UNIVERSITÀ DEGLI STUDI DI SIENA

QUADERNI DEL DIPARTIMENTO
DI ECONOMIA POLITICA E STATISTICA

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Unimodality and equivalence of simple and coalitional strategy-proofness in convex idempotent interval spaces

n. 668 – Dicembre 2012

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November 28, 2012

Abstract

It is shown that simple and coalitional strategy-proofness of a voting rule as defined on the full unimodal domain of a convex idempotent interval space (X, I) are equivalent properties if (X, I) satisfies *interval anti-exchange*, a basic property also shared by a large class of convex geometries including -but not reducing to- trees and linear geometries. Therefore, *strategy-proof location problems in a vast class of networks* fall under the scope of that statement.

It is also established that a much weaker *minimal anti-exchange* property is necessary to ensure equivalence of simple and coalitional strategy-proofness in that setting. An immediate corollary to that result is that such 'unimodal' equivalence fails to hold both in certan median interval spaces including those induced by bounded median semilattices that are not chains, and in certain non-median interval spaces including those induced by Hamming graphs and partial cubes.

MSC 2010 Classification: 05C05, 52021, 52037
JEL Classification number : D71

1 Introduction

It is well-known that nontrivial nondictatorial strategy-proof voting rules can be defined on certain domains of unimodal preference profiles. It is also known that in some of those unimodal domains all the strategy-proof voting rules are coalitionally strategy-proof as well i.e. immune to coalitional manipulations, while in other cases they are not (see e.g. Moulin (1980), Danilov (1994), Nehring, Puppe (2007(b)), Barberà, Berga, Moreno (2010), Savaglio, Vannucci (2012)). Or, to put it in slightly more precise terms, for some full unimodal domains simple (or individual) strategy-proofness and coalitional strategy-proofness turn out to be equivalent properties, while for others that equivalence fails. But then, what is the dividing line between such an equivalence and its failure i.e. under which cases does a full unimodal domain support equivalence of simple/individual and coalitional strategy-proofness?

The present paper purports to address the foregoing open issue in a suitably general setting in order to cover -among others- *strategy-proof location problems in a vast class of networks*. Indeed, the most fitting environment to introduce the general notion of unimodality is perhaps an *interval space*. An interval space is a set X endowed with a suitable *interval function* $I : X^2 \to \mathcal{P}(X)$ mapping each pair of points of X into a subset of X denoting their 'interval' namely the set of points located 'between' them (see e.g. Sholander (1952, 1954), Prenowitz, Jantosciak (1979), Mulder (1980), van de Vel (1993), Coppel (1998), Nebeský (2007), Mulder, Nebeský (2009), Chvátal, Rautenbach, Schäfer (2011)). In particular, interval spaces are said to be *median* if for any three points a, b, c, the intervals of their three pairs have precisely one point in common (their median). A total preorder on a certain interval space is *unimodal* if it has a unique maximum and is such that for any a, b, c of the underlying space, if c lies 'between' a and b then its lower contour must include at least one of the latter. It is well-known that under many relevant specifications of the interval space (including the -median- interval spaces induced by finite chains, by bounded chains of the extended real line, by bounded median semilattices, by bounded undirected trees, by bounded median graphs) and, possibly, under some slight variation on the notion of unimodality, there exist nondictatorial strategy-proof voting rules on unimodal domains even allowing for 'many' -i.e. more than just two- possible outcomes. Moreover, it has also been shown that in a few key cases namely bounded chains (see Moulin (1980)) and bounded undirected trees (see Danilov (1994), Danilov, Sotskov (2002)) *all* the strategy-proof voting rules on full unimodal domains are also *coalitionally strategy-proof*, hence simple (or individual) strategy-proofness and coalitional strategy-proofness turn out to be equivalent properties. But then, to what extent such an equivalence between simple and coalitional strategy-proofness of voting rules on full unimodal domains can be generalized to other interval spaces? Partial information on that issue is provided by Le Breton, Zaporozhets (2009) and Barberà, Berga, Moreno (2010) that identify certain sufficient conditions for equivalence in more general domains under a stricter notion of unimodality (see also Schummer, Vohra(2002) for some relevant results on strategy-proofness in possibly cyclic, non-median graphs). Nehring, Puppe (2007a, 2007b) do not address explicitly that issue but establish some results suggesting equivalence failure for unimodal domains of *linear orders* on some median interval spaces, while Savaglio, Vannucci (2012) show equivalence failure for unimodal domains on the (median) interval spaces induced by bounded distributive lattices that are not chains. However, to the best knowledge of the author, no result is available for general, minimally 'regular' -i.e. convex and idempotent- interval spaces (an interval space is denoted here 'convex' if its intervals are convex in the obvious sense, and 'idempotent' if the degenerate interval between one point and itself reduces to the very same point).

The present paper addresses the issue of equivalence between simple and coalitional strategy-proofness of voting rules on full unimodal domains in such a general setting of minimally 'regular' interval spaces. A sufficient condition for equivalence is provided: it is shown that equivalence holds whenever the interval space satisfies a certain 'Interval Anti-Exchange' property that is sometimes used as a basic axiom to characterize linear geometries among convex geometries (recall that standard Euclidean convex sets amount in fact to a very special subclass of linear geometries), and is shared by all trees. A significant implication of that result for *location problems in networks* is quite clear: *whenever the network is a tree, a linear geometry or indeed any graph whose interval function is convex, idempotent and satisfies Interval Anti-Exchange, any strategy-proof voting rule for the corresponding full unimodal domain is also coalitionally strategy-proof on that domain.*

A much weaker 'Minimal Anti-Exchange' property is also shown to be a *necessary* condition for equivalence of simple and coalitional strategy-proofness of voting rules on full unimodal domains. It follows that, as a consequence, equivalence fails to hold in any median interval space induced by a bounded median semilattice (or bounded median graph) that is not a chain and in a large class of non-median interval spaces, including those induced by Hamming and partial cubes as discussed below. Such an equivalence failure is established by proving the existence of a nontrivial nondictatorial strategy-proof voting rule on the relevant full unimodal domain that admits at least four distinct outcomes in its range and is not immune to coalitional manipulations.

2 Simple and coalitional strategy-proofness on full unimodal domains: equivalent properties or not?

Let us consider a location problem on a network (or graph), or on a suitably ordered structure, to be settled by a voting rule under the assumption that the voters' preferences are unimodal. Then, it is quite natural to focus on strategy-proof voting rules, namely on those rules that are immune to simple/individual manipulations. Moreover, it is also worth asking which -if any- of the available strategy-proof rules are also coalitionally strategy-proof i.e. immune to coalitional manipulations. In particular, 'unimodal' equivalence between simple and coalitional strategy proofness obtains on a certain outcome space whenever all of the voting rules that are strategy-proof on the full unimodal domain of that space turn out to be coalitionally strategy-proof as well.

Those issues have been partially explored in some specific classes of outcome spaces, including some median interval spaces. As mentioned above, median interval spaces are those interval spaces such that the intervals of any three points have precisely one point in common, their *median*. Indeed, some facts about equivalence of simple and coalitional strategy-proofness (or lack of it) on full unimodal domains in some specific median interval spaces are well-known. That is largely due to the circumstance that the structure of strategy-proof voting rules in those spaces is now well understood: in fact, it has been established that strategy-proof voting rules on unimodal domains in median interval spaces can be represented by iterated medians of projections (i.e. dictatorial rules) and constants (see e.g. Moulin (1980), Danilov (1994), Savaglio, Vannucci (2012)). Let us then start with a quick review of the best known classes of examples:

The outcome space is a finite or bounded chain

If (X, I) is the median interval space induced by a bounded chain $(X, \leq)$ -with $I(x, y) = \{z \in X : x \leq z \leq y \text{ or } y \leq z \leq x\ \}$ for all $x, y \in X$- the equivalence-issue is settled by the pioneering work of Moulin (1980), showing that (i) the strategy-proof rules for the full unimodal domain on (X, I) are precisely those which can be represented as iterated medians of projections (i.e. dictatorial rules) and constants, and (ii) all such strategy-proof rules are also coalitionally strategy-proof on the same domain. Thus, simple strategy-proofness and coalitional strategy-proofness are equivalent properties here. In particular, the ordinary (extended) median rule is coalitionally strategy-proof.

The outcome space is a bounded tree

If (X, I) is the median interval space induced by a (discrete) bounded tree (i.e. a bounded connected graph without cycles) -with

$I(x, y) = \{z \in X : z$ lies on the unique shortest path joining x and $y\}$

for all $x, y \in X$-

the equivalence-issue is also settled by Danilov (1994), showing that (i) the strategy-proof rules for the full unimodal domain on (X, I) are precisely those which can be represented as iterated medians of projections (i.e. dictatorial rules) and constants, and (ii) all such strategy-proof rules are also coalitionally strategy-proof on the same domain. Thus, simple strategy-proofness and coalitional strategy-proofness are equivalent properties for full unimodal domains in bounded trees. In particular, the ordinary (extended) median rule is coalitionally strategy-proof.

The outcome space is a bounded distributive lattice (or its covering graph)

If (X, I) is the (median) interval space induced by an arbitrary bounded distributive lattice $(X, \leqslant)$ that is not a chain (as defined by the usual rule $I(x, y) = \{z : x \wedge y \leqslant z \leqslant x \vee y\}$, where $\wedge$ and $\vee$ denote the $\leqslant$-induced g.l.b. and l.u.b. operations), the equivalence-issue is also already settled *in the negative* by Savaglio,Vannucci (2012) showing that (i) the strategy-proof rules for the full unimodal domain on (X, I) are precisely those which can be represented as iterated medians of projections (i.e. dictatorial rules) and constants, and (ii) if $(X, \leqslant)$ is a bounded distributive lattice but is not a chain, then there are strategy-proof voting rules on that domain that are not coalitionally strategy-proof (but see also Nehring, Puppe (2007 (a),(b)) that do not address the equivalence-issue as such, but include some related, remarkable observations and results). Notice that the former equivalence-failure result can be instantly extended to the interval space induced by the covering graph (or Hasse diagram) of $(X, \leqslant)$ itself: the general equivalence-issue is therefore also settled in the negative for the class of all median interval spaces induced by median graphs that are not trees (and a fortiori for the even larger class of interval spaces induced by connected graphs that are not trees).

Namely, simple strategy-proofness and coalitional strategy-proofness are *not* equivalent properties for full unimodal domains in the class of all median interval spaces induced by some arbitrary bounded distributive lattice, or by some arbitrary median graphs. In particular, it can be shown that the ordinary (extended) median rule retains its strategy-proofness on such domains but may be *not coalitionally strategy-proof.* To check the last point, consider for instance the following example borrowed e.g. from Nehring, Puppe (2007 (b)). Take the interval space induced by the Boolean cube $\mathbf{2^3} = (\mathbf{2}^3, \leqslant)$ where

$$\leqslant = \{(y,x) : x \in \mathbf{2}^3 \text{ and } y \in \{1,x\}\} \cup$$
$$\cup \left\{ \begin{array}{c} (x_1,x_4),(x_1,x_5),(x_1,0),(x_2,x_4),(x_2,x_6),(x_2,0), \\ (x_3,x_5),(x_3,x_6),(x_3,0),(x_4,0),(x_5,0),(x_6,0) \end{array} \right\}.$$

Notice that such a (median) interval space $\mathcal{I} = (\mathbf{2}^3, I)$ induced by $(\mathbf{2}^3, \leqslant)$ is defined as follows: $I(1,x_4) = I(x_1,x_2) = \{1,x_1,x_2,x_4\}$, $I(1,x_5) = I(x_1,x_3) = \{1,x_1,x_3,x_5\}$, $I(1,x_6) = I(x_2,x_3) = \{1,x_2,x_3,x_6\}$,
$I(x_1,0) = I(x_4,x_5) = \{x_1,x_4,x_5,0\}$, $I(x_2,0) = I(x_4,x_6) = \{x_2,x_4,x_6,0\}$,
$I(x_3,0) = I(x_5,x_6) = \{x_3,x_5,x_6,0\}$, $I(1,0) = I(x_1,x_6) = I(x_2,x_5) = I(x_3,x_4) = \mathbf{2}^3$, and $I(x,y) = \{x,y\}$ otherwise.

Let $N = \{1,2,3\}$ and consider an $\mathcal{I}$-unimodal preference profile $(\succcurlyeq_1, \succcurlyeq_2, \succcurlyeq_3)$ such that $top(\succcurlyeq_1) = x_1$, $top(\succcurlyeq_2) = x_2$, $top(\succcurlyeq_3) = 0$, and $x_3 \succ_i x_4$ for all $i \in N$. Note that such an $\mathcal{I}$-unimodal preference profile does certainly exist since there is no $x \in \mathbf{2}^3$, $x \neq x_4$ with $x_4 \in I(x_3,x)$. Now, it is immediately checked that the median of the top outcomes of preference profile $(\succcurlyeq_1, \succcurlyeq_2, \succcurlyeq_3)$ is $\mu(x_1,x_2,0) = x_4$ because $I(x_1,x_2) \cap I(x_2,0) \cap I(x_1,0) = \{x_4\}$. However, observe that e.g. $\mu(x_3,x_3,z) = \mu(z,x_3,x_3) = \mu(x_3,z,x_3) = \mu(x_3,x_3,x_3) = x_3$ for any $z \in \mathbf{2}^3$. It follows that the median rule is in fact manipulable by the grand coalition and by any two-player coalition, hence it is clearly not coalitionally strategy-proof.

Let us now move on to a few interesting classes of networks/interval spaces where -to the best of our knowledge- very little is known about the structure of strategy-proof voting rules on the corresponding full unimodal domains.

To begin with, let us consider the class of (convex, idempotent) interval spaces as resulting from the following important class of outcome spaces.

The outcome space is a simplex in an Euclidean convex space

In that case (X, I^E) is the (convex, idempotent) interval space induced by a simplex in an Euclidean convex space in the standard manner, namely $X = \left\{x \in \mathbb{R}_+^m : \sum_{i=1}^m x_i = 1\right\}$, and for all $x, y \in X$,

$I^E(x, y) = \{z \in X : z = \lambda x + (1 - \lambda)y \text{ for some } \lambda \in [0, 1]\}$.

That is clearly *not* a median interval space: in fact, any nondegenerate triangle in X fails to admit a median as defined above. As mentioned above, very little is apparently known about the class of strategy-proof voting rules for the full unimodal domain on (X, I^E), or the existence of nontrivial non-dictatorial strategy-proof voting rules on such domain.

The next class of outcome spaces is clearly less widely used than the previous one, but is also a most remarkable one, since it is one of the few irreducible building blocks of any linear geometry (to be discussed below, in Section 3):

The outcome space is a multi-cross

Let X be a (minimal) multi-cross with centre z i.e. an array of three-point and two-point lines such that z is the middle point of each three-point line, and does not lie on any two-point line (see e.g. Coppel (1998)): thus, the corresponding (convex, idempotent) interval space (X, I) is such that for all $x, y \in X \setminus \{z\}$ either $I(x, y) = \{x, y, z\}$ or $I(x, y) = \{x, y\}$, and $I(x, z) = \{x, z\}$ for all $x \in X$. Clearly, (X, I) is *not* median. To the best of the author's knowledge, no attempt has been made to study the structure of strategy-proof voting rules on the corresponding full unimodal domain, or even to establish whether nontrivial nondictatorial strategy-proof voting rules on that domain are available.

The next two following types of networks are also of considerable interest as models of location problems in a large class of abstract spaces:

The outcome space is a Hamming graph

A Hamming graph can be regarded as a network whose vertices $a \in X^H$ denote words $a = (l_1, ..., l_k)$ of a fixed length k having at each position $i = 1, ..., k$ a letter l_i chosen from a finite alphabet $A_i = \left\{a_{i_1}, ...a_{i_{h_i}}\right\}$ (distinct positions may have distinct alphabets), while edges join any two vertices denoting two distinct words having distinct letters at just one position (see e.g. Mulder (1980)). Equivalently, the vertices of a Hamming graph may be regarded as points of a finite multiattribute space, with edges joining pairs of points that differ for the value of just one attribute. Therefore, the (convex, idempotent) interval space (X^H, I^H) induced by a Hamming graph is defined by the following rule:

$X^H = \Pi_{i=1}^k A_i$, and for all $x, y \in X^H$,

$I^H(x, y) = \left\{z \in X^H \text{:} z \text{ lies on one of the shortest paths joining } x \text{ and } y\right\}$.

Notice that if $h_i \geq 3$ for some $i = 1, ..., k$, a Hamming graph is not triangle-free and, as a consequence, its interval space is *not* median. To the best of the author's knowledge, the structure of strategy-proof voting rules on the full unimodal domain of (X^H, I^H) is still essentially unknown.

The outcome space is a partial cube (or mediatic graph)

A *partial cube* is a network that can be isometrically embedded into the cube induced by some arbitrary set Y (recall that the cube induced by set Y is the graph having the subsets of Y as vertices, with edges joining precisely those subsets $A \subseteq Y, B \subseteq Y$ such that $\#(A \smallsetminus B) \cup (B \smallsetminus A) = 1$). It is now well-known that the class of partial cubes is precisely the class of *mediatic graphs* i.e. those graphs that represent the set of states and transitions associated with procedures satisfying a suitable pair of conditions and denoted as *media* (see e.g. Eppstein, Falmagne, Ovchinnikov (2008)). The convex idempotent interval space induced by a partial cube is defined in the usual way in terms of geodesics i.e. shortest paths and, generally speaking, is not median. Apparently, the structure of strategy-proof voting rules on the full unimodal domain of such spaces has not been the object of any work in the earlier literature.

Finally, let us consider a further class of networks/interval spaces where the existence of nontrivial nondictatorial strategy-proof voting rules for the full unimodal domain has been already established (while the resulting 'unimodal' equivalence-issue between simple and coalitional strategy-proofness has not been addressed yet).

The outcome space is the join of a clique and a chain

Let (X, I) be the interval space induced by a graph that can be decomposed into a clique (or complete graph) and a chain as joined through a common vertex. The latter sort of graph is a special subclass of the class of networks studied by Schummer, Vohra (2002), where it is shown that some nontrivial nondictatorial voting rules exist on a certain unimodal domain on (X, I) (e.g. those rules resulting from the combination of a locally-dictatorial rule that applies whenever the top outcome of the relevant 'clique-dictator' lies on the clique, and a median rule as restricted to the subset of outcomes lying between the top outcome of the 'clique-dictator' and the outcome that is closest to the clique, that applies otherwise). However, the equivalence-issue concerning simple and coalitional strategy-proofness of voting rules on that full unimodal domain has never been addressed in the extant literature.

Thus, the foregoing list includes examples of outcome spaces where the status of available information on the issue concerning equivalence of simple and coalitional strategy-proofness of voting rules for full unimodal domains is quite diverse. In a few of them, the 'unimodal' equivalence-issue has been addressed and settled (either affirmatively, for bounded chains and bounded trees, or negatively, for bounded semilattices - or bounded median graphs - that are not chains). In other cases (i.e. Euclidean simplexes, multi-crosses, Hamming graphs, partial cubes) no general results on the existence of nontrivial nondictatorial strategy-proof voting rules for the full unimodal domain are available in earlier works. In the last case of that list (i.e. joins of one clique and one chain) it is known that nontrivial nondictatorial strategy-proof voting rules for the full unimodal domain do exist but is not known whether equivalence of simple and coalitional strategy-proofness on that domain holds true.

It is therefore quite remarkable that the main results of the present paper (i.e. Theorems 3 and 5) jointly address and settle at once such 'unimodal' equivalence-issue in all of the outcome spaces considered above. Indeed, it is easily checked that (the interval spaces induced by) bounded chains, bounded trees, Euclidean simplexes, multi-crosses, and joins of cliques and chains do satisfy the Interval Anti-Exchange property (to be introduced in the next Section) so that Theorem 3 applies to them to the effect of ensuring that 'unimodal' equivalence holds for strategy-proof voting rules on those spaces. Conversely, it is easily checked that (the interval spaces induced by) bounded semilattices and bounded median graphs which are not chains violate Minimal Anti-Exchange (as defined in the next Section), and the same observation applies to (the non-median interval spaces induced by) Hamming graphs and partial cubes. Therefore, Theorem 5 below applies, establishing that *nontrivial nondictatorial strategy-proof voting rules that are coalitionally manipulable (and admit at least four distinct outcomes) do exist on the full unimodal domains of those interval spaces*: as a consequence, 'unimodal' equivalence fails to hold for such spaces.

Let us then turn to the formal setting and the ensuing analysis.

3 Model and results

Let $N = \{1, .., n\}$ denote the finite population of voters, and $\mathcal{I} = (X, I)$ the **interval space** of alternative outcomes, i.e. X is an arbitrary nonempty set and I an *interval function on* X, namely $I : X^2 \to \mathcal{P}(X)$ is a function that satisfies the following conditions:

I-(i) (**Extension**): $\{x, y\} \subseteq I(x, y)$ for all $x, y \in X$,

I-(ii) (**Symmetry**): $I(x, y) = I(y, x)$ for all $x, y \in X$.

In particular, we also assume that $n \geq 2$ in order to avoid tedious qualifications, and that $\mathcal{I} = (X, I)$ is an *idempotent* interval space namely that

(**Idempotence**): $I(x, x) = \{x\}$ for all $x \in X$

is also satisfied.

A subset $Y \subseteq X$ is $\mathcal{I}$*-convex* iff $I(x, y) \subseteq Y$ for all $x, y \in Y$. For any $Y \subseteq X$, the $\mathcal{I}$*-convex hull* of Y - denoted $co_{\mathcal{I}}(Y)$- is the smallest $\mathcal{I}$-convex superset of Y, namely $co_{\mathcal{I}}(Y) = \bigcap \{A \subseteq X : A \text{ is } \mathcal{I}\text{-}convex \text{ and } A \supseteq Y\}$.

An interval space $\mathcal{I} = (X, I)$ is *convex* if I also satisfies

(**Convexity**): $I(x, y)$ is $\mathcal{I}$-convex for all $x, y \in X$.

Observe that Idempotence and Convexity are indeed mutually independent properties of interval spaces. To confirm that statement, consider interval spaces $\mathcal{I}_1 = (X, I_1)$, $\mathcal{I}_2 = (\{x, y, v, z\}, I_2)$ where $\#X > 1$, $\#\{x, y, v, z\} = 4$, $I_1(a, b) = X$ for all $a, b \in X$, while $I_2(x, y) = \{x, y, z\}$, $I_2(y, z) = \{y, v, z\}$, and $I_2(a, b) = \{a, b\}$ for all $a, b \in X$ such that $\{x, y\} \neq \{a, b\} \neq \{y, z\}$. It is immediately checked that $\mathcal{I}_1$ is convex but not idempotent, while $\mathcal{I}_2$ is idempotent but not convex since $\{y, z\} \subseteq I_2(x, y)$ and $v \in I_2(y, z) \setminus I_2(x, y)$.

Remark 1 An idempotent interval space (X, I) is said to be a **convex geometry** if it also satisfies

(**Peano Convexity**) for all $x, y, v_1, v_2, z \in X$, if $y \in I(x, v_1)$ and $z \in I(y, v_2)$ then there exists $v \in I(v_1, v_2)$ such that $z \in I(x, v)$.

It can be quite easily shown that a convex geometry is in particular a convex interval space (see e.g. Coppel (1998), chpt.2, Proposition 1). The converse however does not hold: to check the latter statement, consider interval space $\mathcal{I}^* = (X, I^*)$ with $X = \{x, y_1, y_2, v_1, v_2, z\}$, $\#X = 6$, and I^* defined as follows: $I^*(x, y_1) = \{x, y_1, v_1\}$, $I^*(x, y_2) = \{x, y_2, v_2\}$, $I^*(v_1, v_2) = \{v_1, v_2, z\}$, $I^*(y_1, z) = \{y_1, v_1, z, \}$, $I^*(y_2, z) = \{y_2, v_2, z, \}$, $I^*(y_2, v_1) = \{y_2, y_1, v_1\}$, and $I^*(a, b) = \{a, b\}$ otherwise. As it is easily checked, $\mathcal{I}^*$ is idempotent and convex by construction. However, it can also be shown (see e.g. Coppel (1998), chpt.2, Proposition 2) that a convex geometry also satisfes the following property:

(*) for all $x, y_1, y_2, v_1, v_2, z \in X$, if $v_1 \in I^*(x, y_1)$, $v_2 \in I^*(x, y_2)$ and $z \in I^*(v_1, v_2)$ then there exists $w \in I^*(y_1, y_2)$ such that $z \in I^*(x, w)$.

Now, it is immediately checked that no such w exists in $\mathcal{I}^*$

for $x, y_1, y_2, v_1, v_2, z \in X$, since by construction $I^*(y_1, y_2) = \{y_1, y_2\}$, and $z \notin I^*(x, y_1) \cup I^*(x, y_2) = \{x, y_1, y_2, v_1, v_2\}$.

Therefore, $\mathcal{I}^*$ fails to satisfy Peano convexity.

It follows that Peano convexity is indeed a *strictly stronger* requirement than convexity of an interval space as previously defined or, equivalently, a convex idempotent interval space amounts to a *generalized convex geometry. Therefore all of the results of the present paper clearly hold in particular when restricting the statements to convex geometries.*

On the other hand, a **linear geometry** is a convex geometry that satisfies **interval anti-exchange** *(* to be defined below) and three further properties called **additivity** (i.e. for all $x, y, z \in X$, if $z \in I(x, y)$ then $I(x, y) = I(x, z) \cup I(z, y)$), **no-branchpoint** (i.e. for all $x, y, z \in X$, if $z \notin I(x, y)$ and $y \notin I(x, z)$ then $I(x, y) \cap I(x, z) = \{x\}$), and the **Pasch-Peano condition** (i.e. for all $x, y, z, u, v \in X$, if $y \in I(x, u)$ and $z \in I(x, v)$ then $I(y, v) \cap I(z, u) \neq \varnothing$: see e.g. Coppel (1998)).

Finally, (undirected) **trees** (i.e. connected graphs without cycles) are also convex geometries if the interval $I(x, y)$ of each pair x, y of vertices is defined as the set of vertices that lie on the (unique) shortest path joining x and y. Characterizations of the interval functions of trees thus defined have been provided by Sholander (1952, 1954) and, most recently, by Chvátal, Rautenbach, Schäfer (2011). It should be noticed that trees do satisfy weak anti-exchange (as defined below) and additivity, but may or may not satisfy the no-branchpoint property and the Pasch-Peano condition (namely, some trees satisfy both of those properties, while others violate both). Conversely, linear geometries may or may not have cycles (we shall introduce below some examples of linear geometries with cycles). Thus, linear geometries and trees are subclasses of convex geometries that partially overlap: there are convex geometries that are both linear geometries and trees (namely, trees with no branchpoints), while other convex geometries are just trees (e.g. trees with branchpoints), just linear geometries (e.g. cliques i.e. complete graphs), or neither (see e.g. the quasi-complete graph presented below under Example 2).

Occasionally, *antisymmetric* interval spaces will also be considered in the sequel. Indeed, an interval space $\mathcal{I} = (X, I)$ is *antisymmetric* if I satisfies

(**Antisymmetry**): for all $x, y, z \in X$, if $x \in I(y, z)$ and $y \in I(x, z)$ then $x = y$.

It can be shown that an important subclass of convex geometries (but by no means all of them) do satisfy antisymmetry (see Coppel (1998)).

Finally, we should also mention that an *idempotent* interval space $\mathcal{I} = (X, I)$ is said to be a **median space** if I satisfies the following

(**Median property**): for all $x, y, z \in X$, $\#(I(x, y) \cap I(y, z) \cap I(x, z)) = 1$.

The common point of the three intervals defined by each pair of any three points x, y, z in a median interval space (X, I) is said to be the *median* of those points, that therefore defines a ternary operation on X.

It is well-known that e.g. the interval spaces induced by trees or median semilattices (including distributive lattices) are median (see Sholander (1952), (1954)).

Let $\succcurlyeq$ denote a total preorder i.e. a reflexive, connected and transitive binary relation on X (we shall denote by $\succ$ and $\sim$ its asymmetric and symmetric components, respectively). Then, $\succcurlyeq$ is said to be unimodal with respect to interval space $\mathcal{I} = (X, I)$ - or $\mathcal{I}$-**unimodal** - if and only if

U-(i) there exists a *unique maximum* of $\succcurlyeq$ in X, its *top* outcome -denoted $top(\succcurlyeq)$- and

U-(ii) for all $x, y, z \in X$, if $z \in I(x, y)$ then $\{u \in X : z \succcurlyeq u\} \cap \{x, y\} \neq \varnothing$.

We denote by $U_{\mathcal{I}}$ the set of all $\mathcal{I}$-*unimodal* total preorders on X. An N-profile of $\mathcal{I}$-unimodal total preorders is a mapping from N into $U_{\mathcal{I}}$. We denote by $U_{\mathcal{I}}^N$ the set of all N-profiles of $\mathcal{I}$-unimodal total preorders.

A **voting rule** for (N, X) is a function $f : X^N \to X$. A voting rule f is (simply) **strategy-proof** *on* $U_{\mathcal{I}}^N$ iff for all $\mathcal{I}$-unimodal N-profiles $(\succcurlyeq_i)_{i\in N} \in U_{\mathcal{I}}^N$, and for all $i \in N$, $y_i \in X$, and $(x_j)_{j\in N} \in X^N$ such that $x_j = top(\succcurlyeq_j)$ for each $j \in N$, $f((x_j)_{j\in N}) \succcurlyeq_i f((y_i, (x_j)_{j\in N\smallsetminus\{i\}}))$. Moreover, a voting rule f is **coalitionally strategy-proof** *on* $U_{\mathcal{I}}^N$ iff for all $\mathcal{I}$-unimodal N-profiles $(\succcurlyeq_i)_{i\in N} \in U_{\mathcal{I}}^N$, and for all $C \subseteq N$, $(y_i)_{i\in C} \in X^C$, and $(x_j)_{j\in N} \in X^N$ such that $x_j = top(\succcurlyeq_j)$ for each $j \in N$, there exists $i \in C$ with $f((x_j)_{j\in N}) \succcurlyeq_i f((y_i)_{i\in C}, (x_j)_{j\in N\smallsetminus C}))$. Finally, a voting rule $f : X^N \to X$ is $\mathcal{I}$-**monotonic** iff for all $i \in N$, $y_i \in X$, and $(x_j)_{j\in N} \in X^N$, $f((x_j)_{j\in N}) \in I(x_i, f(y_i, (x_j)_{j\in N\smallsetminus\{i\}}))$.

We are now ready to state the main results of this paper concerning the equivalence of strategy-proofness and coalitional strategy-proofness of voting rules on the domain of all unimodal profiles. Our results rely on the following lemma that establishes the equivalence between *monotonicity* with respect to an arbitrary convex idempotent interval space $\mathcal{I}$ and *strategy-proofness on the corresponding (full) unimodal domain* $U_{\mathcal{I}}^N$.

Lemma 1 *Let $\mathcal{I} = (X, I)$ be a convex idempotent interval space. Then, a voting rule $f : X^N \to X$ is strategy-proof on the full unimodal domain $U_{\mathcal{I}}^N$ iff it is $\mathcal{I}$-monotonic.*

Proof. Let us assume that $f : X^N \to X$ is *not* $\mathcal{I}$-monotonic: thus, there exist $i \in N$, $x_i' \in X$ and $x_N = (x_i)_{i\in N} \in X^N$ such that $f(x_N) \notin I(x_i, f(x_i', x_{N\smallsetminus\{i\}}))$. Then, consider the total preorder $\succcurlyeq^*$ on X defined as follows: $x_i = top(\succcurlyeq^*)$ and for all $y, z \in X \smallsetminus \{x_i\}$, $y \succcurlyeq^* z$ iff (i) $\{y, z\} \subseteq I(x_i, f(x_i', x_{N\smallsetminus\{i\}})) \smallsetminus \{x_i\}$ or (ii) $y \in I(x_i, f(x_i', x_{N\smallsetminus\{i\}})) \smallsetminus \{x_i\}$ and $z \notin I(x_i, f(x_i', x_{N\smallsetminus\{i\}}))$ or (iii) $y \notin I(x_i, f(x_i', x_{N\smallsetminus\{i\}}))$ and $z \notin I(x_i, f(x_i', x_{N\smallsetminus\{i\}}))$. Clearly, by construction $\succcurlyeq^*$consists of three indifference classes with $\{x_i\}$, $I(x_i, f(x_i', x_{N\smallsetminus\{i\}})) \smallsetminus \{x_i\}$ and $X \smallsetminus I(x_i, f(x_i', x_{N\smallsetminus\{i\}}))$ as top, medium and bottom indifference classes, respectively.

Now, observe that $\succcurlyeq^* \in U_{\mathcal{I}}$. To check that statement, take any $y, z, v \in X$ such that $y \neq z$ and $v \in I(y, z)$ (if $y = z$ then, by Idempotence of I, $v = y = z$ and there is in fact nothing to prove). Also, notice that $\{y, z\} \neq \{x_i\}$ since $y \neq z$, and assume without loss of generality that $y \neq x_i$.

If $\{y, z\} \subseteq I(x_i, f(x_i', x_{N\smallsetminus\{i\}}))$ then, by Convexity of (X, I), $v \in I(x_i, f(x_i', x_{N\smallsetminus\{i\}}))$. Hence, $v \succcurlyeq^* y$ by definition of $\succcurlyeq^*$.

If on the contrary $\{y, z\} \cap (X \smallsetminus I(x_i, f(x_i', x_{N\smallsetminus\{i\}}))) \neq \varnothing$ then take $w \in \{y, z\} \cap (X \smallsetminus I(x_i, f(x_i', x_{N\smallsetminus\{i\}})))$.

Clearly, by definition of $\succcurlyeq^*$again, $v \succcurlyeq^* w$. Since $w \in \{y, z\}$, it follows that the unimodality condition is satisfied again and therefore $\succcurlyeq^* \in U_{\mathcal{I}}$ as claimed.

Also, by assumption $f(x_N) \in X \smallsetminus I(x_i, f(x_i', x_{N\smallsetminus\{i\}}))$ while $f(x_i', x_{N\smallsetminus\{i\}}) \in I(x_i, f(x_i', x_{N\smallsetminus\{i\}}))$ by Extension, whence by construction $f(x_i', x_{N\smallsetminus\{i\}}) \succ^* f(x_N)$. But then, f is *not* strategy-proof on $U_{\mathcal{I}}^N$.

Conversely, let f be monotonic with respect to $\mathcal{I}$. Now, consider any $\succcurlyeq = (\succcurlyeq_j)_{j\in N} \in U_{\mathcal{I}}^N$ and any $i \in N$. By definition of monotonicity $f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}}) \in I(top(\succcurlyeq_i), f(x_i, x_{N\smallsetminus\{i\}}))$ for all $x_{N\smallsetminus\{i\}} \in X^{N\smallsetminus\{i\}}$ and $x_i \in X$. But then, since clearly by definition $top(\succcurlyeq_i) \succcurlyeq_i f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}})$, either $f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}}) = top(\succcurlyeq_i)$ or $f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}}) \succcurlyeq_i f(x_i, x_{N\smallsetminus\{i\}})$ by unimodality of $\succcurlyeq_i$. Hence, $f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}}) \succcurlyeq_i f(x_i, x_{N\smallsetminus\{i\}})$ in any case. It follows that f is indeed strategy-proof on $U_{\mathcal{I}}^N$. ∎

The following property will play a key role in the ensuing analysis

(**Interval Anti-Exchange (IAE)**): for all $x, y, v, z \in X$ such that $x \neq y$, if $x \in I(y, v)$ and $y \in I(x, z)$ then $x \in I(v, z)$.

Since one of the main results of the ensuing analysis will concern convex idempotent interval spaces that satisfy Interval Anti-Exchange, it is worth mentioning here that the class of such spaces is *strictly larger than the class of convex geometries that satisfy Interval Anti-Exchange,* as made clear by the following simple

Example 1 (**A convex idempotent interval space that satisfies Interval Anti-Exchange but is not a convex geometry**)

Take $\mathcal{I} = (X, I)$ with $X = \{x, u, v, y, z\}$, $\#X = 5$, and I as defined by the following rule: $I(x, u) = \{x, u, y\}$, $I(y, v) = \{y, v, z\}$, and $I(a, b) = \{a, b\}$ otherwise. Observe that, by construction, $\mathcal{I}$ is a convex and idempotent interval space. It is also quickly established that $\mathcal{I}$ does satisfy Interval Anti-Exchange: indeed, it is immediately seen that $a \in I(b, c)$ and $b \in I(a, d)$ with $a \neq b$ only hold in $\mathcal{I}$ if one of the following clauses is satisfied: (1) $a = c$, (2) $a = y$,$\{b, c\} = \{x, u\}$, and $b = d$, (3) $a = z$, $\{b, c\} = \{y, v\}$, and $b = d$. Now, if (1) holds then $a \in I(c, d)$ by Extension of I (i.e. by definition of interval space). If (2) holds, $\{c, d\} = \{x, u\}$ hence $a = y \in I(x, u) = I(c, d)$ by construction. If (3) holds, then $\{c, d\} = \{b, c\}$ hence $a \in I(b, c) = I(c, d)$ by hypothesis. Therefore, $\mathcal{I}$ satisfies Interval Anti-Exchange as claimed. However, it can be quite easily shown that $\mathcal{I}$ fails to satisfy Peano Convexity as defined above (see e.g. Coppel (1998), chpt.2), hence it is *not a convex geometry.* It should also be noticed that $\mathcal{I}$ is *not median* (since e.g. $I(x,y) \cap I(y,z) \cap I(x,z) = \varnothing$).

Remark 2 An explanation concerning our terminology is in order here. Recall that a *convexity space* (or *aligned space,* or *convex closure system*) is a pair $(X, \mathcal{C})$ where X is any set and $\mathcal{C}$ is a *convexity* (or *alignment*) on X i.e. a family of subsets of X such that: (i) $\{\varnothing, X\} \subseteq \mathcal{C}$; (ii) $\cap\mathcal{D} \in\mathcal{C}$ for any nonempty $\mathcal{D} \subseteq \mathcal{C}$; (iii) $\cup\mathcal{D} \in\mathcal{C}$ for any nonempty $\mathcal{D} \subseteq \mathcal{C}$ which is *nested* i.e. totally ordered by inclusion. The subsets in $\mathcal{C}$ are by definition the *convex sets* of convexity space $(X, \mathcal{C})$, while for any $Y \subseteq X$ its *convex hull* $co_{\mathcal{C}}(Y)$ in $(X, \mathcal{C})$ is the smallest superset of Y that belongs to $\mathcal{C}$ (observe that $co_{\mathcal{C}}(Y)$ is well-defined for any $Y \subseteq X$ thanks to properties (i) and (ii) of $\mathcal{C}$). It is quite easy to check that the set $\mathcal{C}_{\mathcal{I}}$ of $\mathcal{I}$-convex sets of any interval space $\mathcal{I} = (X, I)$ as defined above provides a particular instance of a *convexity* on X, but generally speaking a convexity on X need not arise in that way (see e.g. van de Vel (1993)).

Indeed, Anti-Exchange is a commonly used label denoting the following property of a convexity space $(X, \mathcal{C})$:

(**Anti-Exchange** *(*AE)): for all $x, y \in X$ and $Y \subseteq X$, if $x \neq y$, $x \in co_{\mathcal{C}}(Y \cup \{y\})$ and $x \notin co_{\mathcal{C}}(Y)$ then $y \notin co_{\mathcal{C}}(Y \cup \{x\})$.

Clearly enough, Anti-Exchange can be in particular regarded as a possible property of any interval space $\mathcal{I} = (X, I)$ by taking $\mathcal{C} = \mathcal{C}_{\mathcal{I}}$, and in that case - by construction - $co_{\mathcal{C}_{\mathcal{I}}}(Y) = co_{\mathcal{I}}(Y)$ as defined above, for all $Y \subseteq X$.

Now, it can be shown that any convex geometry $\mathcal{I} = (X, I)$ that satisfies IAE does also satisfy AE while any interval space that satisfies AE must also satisfy IAE (see e.g. Coppel (1998), where IAE is denoted as 'axiom L2').

It should also be noticed here that if an interval space $\mathcal{I} = (X, I)$ satisfies AE and X is *finite* then the set of its $\mathcal{I}$-convex sets is an instance of an *antimatroid*. Accordingly, let us denote as *antimatroidal* a finite interval space $\mathcal{I} = (X, I)$ that satisfies AE. Thus, clearly, Theorem 3 below also applies in particular to the class of all *antimatroidal convex and idempotent interval spaces.* It is easily checked that the finite convex idempotent interval space introduced above under Example 1 does also satisfy AE and is therefore a representative of that subclass of antimatroidal interval spaces.

The next condition is a considerably weakened version of IAE:

(**Minimal Anti-Exchange** *(MAE)*): for all $x, y, v, z \in X$ such that $x \neq y$, and $v \neq z$ at least one of the following clauses is satisfied: (i) $I(y, v) \cap \{x, z\} \neq \{x, z\}$, (ii) $I(x, z) \cap \{y, v\} \neq \{y, v\}$, (iii) $I(v, z) \cap \{x, y\} \neq \varnothing$, (iv) $I(y, z) \cap \{x, v\} \neq \varnothing$.

Remark 3 It is easily checked that, for an arbitrary interval space $\mathcal{I} = (X, I)$, IAE does indeed entail MAE, while the reverse does not generally hold. To see this, observe that by definition IAE amounts to requiring that for all $x, y, v, z \in X$ such that $x \neq y$, at least one of the following three clauses is satisfied: (i) $x \notin I(y, v)$, (ii) $y \notin I(x, z)$, (iii') $x \in I(v, z)$. Clearly, (iii') entails (iii) whence MAE holds true whenever IAE does. On the other hand, consider interval space $\mathcal{I} = (X, I)$ with $X = \{x, y, v, z\}$, $\#X = 4$, and I as defined by the following rule: $I(x, z) = \{x, y, z\}$, $I(y, v) = \{x, y, v\}$, and $I(a, b) = \{a, b\}$ otherwise. Notice that, by construction, $\mathcal{I}$ is convex and idempotent. Moreover, $I(x, z) \cap \{y, v\} = \{y\} \neq \{y, v\}$ hence $\mathcal{I}$ satisfies MAE. However, $x \in I(y, v)$, $y \in I(x, z)$, and $x \notin I(v, z)$: therefore $\mathcal{I}$ fails to satisfy IAE. Interval space $\mathcal{I}^* = (X, I^*)$ as defined in Remark 1 is another simple example of a convex idempotent interval space that satisfies MAE but fails to satisfy IAE. Indeed, it is easily checked that $\mathcal{I}^*$ is antisymmetric: hence, for any $Y \subseteq X$, its restriction to Y, denoted $\mathcal{I}^*_Y = (Y, I^*_Y)$, is also antisymmetric. It follows that (in view of the proof of Corollary 4 below, establishing that antisymmetric idempotent spaces of cardinality not larger than three must satisfy IAE) $\mathcal{I}^*_Y$ satisfies IAE (hence a fortiori MAE) whenever $\#Y \leq 3$. Thus, it only remains to check for MAE with respect to four *distinct* $x, y, v, z \in X$: but then, since by construction $\#I^*(a, b) \leq 3$ for all $a, b \in X$, *both* clauses MAE(i) and MAE(ii) (that amount precisely to requiring that $I^*(y, v) < 4$ and $I^*(x, z) < 4$) are clearly satisfied, whence MAE holds. On the other hand, recall that by definition $c_1 \in I^*(a, b_1) = I^*(b_1, a)$, $b_1 \in I^*(c_1, b_2)$, and $c_1 \notin I^*(a, b_2)$: therefore $\mathcal{I}^*$ fails to satisfy IAE.

The next lemma provides a remarkable property of $\mathcal{I}$-monotonic voting rules when $\mathcal{I}$ satisfies Interval Anti-Exchange:

Lemma 2 *Let $\mathcal{I} = (X, I)$ be an interval space that satisfies Interval Anti-Exchange, and $f : X^N \to X$ an $\mathcal{I}$-monotonic voting rule. Then, for all $x_N, y_N \in X^N$, $f(x_N) \neq f(y_N)$ entails that $f(x_N) \in I(x_i, y_i)$ for some $i \in N$.*

Proof. Let $x_N, y_N \in X^N$, and $f(x_N) \neq f(y_N)$. Then, by $\mathcal{I}$-monotonicity of f, $f(x_N) \in I(x_i, f(y_i, x_{N \smallsetminus \{i\}}))$ and $f(y_i, x_{N \smallsetminus \{i\}}) \in I(y_i, f(x_N))$ for each $i \in N$. Then, take $i = 1$. If $f(x_N) \neq f(y_1, x_{N \smallsetminus \{1\}})$ then, thanks to Symmetry of I, Interval Anti-Exchange applies, whence $f(x_N) \in I(x_1, y_1)$, and the thesis immediately follows. Let us then suppose that, on the contrary, $f(x_N) = f(y_1, x_{N \smallsetminus \{1\}})$. Next, consider $f(y_1, y_2, x_{N \smallsetminus \{1,2\}})$.

By $\mathcal{I}$-monotonicity of f, $f(y_1, x_{N \smallsetminus \{1\}}) \in I(x_2, f(y_1, y_2, x_{N \smallsetminus \{1,2\}}))$ and $f(y_1, y_2, x_{N \smallsetminus \{1,2\}}) \in I(y_2, f(y_1, x_{N \smallsetminus \{1\}}))$.

If $f(x_N) = f(y_1, x_2, x_{N \smallsetminus \{1,2\}}) \neq f(y_1, y_2, x_{N \smallsetminus \{1,2\}})$ then again, by Interval Anti-Exchange of $\mathcal{I}$, it follows that $f(x_N) = f(y_1, x_2, x_{N \smallsetminus \{1,2\}}) \in I(x_2, y_2)$ as required by the thesis. Thus, assume again that on the contrary $f(x_N) = f(y_1, x_2, x_{N \smallsetminus \{1,2\}}) = f(y_1, y_2, x_{N \smallsetminus \{1,2\}})$. A suitable iteration of the previous argument allows us to establish that either $f(x_N) \in I(x_i, y_i)$ for some $i \in \{1, ..., n-1\}$ or $f(x_N) = f(y_{N \smallsetminus \{n\}}, x_n)$. In the former case the thesis holds. In the latter case, by $\mathcal{I}$-monotonicity of f, $f(x_N) = f(y_{N \smallsetminus \{n\}}, x_n) \in I(x_n, f(y_N))$ and $f(y_N) \in I(y_n, f(y_{N \smallsetminus \{n\}}, x_n))$. Since by hypothesis $f(x_N) \neq f(y_N)$ it follows, by Interval Anti-Exchange of $\mathcal{I}$, that $f(x_N) = f(y_{N \smallsetminus \{n\}}, x_n) \in I(x_n, y_n)$ and the thesis is therefore established. ■

The next Theorem establishes that for convex idempotent interval spaces Interval Anti-Exchange ensures that simple (or individual) strategy-proofness and coalitional strategy-proofness of a voting rule on the full unimodal domain are equivalent properties.

Theorem 3 *Let $\mathcal{I} = (X, I)$ be a convex idempotent interval space that satisfies Interval Anti-Exchange (IAE), and $f : X^N \to X$ a voting rule that is strategy-proof on the full unimodal domain $U_{\mathcal{I}}^N$. Then, f is also coalitionally strategy-proof on $U_{\mathcal{I}}^N$.*

Proof. Indeed, suppose that f is not coalitionally strategy-proof on $U_{\mathcal{I}}^N$. Then, there exist $S \subseteq N$, $(\succcurlyeq_i)_{i \in N} \in U_{\mathcal{I}}^N$, $x_N \in X^N$ and $x'_S \in X^S$ such that for all $i \in S$, $top(\succcurlyeq_i) = x_i$ and $f(x'_S, x_{N \smallsetminus S}) \succ_i f(x_N)$ (where $\succ_i$denotes the asymmetric component of $\succcurlyeq_i$).

Notice that it may be assumed without loss of generality that $S = N$: to see this, consider $f_{x_{N \smallsetminus S}} : X^S \to X$ as defined by the rule $f_{x_{N \smallsetminus S}}(y_S) = f(y_S, x_{N \smallsetminus S})$ for all $y_S \in X^S$ and observe that, by construction, $f_{x_{N \smallsetminus S}}$ is both strategy-proof and *not* coalitionally strategy-proof. Let us then posit $f(x_N) = f(x_S) = u$, and $f(x'_N) = f(x'_S) = v$: by construction, $v \succ_i u$ for all $i \in N$. By Lemma 1 above, f is $\mathcal{I}$-monotonic: therefore, $f(v, x'_{N\smallsetminus\{1\}}) \in I(v, f(x'_N)) = I(v, v)$, whence $f(v, x'_{N\smallsetminus\{1\}}) = v$, by idempotence of $\mathcal{I}$. Similarly, by $\mathcal{I}$-monotonicity of f again, $f(v, v, x'_{N\smallsetminus\{1,2\}}) \in I(v, f(v, x'_2, x'_{N\smallsetminus\{1,2\}})) = I(v, v)$: thus, by idempotence of $\mathcal{I}$ again, $f(v, v, x'_{N\smallsetminus\{1,2\}}) = v$. A suitable iteration of the same argument establishes that $f(v, v, ..., v) = f(x'_N) = v$.

Now, suppose that there exists $i \in N$, such that $f(x_N) = u \in I(x_i, v)$: since $x_i = top(x_i)$ and $v \succ_i u$ by assumption, then $\succcurlyeq_i \notin U_{\mathcal{I}}$, a contradiction. Therefore, $f(x_N) \notin I(x_i, v)$ for each $i \in N$. By Lemma 2 above it follows that $f(x_N) = f(v, ..., v) = f(x'_N)$, a contradiction again, whence the thesis is established. ■

It turns out that *Theorem 3 implies at once that simple/individual and coalitional strategy-proofness on the full $\mathcal{I}$-unimodal domain are equivalent if $\mathcal{I} = (X, I)$ is an antisymmetric idempotent interval space with at most three points,* as made precise by the following Corollary (see Barberà, Berga, Moreno (2010) for a closely related but -strictly speaking- independent result):

Corollary 4 *Let $\mathcal{I} = (X, I)$ be an antisymmetric idempotent interval space such that $\#X \leq 3$, and $f : X^N \rightarrow X$ a voting rule that is strategy-proof on the full unimodal domain $U_{\mathcal{I}}^N$. Then, f is also coalitionally strategy-proof on $U_{\mathcal{I}}^N$.*

Proof. To begin with, notice that if $\#X \leq 3$, then any interval space (X, I) is convex: indeed, recall that in order to be *not* convex an interval space has to include at least two points x, y and two points u, v such that $\{u, v\} \subseteq I(x, y)$ but $I(u, v) \nsubseteq I(x, y)$ whence at least *four* points are needed. It is also immediately checked that *every* antisymmetric idempotent interval space $\mathcal{I} = (X, I)$ with $\#X \leq 3$ does satisfy Interval Anti-Exchange: to see that, take $X = \{x, y, z\}$ and assume that on the contrary there exist $a, b, c, d \in X$ such that $a \neq b$, $a \in I(b, c)$, $b \in I(a, d)$, and $a \notin I(c, d)$. Now, $a \notin I(c, d)$ implies $a \notin \{c, d\}$ hence either $c = d$ or $c = b$ or else $d = b$. If $c = d$ then by antisymmetry $a = b$, a contradiction. If $c = b$ then $a \in I(b, b)$ hence by idempotence $a = b$, a contradiction again. Then, it must be the case that $d = b$ whence $a \in I(d, c) = I(c, d)$, a contradiction again. But then, Theorem 3 applies and the proof is complete. ■

It should also be emphasized that Theorem 3 above amounts to a considerable generalization of the previous results on equivalence of simple and coalitional strategy-proofness due to Moulin (1980) and Danilov (1994), concerning (bounded) chains and trees, respectively. Clearly, Theorem 3 applies to all trees and linear geometries. However, its scope is much wider than trees or linear geometries: it clearly includes interval spaces that are induced by the geodesics of some graphs with cycles but are not linear geometries. To check the latter statement just consider the following simple

Example 2 (**A convex geometry that satisfies Interval Anti-Exchange but is not a linear geometry**)

Take an idempotent interval space $\mathcal{I} = (X, I)$ with $X = \{x, y, v, z\}$, $\#X = 4$, and I as defined by the following rule: $I(x, z) = I(z, x) = X$ and $I(a, b) = \{a, b\}$ for any $a, b \in X$ such that $\{a, b_1\} \neq \{x, z\}$ (see Coppel (1998)). Indeed, $\mathcal{I}$ is the interval space induced by the *quasi-complete graph* with vertex set X obtained by removing edge xz from the complete graph on X. It is readily checked that interval space $\mathcal{I}$ is convex (and, in fact, a convex geometry: to confirm the latter statement, suppose that $c \in I(a, b_1)$, $d \in I(c, b_2)$; if $\{a, b_1\} \neq \{x, z\}$ then $c \in \{a, b_1\}$, whence $c \in I(a, b_1)$; if $\{a, b_1\} = \{x, z\}$ then $c \in I(a, b_1) = X$ hence in any case Peano convexity holds). Interval space $\mathcal{I}$ also satisfies the no-branchpoint property: if $c \notin I(a, b)$ and $b \notin I(a, c)$ then clearly $b \neq c$ and $\{a, b\} \neq \{x, z\} \neq \{a, c\}$ hence $I(a, b) \cap I(a, c) = \{a\}$). However, $\mathcal{I}$ cannot possibly be induced by any linear geometry or tree since I does not satisfy the additivity property: e.g. $y \in I(x, z)$, while $I(x, y) \cup I(y, z) = \{x, y, z\} \neq X = I(x, z)$. Also, $\mathcal{I}$ is *not* a *median interval space (notice that e.g.* $I(x, y) \cap I(x, v) \cap I(y, v) = \varnothing$). On the other hand, it is straightforward to verify that $\mathcal{I}$ satisfies IAE: suppose $a \neq b$, $a \in I(b, c)$, $b \in I(a, d)$. If $\{a, d\} \neq \{x, z\}$ then $a \in I(b, c)$ entails $a = c$: therefore $a \in I(a, d) = I(c, d)$. If $\{b, c\} \neq \{x, z\}$, then $b \in I(a, d)$ entails $b = d$ hence $a \in I(b, c) = I(c, d)$. If on the contrary $\{a, d\} = \{x, z\} = \{b, c\}$ then it must be the case that $a = c$ and $b = d$ whence $a \in I(a, b) = I(c, d)$. In any case $a \in I(c, d)$ and IAE is therefore satisfied.

Concerning linear geometries, that fall entirely under the scope of Theorem 3, it should be emphasized that they cover a wealth of interesting structures. To begin with, it should be recalled here that Euclidean convex sets can be shown to reduce to linear geometries with three special properties namely *denseness, unendingness,* and *completeness* (see Coppel (1998)).

Moreover, both chains and trees with the no-branchpoint property are special instances of linear geometries. But, as a matter of fact, the class of linear geometries is much wider than that. To mention just a pair of very simple interesting examples, consider the interval space $\mathcal{I}' = (X, I')$ induced by the clique or complete graph on X (i.e. with $I'(x,y) = \{x,y\}$ for all $x, y \in X$), and the interval space $\mathcal{I}'' = (Y, I'')$ induced by the join or linear sum of a clique and a chain (see Section 2 above): it is readily checked that both of them are indeed linear geometries (and neither of them is a median interval space). Interval space $\mathcal{I}''$ is particularly interesting in the present connection, since the results provided by Schummer, Vohra (2002) imply the existence of nontrivial nondictatorial voting rules on the full unimodal domain $U^N_{\mathcal{I}''}$(e.g. those resulting from the combination of a clique-related, locally-dictatorial rule that applies whenever the top outcome of the appointed 'clique-dictator' lies on the clique, and a median rule as restricted to the subset of outcomes lying between the top outcome of the 'clique-dictator' and the outcome that is closest to the clique, that applies otherwise). Then, Theorem 3 above does indeed imply in turn that even all such nontrivial nondictatorial strategy-proof voting rules are also coalitionally strategy-proof on $U^N_{\mathcal{I}''}$.

We conclude with a partial converse result. Namely, a convex idempotent interval space $\mathcal{I}$ ensures equivalence of simple and coalitional strategy-proofness on the full unimodal domain only if it also satisfies Minimal Anti-Exchange, as established by the following:

Theorem 5 *Let $\mathcal{I} = (X, I)$ be a convex and idempotent interval space such that every voting rule $f : X^N \rightarrow X$ which is strategy-proof on the full unimodal domain $U^N_{\mathcal{I}}$ is also coalitionally strategy-proof on $U^N_{\mathcal{I}}$. Then, $\mathcal{I} = (X, I)$ satisfies Minimal Anti-Exchange (MAE).*

Proof. Indeed, suppose $\mathcal{I}$ does not satisfy MAE. Then, there exist $x, y, v, z \in X$ such that $x \neq y$, $v \neq z$, $x \in I(y,v)$, $y \in I(x,z)$, $v \in I(x,z)$, $z \in I(y,v)$, $I(v,z) \cap \{x,y\} = \varnothing$ and $I(y,z) \cap \{x,v\} = \varnothing$ (notice that by definition of I it follows at once that $\#X \geq 4$). But then, consider the following total preorder $\succcurlyeq^*$ on $\{x,y,v,z\}$:

$\succcurlyeq^* = \{(v,z),(v,x),(v,y),(z,x),(z,y),(x,y),(y,x),(x,x),(y,y),(v,v),(z,z)\}$, namely $v \succ^* z \succ^* x \sim^* y$.

Notice that by construction $\mathcal{I}$-unimodality of $\succcurlyeq^*$ only requires that both $x \notin I(v,z)$ and $y \notin I(v,z)$. Thus, $\succcurlyeq^*$ is $\mathcal{I}$-unimodal (and it can be extended to an $\mathcal{I}$-unimodal total preorder $\succcurlyeq'$on X with the same top element as $\succcurlyeq^*$: as any new element w is introduced one should just consider the join Y of all intervals $I(a,b)$ such that $w \in I(a,b)$, and make sure that w is indifferent to the lowest-ranked element(s) of Y; therefore, we can assume without loss of generality that $X = \{x,y,v,z\}$).

Next, consider another total preorder $\succcurlyeq'$on $\{x,y,v,z\}$:

$\succcurlyeq' = \{(y,z),(y,x),(y,v),(z,x),(z,v),(x,v),(v,x),(x,x),(y,y),(v,v),(z,z)\}$,

namely $y \succ' z \succ' x \sim' v$. Clearly, $\mathcal{I}$-unimodality of $\succcurlyeq'$ only requires that $x \notin I(y,z)$ and $v \notin I(y,z)$. Thus, $\succcurlyeq'$is also $\mathcal{I}$-unimodal.

Then, consider the class of all voting rules $f' : X^N \to X$ such that for all $\mathbf{u} = u_{N\smallsetminus\{1,2\}} \in X^{N\smallsetminus\{1,2\}}$

$f'(v,y,\mathbf{u}) = x$, and $f'(z,z,\mathbf{u}) = z$.

Let us now show that there exists a voting rule f in that class which is $\mathcal{I}$-monotonic. To see that, observe that $\mathcal{I}$-monotonicity of f amounts precisely to conditions (a)-(l) as listed below: for all $\mathbf{u} \in X^{N\smallsetminus\{1,2\}}$,

(a) $f(x,x,\mathbf{u}) \in I(x,f(y,x,\mathbf{u})) \cap I(x,f(v,x,\mathbf{u})) \cap I(x,f(z,x,\mathbf{u})) \cap$

$\cap I(x,f(x,y,\mathbf{u})) \cap I(x,f(x,v,\mathbf{u})) \cap I(x,f(x,z,\mathbf{u}))$ hence positing $f(x,x,\mathbf{u}) = x$ is clearly consistent with (a);

(b) $f(y,y,\mathbf{u}) \in I(y,f(x,y,\mathbf{u})) \cap I(y,f(v,y,\mathbf{u})) \cap I(y,f(z,y,\mathbf{u})) \cap$

$\cap I(y,f(y,x,\mathbf{u})) \cap I(y,f(y,v,\mathbf{u})) \cap I(y,f(z,y,\mathbf{u}))$ hence positing $f(y,y,\mathbf{u}) = y$ is clearly consistent with (b) (and (a));

(c) $f(v,v,\mathbf{u}) \in I(v,f(x,v,\mathbf{u})) \cap I(v,f(y,v,\mathbf{u})) \cap I(v,f(z,v,\mathbf{u})) \cap$

$\cap I(v,f(v,x,\mathbf{u})) \cap I(v,f(y,v,\mathbf{u})) \cap I(y,f(z,y,\mathbf{u}))$ hence positing $f(v,v,\mathbf{u}) = v$ is similarly consistent with the whole of (a),(b) and (c);

(d) $f(x,y,\mathbf{u}) \in I(x,f(y,y,\mathbf{u})) \cap I(x,f(v,y,\mathbf{u})) \cap I(x,f(z,y,\mathbf{u})) \cap$

$\cap\ I(y,f(x,x,\mathbf{u})) \cap I(y,f(x,v,\mathbf{u})) \cap I(y,f(x,z,\mathbf{u}))$ hence it must be the case that $f(x,y,\mathbf{u}) = x$ since by construction $I(x,f(v,y,\mathbf{u})) = I(x,x) = \{x\}$ (also notice that since by construction $x \in I(y,v)$ that value is certainly consistent with the whole of (a),(b),(c),(d) if $\{f(x,v,\mathbf{u}),f(x,z,\mathbf{u})\} \subseteq \{x,v\}$: so let us assume the latter inclusion as well);

(e) $f(v,x,\mathbf{u}) \in I(v,f(x,x,\mathbf{u})) \cap I(v,f(y,x,\mathbf{u})) \cap I(v,f(z,x,\mathbf{u})) \cap$

$\cap\ I(x,f(v,y,\mathbf{u})) \cap I(x,f(v,v,\mathbf{u})) \cap I(x,f(v,z,\mathbf{u}))$ hence $f(v,x,\mathbf{u}) = x$ since $I(x,f(v,y,\mathbf{u})) = I(x,x) = \{x\}$ (notice that that value is certainly consistent with the whole of (a),(b),(c)(d),(e) if $\{f(y,x,\mathbf{u}),f(z,x,\mathbf{u})\} \subseteq \{x,y\}$ as well: then, let us also assume that inclusion);

(f) $f(y,v,\mathbf{u}) \in I(y,f(x,v,\mathbf{u})) \cap I(y,f(v,v,\mathbf{u})) \cap I(y,f(z,v,\mathbf{u})) \cap$

$\cap\ I(v,f(y,x,\mathbf{u})) \cap I(v,f(y,y,\mathbf{u})) \cap I(v,f(y,z,\mathbf{u}))$ (notice that, therefore, positing $f(y,v,\mathbf{u}) = f(x,v,\mathbf{u}) = f(z,v,\mathbf{u}) = v$ is consistent with (a),(b),(c),(d),(e),(f) as introduced above);

(g) $f(y,z,\mathbf{u}) \in I(y,f(x,z,\mathbf{u})) \cap I(y,f(v,z,\mathbf{u})) \cap I(y,f(z,z,\mathbf{u})) \cap$

$\cap\, I(z,f(y,x,\mathbf{u})) \cap I(z,f(y,y,\mathbf{u})) \cap I(z,f(y,v,\mathbf{u}))$ hence, $f(y,z,\mathbf{u}) = z$ and $f(x,z,\mathbf{u}) = v$ are jointly consistent with (a),(b),(c),(d),(e),(f),(g) since by assumption $z \in I(y,v)$.

(h) $f(v,z,\mathbf{u}) \in I(v,f(x,z,\mathbf{u})) \cap I(v,f(y,z,\mathbf{u})) \cap I(v,f(z,z,\mathbf{u})) \cap$

$\cap\, I(z,f(v,x,\mathbf{u})) \cap I(z,f(v,y,\mathbf{u})) \cap I(z,f(v,v,\mathbf{u}))$: observe that, since $v \in I(x,z)$, $f(v,z,\mathbf{u}) = v$ is indeed consistent with (a),(b),(c),(d),(e),(f),(g),(h) as introduced above;

(i) $f(z,y,\mathbf{u}) \in I(z,f(x,y,\mathbf{u})) \cap I(z,f(y,y,\mathbf{u})) \cap I(z,f(v,y,\mathbf{u})) \cap$

$\cap\, I(y,f(z,x,\mathbf{u})) \cap I(y,f(z,v,\mathbf{u})) \cap I(y,f(z,z,\mathbf{u}))$ hence $f(z,y,\mathbf{u}) = y$ is consistent with (a),(b),(c),(d),(e),(f),(g),(h),(i)

since $y \in I(x,z) = I(z,f(x,y,\mathbf{u})) = I(z,f(v,y,\mathbf{u}))$;

(l) $f(z,v,\mathbf{u}) \in I(z,f(x,v,\mathbf{u})) \cap I(z,f(y,v,\mathbf{u})) \cap I(z,f(v,v,\mathbf{u})) \cap$

$\cap\, I(v,f(z,x,\mathbf{u})) \cap I(v,f(z,y,\mathbf{u})) \cap I(v,f(z,z,\mathbf{u}))$ hence in view of (e) $f(z,v,\mathbf{u}) = z$ and $f(z,x,\mathbf{u}) = y$ are jointly consistent with (a),(b),(c),(d),(e),

(f),(g),(h),(i),(l) since $z \in I(y,v) = I(v,f(z,x,\mathbf{u})) = I(v,f(z,y,\mathbf{u}))$;

Thus, we have just shown that there indeed exists a voting rule f that satisfies all of the requirements (a)-(l) above, and is therefore $\mathcal{I}$-monotonic, while at the same time being such that for all $\mathbf{u} = u_{N \smallsetminus \{1,2\}} \in X^{N \smallsetminus \{1,2\}}$

$f(v,y,\mathbf{u}) = x$, and $f(z,z,\mathbf{u}) = z$.

Now, take any profile $(\succcurlyeq_i)_{i \in N} \in U_{\mathcal{I}}^N$ of $\mathcal{I}$-unimodal total preorders on X such that $\succcurlyeq_1 = \succcurlyeq^*$and $\succcurlyeq_2 = \succcurlyeq'$.

Then, by construction, $top(\succcurlyeq_1) = v$, $top(\succcurlyeq_2) = y$, $z \succ_1 x$, $z \succ_2 x$, $f(v,y,(top(\succcurlyeq_i)_{i \in N \smallsetminus \{1,2\}})) = x$, and $f(z,z,(top(\succcurlyeq_i)_{i \in N \smallsetminus \{1,2\}})) = z$. It follows that f is *not* coalitionally strategy-proof, yet in view of Lemma 1 f is (individually) strategy-proof, a statement that contradicts our general hypothesis. ■

One of the simplest examples of a convex idempotent space that fails to satisfy MAE is the interval space (X,I) induced by the Boolean lattice $\mathbf{2^2} = (\{0,1,x,y\},\vee,\wedge)$ by taking $X = \{0,1,x,y\}$ and defining I by the rule $I(a,b) = \{c \in X : a \wedge b \leqslant c \leqslant a \vee b\}$ where $u \leqslant v$ if and only if $u = u \wedge v$. Indeed, the results of Savaglio, Vannucci (2012) imply equivalence failure in such an interval space (and, more generally, in any interval space induced by a bounded median semilattice that is not a chain). Now, as it is well-known, the interval spaces thus induced by distributive lattices are another prominent class of median interval spaces (along with the interval spaces of trees). Notice however that since both Hamming graphs and partial cubes typically include cubes, their (non-median) interval spaces also violate MAE and therefore -precisely as the interval space of a Boolean distributive lattice $\mathbf{2}^{\mathbf{K}}$ with $\mathbf{K} > 1$- admit nontrivial nondictatorial strategy-proof voting rules (such as rule f as defined in the proof of Theorem 5) that are *not* coalitionally strategy-proof. Indeed, in view of the proof of Theorem 5 (and as also suggested by Corollary 4 above), if a given convex idempotent interval space fails to satisfy MAE then there exists a nontrivial nondictatorial strategy-proof voting rule on the full unimodal domain of that space that admits at least *four* distinct outcomes, and is manipulable by some coalitions. Therefore Theorem 5 confirms that, generally speaking, 'unimodal' equivalence of simple and coalitional strategy-proofness fails to hold in certain important classes of interval spaces, both median and non-median.

4 Concluding remarks

It should be emphasized that the sufficient condition for equivalence of simple and coalitional strategy-proofness of voting rules on full unimodal domains that has been established in the present paper is in fact quite general. As repeatedly mentioned above, *Interval Anti-Exchange* (*IAE*) is shared by all trees and indeed by all linear geometries but is characteristic of a much larger class of convex idempotent interval spaces. Therefore, our results provide significant information concerning problems of strategy-proof location in a vast class of networks.

We have also established that any convex idempotent interval space where the foregoing 'unimodal' equivalence obtains must satisfy *Minimal Anti-Exchange (MAE),* which in turn implies that such equivalence fails to hold in certain convex idempotent interval spaces, both median and non-median (and that such spaces admit the existence of nontrivial nondictatorial strategy-proof voting rules with at least four distinct outcomes on their full unimodal domains). It remains to be seen whether or not some convex, idempotent interval spaces that satisfy MAE while violating IAE do also support such an equivalence of simple and coalitional strategy-proofness of voting rules on full unimodal domains.

References

[1] Barberà S., D. Berga, B. Moreno (2010): Individual versus group strategy-proofness: When do they coincide?, *Journal of Economic Theory* 145, 1648-1674.

[2] Chvátal V., D. Rautenbach, P.M. Schäfer (2011): Finite Sholander trees, trees, and their betweenness, arXiv:1101.2957v1 [math.CO]

[3] Coppel W.A. (1998): *Foundations of convex geometry.* Cambridge University Press, Cambridge UK.

[4] Danilov V.I. (1994): The structure of non-manipulable social choice rules on a tree, *Mathematical Social Sciences* 27, 123-131.

[5] Danilov V.I., A.I. Sotskov (2002): *Social choice mechanisms.* Springer, Berlin.

[6] Eppstein D., J.-C. Falmagne, S. Ovchinnikov (2008): *Media Theory.* Springer, Berlin Heidelberg.

[7] Le Breton M., V. Zaporozhets (2009): On the equivalence of coalitional and individual strategy-proofness properties, *Social Choice and Welfare* 33, 287-309.

[8] Moulin H. (1980): On strategy-proofness and single peakedness, *Public Choice* 35, 437-455.

[9] Mulder H.M. (1980): *The interval function of a graph.* Mathematical Centre Tracts 132, Amsterdam.

[10] Mulder H.M., L. Nebeský (2009): Axiomatic characterization of the interval function of a graph, *European Journal of Combinatorics* 30, 1172-1185.

[11] Nebeský L. (2007): The interval function of a connected graph and road systems, *Discrete Mathematics* 307, 2067-2073.

[12] Nehring K., C. Puppe (2007 (a)): The structure of strategy-proof social choice - Part I: General characterization and possibility results on median spaces, *Journal of Economic Theory* 135, 269-305.

[13] Nehring K., C. Puppe (2007 (b)): Efficient and strategy-proof voting rules: a characterization, *Games and Economic Behavior,* 59, 132-153.

[14] Prenowitz W., J. Jantosciak (1979): *Join geometries. A theory of convex sets and linear geometry*. Springer, New York.

[15] Savaglio E., S. Vannucci (2012): Strategy-proofness and unimodality in bounded distributive lattices, Working Paper DEPS 642, June 2012, Siena.

[16] Schummer J., R.V. Vohra (2002): Strategy-proof location on a network, *Journal of Economic Theory* 104, 405-428.

[17] Sholander M. (1952): Trees, lattices, order, and betweenness, *Proceedings of the American Mathematical Society* 3, 369-381.

[18] Sholander M. (1954): Medians and betweenness, *Proceedings of the American Mathematical Society* 5, 801-807.

[19] van de Vel M.L.J. (1993): *Theory of convex structures*. North Holland, Amsterdam.