyper-relation $\preceq := \preceq_f$ is transitive. That is $A \preceq B$ and $B \preceq C$ imply $A \preceq C$. Because of the union axiom **U**, we can assume that A is a singleton. Thus $A = \{a\}$; obviously we can assume that $a \notin C \cup B$.

Let us set $Y = B - C$. Since for every $y \in Y$ there holds $y \preceq C$ and $y \notin C$, we have $y \notin f(y \cup C)$ for every $y \in Y$. Suppose that $a \not\preceq C$. Then $a \in f(a \cup C)$, and from the axiom **W** we have $a \in f(a \cup Y \cup C) = f(a \cup B)$. The latter means that $a \not\preceq B$ that contradicts the assumption that $a \preceq B$. $\square$

Now, we use Proposition 6 in order to get a representation of a closed choice function as the union of binary choice functions.

Let $\leq$ be a pre-order, let $\preceq$ be a hyper-relation associated to $\leq$, and let $f = \varphi(\preceq)$. Then $a \in f(A) \Leftrightarrow$ does not true $a \preceq A - a \Leftrightarrow$ does not true that there exists $x \in A - a$ such that $a \leq x \Leftrightarrow$ for any $x \in A - a$ does not true $a \leq x \Leftrightarrow$ if $a \leq x$ for some $x \in A$, then $x = a$.

This means that a maximal element of $\leq$ within A belongs to $f(A)$ if and only if A does not contain elements being equivalent to this maximal element.

For a pre-order $\leq$, let us define a relation $<$ on X by the rule:

$$x < y \text{ if and only if } x \neq y \text{ and } x \leq y,$$

Then the implication 'if $a \leq x$ for some $x \in A$, then $x = a$ ' reads as follow: for any $x \in A$, the relation $a < x$ is not the case.

Such a defined relation $<$ has an internal characterization. Namely, let us say that a relation $<$ on X is *para-transitive*, if $<$ is irreflexive, and from $x < y < z$ and $x \neq z$ it follow that $x < z$. It is obvious, that if $<$ is defined as above by a pre-order $\leq$, then $<$ is para-transitive. Vice versa, let $<$ be a para-transitive hyper-relation. Then define a relation $\leq$ by the rule

$$x \leq y, \text{ if } x < y \text{ or } x = y.$$

Such a defined relation $\leq$ is reflexive (obviously) and transitive. In fact, let $x \leq y \leq z$. Then if $x = y$ or $y = z$, then $x \leq z$. If $x \neq y$ and $y \neq z$, then $x < y < z$, and, due to para-transitivity, we have $x < z$ and hence $x \leq z$. Thus, summing up, we get

Corollary. *Any closed (and no-dummy) choice function can be represented as the union of choice functions rationalizable by para-transitive relations.*

6. Meet-linear hyper-relations and Plott choice functions

In this section we assume that X is a finite set. For a linear order $<$ on X, the a hyper-relation associated to $<$ (see Example 8) is said to be *quasi-linear* hyper-relation. We consider here the case with no-dummies, while the case with dummies requires only a rather simple modification.

In this section we consider a subclass of the transitive decent hyper-relations which can be represented as the intersections of the quasi-linear hyper-relation. This subclass was considered in several papers (see the survey [3]). Nerhing and Puppe in [13] establish a relation between such a subclass and the Plott choice functions. In [5] this class is studied under the name *framed dependency relations*. Let us recall that a *Plott choice function* is a choice function that satisfies the so-called *path-independence* property, i.e.: for any A and B

- **PI** $\quad f(A \cup B) = f(f(A) \cup B)$

is a Plott function. It is easy to check that any no-dummy Plott function is non-empty valued.

Let us say that a hyper-relation satisfies *Lattice equivalence* if, for any equivalent sets A and B, that is $A \preceq B$ and $B \preceq A$, the following axiom holds

- **LE** $\quad A \cup B \preceq A \cap B.$

Note, that, for a transitive hyper-relation which satisfies **Mon**, the axiom **LE** implies equivalence $A \cap B$ and $A \cup B$.

Theorem 3. *Let $\preceq$ be a transitive decent no-dummy hyper-relation and let $f := \varphi(\preceq)$ be the corresponding choice function. Then the following are equivalent*

(1) *$\preceq$ has a representation as the intersection of quasi-linear hyper-relation;*
(2) *$\preceq$ satisfies the axiom **LE**;*
(3) *f is a Plott function.*

Proof. Because the axiom **LE** is stable under intersection, that is the intersection of hyper-relations satisfying **LE** also satisfies **LE**, we get the implication 1) $\Rightarrow$ 2).

For the implication 2) $\Rightarrow$ 3), recall that, for $a \in A$, $a \notin f(A)$ if and only if $a \preceq A - a$. Because of the union axiom **U** this implies $A \preceq A - a$, or equivalently, A is equivalent to $A - a$. Thus, $f(A)$ is equal to the intersection of all $A - a$ which are equivalent to A. This and **LE** imply $A \preceq f(A)$.

Thus, we have to show that f satisfies **H** and the outcast axiom **O** (for equivalence between **PI** and these two axioms see, for example, [4, 12]). **H** is satisfied because $\preceq$ is decent. Recall the outcast axiom[5]

- **O** For A and B, if $f(A) \subseteq B \subseteq A$ then $f(A) = f(B)$.

The inclusion $f(A) \subseteq f(B)$ follows from the heritage axiom **H**. Thus, to establish 2) $\Rightarrow$ 3) it remains to verify the inclusion $f(B) \subseteq f(A)$. Suppose on the contrary that there exists and element $b \in f(B)$ such that $b \notin f(A)$. Then $f(A) \subseteq B - b$. Since B is equivalent A and the latter set is equivalent to $f(A)$, due to **Cont** it follows $b \preceq f(A)$. Then, due to **Ext**, we get $b \preceq B - b$, that is equivalent to $b \notin f(B)$. A contradiction. Thus f is a Plott function.

The implication 3) $\Rightarrow$ 1). It is known (see [1, 4, 13]) that a Plott function f is equal to the union of a collection linear Plott functions (a choice function rationalizable by a linear order is a *linear Plott function*). Therefore, the hyper-relation $\preceq$ is equal to the intersection of the corresponding quasi-linear hyper-relations. □

It is worth noticing that the equality in the axiom **PI** might be relaxed. Namely, we have the following:

Proposition 7. *A choice function f is a Plott function if and only if, for any A and $B \in 2^X$, there holds*

$$f(f(A) \cup B) \subseteq f(A \cup B) \subseteq f(A) \cup B. \tag{6.1}$$

Proof. Obviously, **PI** implies (6.1). Let us check the reverse implication. Let us first check validity of axioms **H**: Let $A \subseteq B$. Then denote by $C = B - A$ and apply the second inclusion in (6.1) to sets A and C. We get

$$f(B) = f(A \cup C) \subseteq f(A) \cup C.$$

5) In the language of essential elements, this axiom takes the form: if $A \preceq A - x$ and $A \preceq A - y$, then $A \preceq A - x - y$

Hence $f(B) \cap A \subseteq (f(A) \cup C) \cap A = f(A)$, the latter equality holds because $A \cap C = \emptyset$. Therefore, the heritage axiom **H** holds true.

Because of this, since $f(A) \cup B \subset A \cup B$, we get

$$f(A \cup B) \cap (f(A) \cup B) \subseteq f(f(A) \cup B).$$

This inclusion together with (6.1) give $f(A \cup B) = f(f(A) \cup B)$, that is **PI**. □

In Remark 2 after Proposition 1 (Section 3), we defined two other mappings from choice functions to hyper-relations $f \mapsto \preceq'_f$ and $f \mapsto \preceq''_f$. It turns out that for the subclass of Plott functions, these mappings coincide with the main mapping $f \mapsto \preceq_f$ and with mapping from [5]. Namely, we have the following

Proposition 8. *Let f be a Plott function. Then, for any sets A and B, the following are equivalent*

(1) $A \preceq_f B$;
(2) $A \preceq'_f B$;
(3) $A \preceq''_f B$ *(that is $f(A \cup B) \subseteq B$)*;
(4) $f(A \cup B) = f(B)$.

Proof. The implication 1) $\Longrightarrow$ 2) always true.

Let us check 2) $\Longrightarrow$ 3). Because of **PI**, we have $f(A \cup B) = f(f(A) \cup B)$. Let us consider the set $f(A) \cup B$ as the union of sets $a \cup B$, while a runs over the set $f(A)$. Then, due to **PI**, we have

$$f(f(A) \cup B) = f(\cup_{a \in f(A)} f(a \cup B)).$$

Because of 2), for any $a \in f(A)$, we have $f(a \cup B) \subseteq B$. Hence due to the union axiom **U**, there holds $\cup_{a \in f(A)} f(a \cup B) \subseteq B$, from that follows $f(A \cup B) \subseteq B$.

The implication 3) $\Longrightarrow$ 4): $f(A \cup B) = f((A \cup B) \cup B) = f(f(A \cup B) \cup B) = f(B)$ (the latter equality holds due to $f(A \cup B) \subseteq B$.

Finally, the implication 4) $\Longrightarrow$ 1). Let us verify, that, for any $a \in A$, there holds $f(a \cup B) \subseteq B$. In fact,

$$f(a \cup B) = f(a \cup f(B)) = f(a \cup f(A \cup B)) = f(a \cup A \cup B) = f(A \cup B) \subseteq B.$$

□

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