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Abstract

The β-effectivity function of a strategic game form G describes the decision power of coalitions under G as contingent on the ability of each coalition to predict the behaviour of the complementary coalition. An effectivity function E is β-playable if there exists a strategic game form G such that E is the β-effectivity function of G.

It is shown that whenever the player set and the outcome set are *finite* an effectivity function E is β-playable if and only if E is both outcome-monotonic and polar-superadditive. Moreover, the underlying strategic game form only needs 'small' strategy spaces, whose size is linear in the size of the monotonic co-basis of E.

As a by-product of that result, a few new characterizations of *tight* finite effectivity functions are also obtained.

JEL Classification Numbers: C62, C70, D72.

1 Introduction

Both strategic and coalitional game forms purport to describe the relevant 'rules of the game', but to a different degree of detail. Strategic game forms provide information about 'who can do what *and* by which means/actions', while coalitional game forms just provide information about 'who can do what'. Effectivity functions are the main components of coalitional game forms, describing the events in outcome space that coalitions are able to enforce. An effectivity function -and the associated coalitional game form- can be attached to a strategic game form in several ways, but the two most widely known and used are by far the α-rule and the β-rule. The α-rule -resulting in the α-effectivity function of the given strategic game form- declares an event A enforceable by a certain coalition S if there exists a strategy profile for its players that will invariably result in an outcome that is consistent with event A, independently of the strategies chosen by other players. By contrast, the β-rule –resulting in the β-effectivity function of the given strategic game form- declares an event A enforceable by a certain coalition S if for each strategy profile chosen by members of the complementary coalition there exists a strategy profile for players of S that will result in an outcome that is consistent with event A.

That observation immediately raises the converse *representation* issue, namely: what effectivity functions are *α-playable* i.e. can be represented as the α-effectivity function of an underlying strategic game form? What effectivity functions are *β-playable* i.e. can be represented as the β-effectivity function of an underlying strategic game form? What effectivity functions are tight i.e. are both *α-playable* and *β-playable* with respect to the *same* strategic game form?

Such a representation problem has been quite extensively studied and solved for finite α-playable EFFs (see e.g. Moulin (1983), Otten, Borm, Storcken and Tijs (1995), Peleg (1998), Boros, Elbassioni, Gurvich and Makino (2010)) and for finite *tight EFFs i.e.* β-playable EFFs that are also α-playable with respect to the same strategic game form (see Moulin (1983)), *but it has been virtually ignored for finite β-playable EFFs* (clearly, an EFF E may be β-playable but *not* α-playable).

The main aim of the present note is to fill this gap in the literature by providing a simple characterization of the entire class of finite β-playable EFFs . We also obtain, as a by-product, a few alternative characterizations *of* ***tight*** *finite effectivity functions.*

2 Characterizing β-playable effectivity functions

A **strategic game form** is an array $G = (I, X, (S_i)_{i\in I}, g)$ where I, X, S_i are nonempty sets denoting the player set, the outcome set, and the strategy set of player $i \in I$, respectively, and $g \in X^{\Pi_{i\in I} S_i}$ denotes the (surjective) *strategic outcome function* of G. A **coalitional game form** is an array $\Gamma = (I, X, E)$ where I, X are nonempty sets denoting the player set and the outcome set, and $E \in 2^{2^I \times 2^X}$ -with $2 := \{0, 1\}$, $E(K, X) = E(I, A) = 1$ and $E(K, \emptyset) = E(\emptyset, A) = 0$ for any $K \subseteq I$ and $A \subset X$- denotes the **effectivity function** *(EFF)* of Γ . Indeed, in view of the natural bijection between $2^I \times 2^X$ and $2^{I\cup X}$, an effectivity function $E \in 2^{2^I \times 2^X}$ may be equivalently regarded as a **Boolean function** $E \in 2^{2^{I\cup X}}$ (see Crama, Hammer (2011) for a thorough, up-to-date treatment of Boolean functions). If $E, E' \in 2^{2^{I\cup X}}$ we also write $E \leqslant E'$ whenever $E(S) \leq E'(S)$ for each $S \in 2^X$.

Whenever I and X are clearly fixed, it is not uncommon -and totally innocent- to identify G and Γ with g and E.

While it turns out that effectivity functions (and coalitional game forms) can be attached to strategic game forms by several distinct rules, the two following ones, first introduced by Moulin and Peleg (1982), have played a prominent role in the extant literature:

(α-rule) for any $K \subseteq I$, $A \subseteq X$, $E_g^\alpha(K, A) = 1$ iff there exists $x_K \in \Pi_{i\in K} S_i$ such that $g(x_K, x_{I\setminus K}) \in A$ for each $x_{I\setminus K} \in \Pi_{i\in I\setminus K} S_i$: E_g^α is the α-EFF of g.

(β-rule) for any $K \subseteq I$, $A \subseteq X$, $E_g^\beta(K, A) = 1$ iff for each $x_{I\setminus K} \in \Pi_{i\in I\setminus K} S_i$ there exists $x_K \in \Pi_{i\in K} S_i$ such that $g(x_K, x_{I\setminus K}) \in A$: E_g^β is the β-EFF of g.

An effectivity function E is α-**playable** iff there exists a strategic game form $G = (I, X, (S_i)_{i\in I}, g)$ such that $E = E_g^\alpha$, and β-**playable** iff there exists a strategic game form $G = (I, X, (S_i)_{i\in I}, g)$ such that $E = E_g^\beta$. Furthermore, an effectivity function E is **tight** iff there exists a strategic game form $G = (I, X, (S_i)_{i\in I}, g)$ such that $E = E_g^\alpha = E_g^\beta$.

The following properties of EFFs will play a crucial role in the ensuing analysis: an EFF $E \in 2^{2^{I\cup X}}$ is

outcome-monotonic iff for each $B \subseteq B' \subseteq X$,

$$E(K, B) = 1 \quad \text{implies } E(K, B') = 1;$$

player-monotonic iff for each $K \subseteq K' \subseteq I$,

$$E(K,B)=1 \text{ implies } E(K',B)=1;$$

monotonic iff it is both outcome-monotonic and player-monotonic;

superadditive iff for each $B \subseteq B' \subseteq X, K \subseteq K' \subseteq I$: $E(K,B) = E(K',B') = 1$ and $K \cap K' = \emptyset$ imply $E(K \cup K', B \cap B') = 1$;

polar-superadditive iff for each $B \subseteq B' \subseteq X, K \subseteq K' \subseteq I : E(K,B) = E(K',B') = 0$ and $K \cup K' = I$ imply $E(K \cap K', B \cup B') = 0$;

k-**superadditive** (for any finite integer $k \geq 2$) iff for each

$\{(K_j, B_j) : j \in J'\} \subseteq E^{-1}(1)$ such that $|J'| \leq k$, if $K_j \cap K_{j'} = \emptyset$ for any $j, j' \in J', j \neq j'$, then $E(\cup_{j\in J'} K_j, \cap_{j\in J'} B_j) = 1$;

fully superadditive iff it is $k-$*superadditive* for each k, $2 \leq k \leq |E^{-1}(1)|$;

k-**polar-superadditive** (for any finite integer $k \geq 2$) iff for each

$\{(K_h, B_h) : h \in H'\} \subseteq E^{-1}(0)$ such that $|H'| \leq k$, if $K_h \cup K_{h'} = I$ for any $h, h' \in H', h \neq h'$, then $E(\cap_{h\in H'} K_h, \cup_{h\in H'} B_h) = 0$;

fully polar-superadditive iff it is $k-$*polar-superadditive* for each k, $2 \leq k \leq |E^{-1}(0)|$;

upper acyclic iff for each $\{(K_j, B_j) : j \in J'\} \subseteq E^{-1}(1)$, if $K_j \cap K_{j'} = \emptyset$ for any $j, j' \in J', j \neq j'$, then $\cap_{j\in J'} B_j \neq \emptyset$;

upper polar-acyclic iff for each $\{(K_h, B_h) : h \in H'\} \subseteq E^{-1}(0)$, if $K_h \cup K_{h'} = I$ for any $h, h' \in H', h \neq h'$, then $\cup_{h\in H'} B_h \neq X$;

regular iff for each $A, B \subseteq X, K \subseteq I$: $E(K,A) = 1$ and $E(I \setminus K, B) = 1$ imply $A \cap B \neq \emptyset$;

maximal iff for each $A \subseteq X, K \subseteq I$: $E(K,A) = 0$ implies that there exists $B \subseteq X$ such that $E(I \setminus K, B) = 1$ and $A \cap B = \emptyset$.

Remark 1 *Polar-superadditivity is also used by Abdou and Keiding (1991) under the label '*-superadditivity', and by Gurvich (1992) where it is also denoted as 'transadditivity'. Upper acyclic effectivity functions are introduced and used by Abdou and Keiding (1991), and by Otten et al.(1995). They are also used by Boros et al. (2010) where they are denoted as 'weakly superadditive' effectivity functions. Notice that in some of the literature either EFFs are* not *defined over the empty set (see e.g. Gärdenfors (1981), Peleg (1984), Otten et al. (1995)) or the normalization $E(\emptyset) = \emptyset$ is preferred (see e.g. Moulin and Peleg (1982), Moulin (1983), Abdou and Keiding (1991), Peleg and Peters (2010)). The normalization $E(\emptyset) = \{X\}$ adopted here is also used by Peleg (1998) and by Boros et al.(2010), and will be shown below* *to be in fact very convenient. Both normalizations are explicitly considered by Gurvich (1992).*

The foregoing properties are not independent. In particular, it is easily checked the following

Claim 2 *(i) If EFF E is superadditive then it is also player-monotonic, regular and k-superadditive (for any finite integer $k \geq 2$). If, moreover, the player set I is finite, then a superadditive EFF E is also fully superadditive and upper acyclic; (ii) if EFF E is polar-superadditive then it is also player-monotonic, maximal and k-polar-superadditive (for any finite integer $k \geq 2$). If, moreover, both the player set I and the outcome set X are finite, then a polar-superadditive E is also fully polar-superadditive and upper polar-acyclic.*

Proof. (i) Let E be superadditive. If $E(K,A) = 1$ and $K \subseteq K'$ then $K \cap (K' \smallsetminus K) = \varnothing$ while of course $E(K' \smallsetminus K, X) = 1$. Thus, by superadditivity, $E(K', A) = 1$ hence player-monotonicity holds. Also, if $A, B \subseteq X, K \subseteq I$ are such that $E(K,A) = 1$ and $E(I \setminus K, B) = 1$ then by superadditivity $E(I, A \cap B) = 1$ whence $A \cap B \neq \varnothing$, and regularity holds. Next, consider any family $\{(K_h, B_h) : h \in H\} \subseteq E^{-1}(1)$ such that $K_h \cap K_{h'} = \varnothing$ for any $h, h' \in H$, $h \neq h'$. Therefore, by superadditivity $E(K_h \cup K_{h'}, B_h \cap B_{h'}) = 1$ for any $h, h' \in H$. Then take any $h'' \in H \setminus \{h, h'\}$. Clearly, $(K_h \cup K_{h'}) \cap K_{h''} = (K_h \cap K_{h''}) \cup (K_{h'} \cap K_{h''}) = \varnothing$, hence by superadditivity again $E(K_h \cup K_{h'} \cup K_{h''}, B_h \cap B_{h'} \cap B_{h''}) = 1$. It follows that, by induction, E is k-superadditive for any finite integer $k \geq 2$. If, moreover, I is finite, any family $\{(K_h, B_h) : h \in H'\} \subseteq E^{-1}(1)$ such that $K_h \cap K_{h'} = \varnothing$ for any $h, h' \in H', h \neq h'$ must be finite. Thus, E is also fully superadditive. But then, take any $\{(K_j, B_j) : j \in J'\} \subseteq E^{-1}(1)$, such that $K_j \cap K_{j'} = \emptyset$ for any $j, j' \in J', j \neq j'$. Then, by full superadditivity of E, $E(\cup_{h \in H} K_h, \cap_{h \in H} B_h) = 1$, whence $\cap_{h \in H} B_h \neq \varnothing$. It follows that E is upper acyclic.

(ii) Let EFF E be polar-superadditive. To see that player-monotonicity holds, take any $K, K' \subseteq I, B \subseteq X$ such that $E(K', B) = 0$ and $K \subseteq K'$, and consider $I \setminus (K' \setminus K)$. Since E is an EFF, $E(I \setminus (K' \setminus K), \emptyset) = 0$. Moreover, $(I \setminus (K' \setminus K)) \cup K' = I$ hence by polar superadditivity $E(K, B) = E(I \setminus (K' \setminus K)) \cap K', B) = 0$. To check maximality, suppose that on the contrary there exist $A \subseteq X, K \subseteq I$ such that $E(K, A) = 0$ and for all $B \subseteq X$, if $A \cap B = \emptyset$ then $E(I \setminus K, B) = 0$: thus, in particular, $E(I \setminus K, X \smallsetminus A) = 0$ hence by polar-superadditivity $E(\varnothing, X) = E(K \cap (I \smallsetminus K), A \cup (X \smallsetminus A)) = 0$, a contradiction. Now, consider any family $\{(K_h, B_h) : h \in H\} \subseteq E^{-1}(0)$ such that $K_h \cup K_{h'} = I$ for any $h, h' \in H$, $h \neq h'$. Therefore, by polar superadditivity $E(K_h \cap K_{h'}, B_h \cup B_{h'}) = 0$ for any $h, h' \in H$. Then take any $h'' \in H \setminus \{h, h'\}$. Clearly, $(K_h \cap K_{h'}) \cup K_{h''} = (K_h \cup K_{h''}) \cap (K_{h'} \cup K_{h''}) = I$, hence by polar-superadditivity again $E(K_h \cap K_{h'} \cap K_{h''}, B_h \cup B_{h'} \cup B_{h''}) = 0$. It follows that, by induction, E is k-superadditive for any finite integer $k \geq 2$. Moreover, if I and X are finite then for any $\{(K_h, B_h) : h \in H'\} \subseteq E^{-1}(0)$, $|H'| = k$ for some finite integer k is hence, by k-polar superadditivity, $E(\cap_{h \in H} K_h, \cup_{h \in H} B_h) = 0$ i.e. E is fully polar-superadditive. It also follows that $\cup_{h \in H} B_h \neq X$ hence E is upper polar-acyclic as well. ■

Remark 3 *Notice that the normalization $E(\varnothing, X) = 1$ plays a key role in the foregoing proof that polar-superadditivity implies maximality. In fact, under the alternative choice of normalization $E(\varnothing, X) = 0$, maximality can only hold in a weaker form, for K, A, $\varnothing \neq K \subset I$, $\varnothing \neq A \subset X$. Some parts of the foregoing claim -and of Proposition 3 below- can be found in Gurvich (1991), where they are stated without proof. Notice, however, that Gurvich (1991) employs both a weaker notion of regularity and a stronger notion of maximality than those adopted in the present paper: thus, some care is needed when making comparisons between the content of Claim 2 above -and Proposition 3 below- and their relevant counterparts in Gurvich (1991).*

The following simple, useful proposition can also be easily established:

Proposition 4 *(i) If EFF E is superadditive and maximal then it is also polar-superadditive; (ii) if EFF E is polar-superadditive and regular then it is also superadditive.*

Proof. (i) Suppose E is superadditive and maximal but not polar-superadditive i.e. there exist $K_1, K_2 \subseteq I$, $A, B \subseteq X$ such that $E(K_1, A) = E(K_2, B) = 0$, $K_1 \cup K_2 = I$ and $E(K_1 \cap K_2, A \cup B) = 1$. Thus, by maximality, $E(I \smallsetminus K_1, X \smallsetminus A) = E(I \smallsetminus K_2, X \smallsetminus B) = 1$. Since $(I \smallsetminus K_1) \cap (I \smallsetminus K_2) = I \smallsetminus (K_1 \cup K_2) = \varnothing$, it follows that superadditivity entails $E(I \smallsetminus (K_1 \cap K_2), X \smallsetminus (A \cup B)) = E((I \smallsetminus K_1) \cup (I \smallsetminus K_2), (X \smallsetminus A) \cap (X \smallsetminus B)) = 0$. However, by regularity as implied by superadditivity of E (in view of Claim 2 (i)), $E(K_1 \cap K_2, A \cup B) = 1$ entails $E(I \smallsetminus (K_1 \cap K_2), X \smallsetminus (A \cup B)) = 0$, a contradiction.

(ii) Suppose E is polar-superadditive and regular but not superadditive i.e. there exist $K_1, K_2 \subseteq I$, $A, B \subseteq X$ such that $E(K_1, A) = E(K_2, B) = 1$, $K_1 \cap K_2 = \varnothing$ and $E(K_1 \cup K_2, A \cap B) = 0$. Thus, by regularity, $E(I \smallsetminus K_1, X \smallsetminus A) = E(I \smallsetminus K_2, X \smallsetminus B) = 0$. Since $(I \smallsetminus K_1) \cup (I \smallsetminus K_2) = I \cap (K_1 \cap K_2) = I$, it follows that polar-superadditivity entails $E(I \smallsetminus (K_1 \cup K_2), X \smallsetminus (A \cap B)) = E((I \smallsetminus K_1) \cap (I \smallsetminus K_2), (X \smallsetminus A) \cup (X \smallsetminus B)) = 0$. However, by maximality as implied by polar-superadditivity of E (in view of Claim 2 (ii)), $E(K_1 \cup K_2, A \cap B) = 0$ entails $E(I \smallsetminus (K_1 \cup K_2), X \smallsetminus (A \cap B)) = 1$, a contradiction.
■

The following notions will also play a key role in the ensuing analysis.

For any EFF E it can be defined its (monotonic) *basis*

$$\mathcal{B}(E) := \{(K_j, B_j) : j \in J\} = \left\{ \begin{array}{c} (K, B) : E(K, B) = 1 \text{ and } E(K', B') = 0 \\ \text{for each } (K', B') \neq (K, B) \text{ such that} \\ K' \subseteq K \text{ and } B' \subseteq B \end{array} \right\}$$

and its (monotonic) *co-basis*

$$\mathcal{B}^*(E) := \{(K_h, B_h) : h \in H\} = \left\{ \begin{array}{c} (K, B) : E(K, B) = 0 \text{ and } E(K', B') = 1 \\ \text{for each } (K', B') \neq (K, B) \text{ such that} \\ K \subseteq K' \text{ and } B \subseteq B' \end{array} \right\}.$$

Clearly, if E is *monotonic* then it is uniquely defined both by its basis and by its co-basis, indeed

(i) $E(K, B) = 1$ if and only if there exists $(\widetilde{K}, \widetilde{B}) \in \mathcal{B}(E)$ such that $\widetilde{K} \subseteq K$ and $\widetilde{B} \subseteq B$

and

(ii) $E(K, B) = 0$ if and only if there exists $(K^*, B^*) \in \mathcal{B}^*(E)$ such that $K \subseteq K^*$ and $B \subseteq B^*$

Therefore, one may safely represent any *monotonic* EFF E either by its basis $\mathcal{B}(E) = \{(K_j, B_j) : j \in J\}$ as indexed by $J = J(E)$ or equivalently by its co-basis $\mathcal{B}^*(E) = \{(K_h, B_h) : h \in H\}$ as indexed by $H = H(E)$.

It is well-known and easily checked that both E_g^α and E_g^β are *monotonic*, and E_g^α is also *superadditive*. Conversely, it was established by Moulin (1983) that if $E \in 2^{2^I \times 2^X}$ (with *finite* I and X) is a monotonic and superadditive EFF then there exists a strategic game form g such that $E = E_g^\alpha$. Otten et al. (1995) and Peleg (1998) subsequently extended that result to the case of an *infinite* A. Unfortunately, all the relevant proofs in the foregoing works rely on a g with *huge* strategy spaces (namely, for each $i \in I$, the size of S_i is *doubly exponential* in the size of $\mathcal{B}(E)$). However, the following remarkable result due to Boros et al. (2010) establishes that a strategic game form with a much smaller strategy space -*linear* in $|\mathcal{B}(E)|$- is also sufficient to ensure α-playability of an outcome-monotonic, superadditive effectivity function, namely

Theorem 5 *(Boros et al. (2010)): An EFF E with finite I, X is outcome-monotonic and superadditive if and only if there exists a strategic game form $G = (I, X, (S_i)_{i \in I}, g)$ such that $E = E_g^\alpha$ and*
$|S_i| = |\{j \in J : E(K_j, B_j) = 1 \text{ and } i \in K_j\}| + |X|$ for each $i \in I$.

Remark 6 *Actually, Boros et al. (2010) do not distinguish between monotonicity and the weaker outcome-monotonicity property, and state their result -in a slightly redundant way- in terms of 'monotonic and superadditive EFFs'. The strategic game form $G = (I, X, (S_i)_{i \in I}, g)$ with $|\mathcal{B}(E)|$-linear strategy spaces which does the job is defined as follows: first, take $\mathcal{B}(E) := \{(K_j, B_j) : j \in J\}$ and for each $i \in I$ and $j \in J$ such that $i \in K_j$ posit $y_i^j = (i, K_j)$, and $Y_i^{\mathcal{B}(E)} = \{y_i^j : j \in J\}$ and $\mathbb{Z}_X = \{0, 1, 2, .., z_k, ..\}$ such that $|\mathbb{Z}_X| = |X|$, and define $S_i = Y_i^{\mathcal{B}(E)} \cup \mathbb{Z}_X$ (the strategy set of player i). For any strategy profile $s = (s_i)_{i \in I} \in \prod_{i \in I} S_i$ and any $K \subseteq I$, a strategy subprofile $(s_i)_{i \in I \setminus K} \in \prod_{i \in I \setminus K} S_i$ is said to be* proper *if there exists $j \in J$ such that $K = K_j$ and $s_i = y_i^j \in Y_i^{\mathcal{B}(E)}$ for each $i \in K$.*

Next, take $g : \prod_{i \in I} S_i \rightarrow X$ such that for any $s = (s_i)_{i \in I} \in \prod_{i \in I} S_i$ and any $K \subseteq I$,

if $(s_i)_{i \in K}$ is proper and $K = K_j$ then $g((s_i)_{i \in I}) \in B_j$.

It should also be emphasized here that -in view of Claim 2 (i)- Theorem 3 above can be easily extended to the case of a countable outcome set.

As mentioned above, the corresponding issue concerning *general* conditions for β-playability of (finite) effectivity functions has been virtually ignored. To be sure, Moulin (1983) has an early result characterizing -for *finite* I and X- the class of *tight effectivity functions* by the combination of *monotonicity, superadditivity and maximality.* However, generally speaking, a β-playable effectivity function need not be α-playable or *tight* (and conversely an α-playable effectivity function need not be β-playable). To check that statement, observe that (i) the α-effectivity function of a strategic game form is -by construction- monotonic and regular, (ii) the β-effectivity function of a strategic game form is -by construction- monotonic and maximal, and consider the following

Example 7 *Let $|I| \geq 2, |X| \geq 2$, and $E \in 2^{2^{I \cup X}}$ be defined as follows: $E(K, \emptyset) = E(\emptyset, A) = 0$ for each $K \subseteq N$ and each $A \subset X$, and $E(K, A) = E(\emptyset, X) = 1$ for each K, A such that $\emptyset \neq K \subseteq I$, $\emptyset \neq A \subseteq X$. It is immediately checked that E is a well-defined effectivity function that is* not *regular (indeed, take $x, y \in X$, $x \neq y$, and $K, K' \subseteq I$ such that $\emptyset \neq K \neq I \neq K' \neq \emptyset$ and $K \cup K' = I$: then, $E(K, \{x\}) = E(K', \{y\}) = 1$). Next, consider $E' \in 2^{2^{I \cup X}}$ defined as follows: $E'(K, A) = 0$ for each $K \subset I$ and $A \subset X$, and $E'(I, A) = E(K, X) = 1$ for each $K \subseteq I$ and $\varnothing \neq A \subseteq X$. Clearly, E' is a well-defined effectivity function that is* not *maximal (indeed, $E'(K, A) = E'(K', B) = 0$ for any $K, K' \subseteq I$, $A, B \subseteq X$ such that $\emptyset \neq K \neq I \neq K' \neq \emptyset$, $K \cup K' = I$, $\emptyset \neq A \neq X \neq B \neq \emptyset$ and $A \cup B = X$). Now, consider the strategic game form $(I, X, (S_i)_{i \in N}, g_{pl})$ modeling the 'proportional lottery' (or 'random dictatorship') voting mechanism (see e.g. Danilov and Sotskov (2002), Vannucci (2008)), where for all $i \in I$, $S_i = X \times \left\{z \in \mathbb{Z}_+ : z \leq 10^k\right\}$ for some positive integer k, and $g_{pl}((s_i)_{i \in I}) = x(s_{i^*})$ with $i^* = \sum_{i \in I} z(s_i)(\mathrm{mod}\, |I|)$, for all $(s_i)_{i \in I} \in \Pi_{i \in I} S_i$. It is immediately checked that $E^{\alpha}_{g_{pl}} = E'$ and $E^{\beta}_{g_{pl}} = E$. Hence E' is α-playable but not β-playable, while E is β-playable but not α-playable.*

It turns out that the following β-counterpart to Theorem 5 holds:

Theorem 8 *An EFF E with finite I and X is outcome-monotonic and polar-superadditive if and only if there exists a strategic game form $g \in X^{\Pi_{i \in I} S_i}$ such that $E = E^{\beta}_g$ and $|S_i| = |\{h \in H : E(K_h, B_h) = 0$ and $i \notin K_h\}| + |X|$ for each $i \in I$.*

Proof. First, notice again that if $E = E^{\beta}_g$ for some strategic game form g then E is obviously outcome-monotonic (indeed, monotonic) by construction. To see that in that case E is also polar-superadditive, observe that for any $A, B \subseteq X$ and any $K, K' \subseteq I$ if $E^{\beta}_g(K, A) = E^{\beta}_g(K', B) = 0$ then by definition there exist $s_{I \setminus K} \in \prod_{i \in I \setminus K} S_i$ and $t_{I \setminus K'} \in \prod_{i \in I \setminus K'} S_i$ such that for all $s_K \in \prod_{i \in K} S_i$ and for all $t_{K'} \in \prod_{i \in K'} S_i$, $g(s) \notin A$ and $g(t) \notin B$, whence by definition again $E_g^\alpha(I \setminus K, X \setminus A) = E_g^\alpha(I \setminus K', X \setminus B) = 1$. Moreover, if $K \cup K' = I$ then $(I \setminus K) \cap (I \setminus K') = I \setminus (K \cup K') = \varnothing$. But then, since E_g^α -as it is well-known and immediately checked- is superadditive (for any strategic game form g), it follows that $E_g^\alpha((I \setminus K) \cup (I \setminus K'), (X \setminus (A \cup B))) = E_g^\alpha((I \setminus (K \cap K'), (X \setminus (A \cup B))) = 1$ thus by definition $E_g^\beta(K \cap K', A \cup B) = 0$, hence polar superadditivity of E_g^β holds.

Conversely, assume E is polar-superadditive and outcome-monotonic (hence monotonic, in view of Claim 2 (ii)), and let

$\{(K_h, B_h) : h \in H\} := \mathcal{B}^*(E)$ be the co-basis of E as defined above.

Now, for any $i \in I$ and any $h \in H$ such that $i \notin K_h$ posit $y_i^h = (i, K_h)$, $Y_i^{\mathcal{B}^*(E)} = \left\{y_i^h : h \in H\right\}$ and $\mathbb{Z}_X = \{0, 1, 2, .., z_k, ..\}$ such that $|\mathbb{Z}_X| = |X|$, and define $S_i = Y_i^{\mathcal{B}^*(E)} \cup \mathbb{Z}_X$ (the strategy set of player i).

For any strategy profile $s = (s_i)_{i \in I} \in \prod_{i \in I} S_i$ and any $K \subseteq I$, a strategy subprofile $(s_i)_{i \in I \setminus K} \in \prod_{i \in I \setminus K} S_i$ is said to be *co-proper* if there exists $h \in H$ such that $K = K_h$ and $s_i = y_i^h \in Y_i^{\mathcal{B}^*(E)}$ for each $i \in I \setminus K$.

Next, take $g : \prod_{i \in I} S_i \to X$ such that for any $s = (s_i)_{i \in I} \in \prod_{i \in I} S_i$ and any $K \subseteq I$,

if $(s_i)_{i \in I \setminus K}$ is co-proper and $K = K_h$ then $g((s_i)_{i \in I}) \notin B_h$.

To see that such a requirement is well-defined for any polar-superadditive E, notice that for any $s = (s_i)_{i \in I} \in \prod_{i \in I} S_i$ and any $K, K' \subseteq I$ such that both $(s_i)_{i \in K}$ and $(s_i)_{i \in K'}$ are co-proper, if $K \neq K'$ then, by construction, $K \cap K' = \varnothing$ i.e. $(I \setminus K) \cup (I \setminus K') = I$.

Therefore, since E is polar-superadditive and outcome-monotonic hence in particular upper polar-acyclic, it follows that for any $s = (s_i)_{i \in I} \in \prod_{i \in I} S_i$, if $H'(s) = \left\{h \in H : (s_i)_{i \in I \setminus K_h} \text{ is co-proper}\right\}$, then $\cup_{h \in H'(s)} B_h \neq X$.

As a consequence, it must be the case that for each $s \in \prod_{i \in I} S_i$, $g(s) \in X \setminus \cup_{h \in H'(s)} B_h \neq \varnothing$.

In particular, let us denote $I_{\mathbb{Z}_X}(s) = \{i \in I : s_i \in \mathbb{Z}_X\}$. Then posit $X \setminus \cup_{h \in H'(s)} B_h = \left\{x_{\sigma(0)}, ..., x_{\sigma(m)}\right\}$ where σ is a suitable permutation of X and m is a non-negative integer, and $g(s) = x_{\sigma(m^*)}$ with $m^* = (\sum_{i \in I_{\mathbb{Z}_X}(s)} s_i) \bmod m$.

Now, take any (K_h, B_h) such that $E(K_h, B_h) = 0$ and choose $s_i^* = y_i^h$ for each $i \in I \setminus K_h$. Thus, $(s_i^*)_{i \in I \setminus K_h}$ is co-proper hence by construction $g(s) \notin B_h$ for any $s \in \prod_{i \in I} S_i$ with $(s_i)_{i \in I \setminus K_h} = (s_i^*)_{i \in I \setminus K_h}$. It follows that, by definition, $E_g^{\beta}(K_h, B_h) = 0$ i.e. $E_g^{\beta} \leqslant E$.

To check the reverse inequality, let $K \subseteq I, B \subseteq X$ be such that $E_g^{\beta}(K, B) = 0$, namely there exists $x_{I \setminus K} \in \Pi_{i \in I \setminus K} S_i$ such that $g(x_K, x_{I \setminus K}) \notin B$ for each $x_K \in \Pi_{i \in K} S_i$. Let $H'(x_{I \setminus K}) = \left\{h \in H : K \subseteq K_h \text{ and } x_{I \setminus K_h} \text{ is co-proper}\right\}$. We may assume without loss of generality that $x_{I \setminus K}$ itself is co-proper.

Clearly, by construction, $g(x_K, x_{I \setminus K}) \notin B_{K_h}$ and therefore $E_g^{\beta}(K_h, B_h) = E(K_h, B_h) = 0$ for any $h \in H'(x_{I \setminus K})$. But then, polar-superadditivity of E implies $E(K, \cup_{h \in H'(x_{I \setminus K})} B_{K_h}) = 0$. Since by construction $B \subseteq \cup_{h \in H'(x_{I \setminus K})} B_{K_h}$, it follows that $E(K, B) = 0$ by monotonicity of E whence $E \leqslant E_g^{\beta}$ as well.
■

It should also be emphasized that the proof of Theorem 8 relies heavily on *finiteness* of I *and* X, and does not extend- say- to the case of a countable outcome set.

Remark 9 *Observe that outcome-monotonicity is in fact independent of both superadditivity and polar-superadditivity. To see this, take* $I = \{1, 2, 3\}$, X *with* $|X| \geq 3$ *and suppose* X *is endowed with a total order* $\leq$. *Then consider the strategic game form* $G_\mu = (I, X, (S_i = X)_{i \in I}, \mu)$ *where* $\mu : X^3 \to X$ *denotes the median (ternary) operation on* X. *Then, define the function* $E_{ex(\mu)} : \mathbf{2}^{I \cup X} \to \mathbf{2}$ *as defined by the following rule: for all* $K \subseteq I, A \subseteq X$, $E_{ex(\mu)}(K, A) = 1$ *iff there exists* $x_K \in X^K$ *such that* $\left\{\mu(x_K, y_{I \smallsetminus K}) : y_{I \smallsetminus K} \in X^{I \smallsetminus K}\right\} = A$. *Clearly, by construction,* $E_{ex(\mu)}(K, A) = 1$ *iff* $|K| \geq 2$ *and* $|A| = 1$, *or* $|K| \leq 1$ *and* $A = X$. *Thus,* E *is not outcome-monotonic. However, it is easily checked that* E *is both superadditive and polar-superadditive: indeed, suppose that* $E_{ex(\mu)}(K, A) = E_{ex(\mu)}(K', B) = 1$. *Then -by construction of* $E_{ex(\mu)}$- $K \cap K' \neq \varnothing$, *hence superadditivity trivially holds. Moreover, suppose that* $E_{ex(\mu)}(K, A) = E_{ex(\mu)}(K', B) = 0$ *and* $K \cup K' = I$. *Then, either* $|K| \geq 2$ *or* $|K'| \geq 2$ *(or both). If* $|K| \geq 2$ *and* $|K'| \geq 2$ *then it must be the case that* $A = B = \varnothing$: *therefore* $E_{ex(\mu)}(K \cap K', A \cup B) = E_{ex(\mu)}(K \cap K', \varnothing) = 0$. *If* $|K| \geq 2$ *and* $|K'| \leq 1$ *(*$|K'| \geq 2$ *and* $|K| \leq 1$, *respectively) then* $A = \varnothing$ *and* $B \neq X$ *(*$A \neq X$ *and* $B = \varnothing$, *respectively), hence in any case* $A \cup B \neq X$ *and* $|K \cap K'| \leq 1$: *thus,* $E_{ex(\mu)}(K \cap K', A \cup B) = 0$, *and polar-superadditivity also holds.*

As a straightforward corollary to Theorems 5 and 8, a few characterizations of *tight* effectivity functions with *finite* player and outcome set are also obtained, namely

Corollary 10 *Let E be an effectivity function with finite I and X. Then, the following statements are equivalent:*

(i) E is outcome-monotonic, superadditive and polar-superadditive;

(ii) E is outcome-monotonic, polar-superadditive and regular;

(iii) E is outcome-monotonic, superadditive and maximal;

(iv) E is tight.

Proof. $(i) \Longrightarrow (ii)$ It follows immediately from Claim 2-(i).

$(ii) \Longrightarrow (iii)$ It follows immediately from Claim 2-(ii) and Proposition 3-(ii).

$(iii) \Longrightarrow (ii)$ It follows immediately from Claim 2-(i) and Proposition 3-(i).

$(iii) \Longrightarrow (iv)$ If E is outcome-monotonic, superadditive and maximal then it is also monotonic (in view of Claim 2-(i)), hence by Theorem 5 above it is α-playable i.e. there exists a strategic game form (with outcome function) g such that $E = E_g^\alpha$. Now, let $K \subseteq I, A \subseteq X$ be such that $E_g^\beta(K, A) = 1$, namely for each $y_{I \smallsetminus K} \in X^{I \smallsetminus K}$ there exists $x_K \in X^K$ such that $g(x_K, y_{I \smallsetminus K}) \in A$: thus, by definition of E_g^α, $E_g^\alpha(I \smallsetminus K, X \smallsetminus A) = 0$ whence, by maximality and monotonicity, $E_g^\alpha(K, A) = 1$. Since obviously $E_g^\alpha \leqslant E_g^\beta$ for any strategic outcome function g, it follows that $E_g^\alpha = E_g^\beta$. Therefore, $E = E_g^\alpha = E_g^\beta$ i.e. E is tight.

$(iv) \Longrightarrow (i)$ Trivial: indeed, if there exists a strategic game form such that $E = E_g^\alpha = E_g^\beta$ then E is obviously monotonic. Moreover, let $E(K, A) = E(K', B) = 1$ with $K \cap K' = \varnothing$. Then, since $E = E_g^\alpha$ there exist $x_K \in X^K$ and $x_{K'} \in X^{K'}$ such that for each $y_{I \smallsetminus K} \in X^{I \smallsetminus K}$ and each $y_{I \smallsetminus K'} \in X^{I \smallsetminus K'}$: $g(x_K, y_{I \smallsetminus K}) \in A$ and $g(x_{K'}, y_{I \smallsetminus K'}) \in B$. Therefore, $g(x_K, x_{K'}, y_{I \smallsetminus (K \cup K')}) \in A \cap B$ i.e. $E(K \cup K', A \cap B) = 1$ and superadditivity follows. Also $E = E_g^\beta$ entails polar-superadditivity of E as established by the first part of the proof of Theorem 8 above. ■

Notice that the equivalence between statements (iii) and (iv) is essentially due to Moulin (1983) (see also Otten et al. (1995)).

3 Concluding remarks

A characterization of finite β-playable effectivity functions has been provided. That characterization parallels to a considerable extent the recent characterization of finite α-playable effectivity functions due to Boros et al (2010), with polar-superadditivity (as opposed to superadditivity) and a particular normalization condition for the empty coalition jointly playing a pivotal role in it. However, the present characterization of β-playable finite effectivity functions does not apply to countable outcome sets.

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