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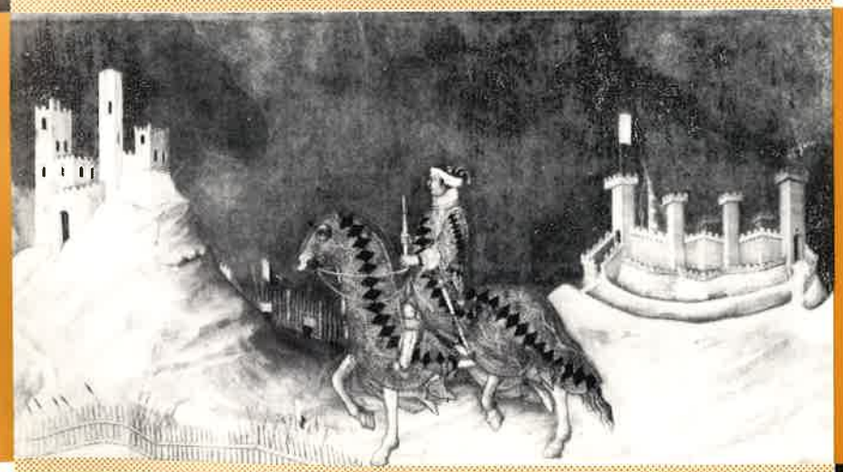
QUADERNI DELL'ISTITUTO DI ECONOMIA

Massimo Di Matteo

WARRANTED, NATURAL, AND ACTUAL RATES OF GROWTH : REFLECTIONS OF A PERPLEX



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Massimo Di Matteo

WARRANTED, NATURAL AND ACTUAL RATES OF GROWTH: REFLECTIONS OF A PERPLEX

Siena, giugno 1987

Massimo Di Matteo is Associate Professor of Economics at the Institute of Economics, Faculty of Economics and Banking, University of Siena.

Warranted, Natural, and Actual Rates of Growth: Reflections of a Perplex.[1]

0.In my opinion the advice of Harrod of thinking dynamically expressed in his famous 1939 paper has been a great step forward in the development of economic analysis. Certainly "thinking dynamically" should not have been struck people accustomed to think within the classical tradition. Smith,Ricardo and Marx were rather advanced in that field. It was a shock in those "years of high theory" when, with the exception of Schumpeter and few others, no one gave serious attention to dynamic analysis.There were pieces of theory that were intrinsically dynamic (the oustanding example being the General Theory) but they were treated with the static apparatus and so could not show all their potentialities before some sort of machinery had been invented.

It is exactly the merit of Harrod to emphasize such a need for dynamic analysis and to build the first concepts in that direction. It was his vision of the capitalistic system that supplied him with the right questions: whether he gave also the right answers It is another matter and it is one of the purposes of this paper to say something on the topic. Harrod's first book on the trade cycle (Harrod 1936b) already shew his preanalytical ideas about the dynamic of capitalism. His ideas remained essentially unchanged troughout all his life as one can judge from his latest book (see Harrod 1973).

Harrod thought he had discovered the very fundamental principles of dynamics by building upon Keynesian,or more precisely, Kahnian ideas. It is only in this light that one can understand (which does not mean one can justify) his stubbornnes in refusing to introduce lags into the analysis even at a later stage. More than that he always spoke of the

[1] I am deeply indebted to prof. Tonveronachi and to dr. Caminati for invaluable suggestions on an earlier draft. Financial support from the Italian Ministry of Education (fondi 60%) is gratefully acknowledged.

instantaneous rate of change of output using implicitly differential equations rather than difference equations. He recognized however that the multiplier process cannot be really instantaneous so that a period could be conveniently defined in terms of the time required by the multiplier process to work. Richard Goodwin himself started his analyses without introducing lags though in the end he completed his models with them.

If we look at the three concepts mentioned in the title of this paper with today's eyes, we find that many objections can be levied against them and their relationships (and some of them will be sketched below). However I think that Harrod's personal vision of the characterizion of our economic system still commands some interest. I will try to scrutinize Harrod's machinery a bit more closely than is usually done with the purpose of clarifying the nature and the usefulness of the concepts of warranted and natural rates of growth and their interrelations.

1.Very briefly the unifying theme behind Harrod's works in the field of dynamics is the recognition that the economic system is unstable in the small and stable in the large, a conception that Richard Goodwin developed at length in his works with a higher degre of awareness. This way of looking at the capitalistic system seems the only one in accordance to facts and that can permit at least in principle a unified treatment of cycles and growth that both Harrod and Goodwin advocated. In passing it can be noticed that in this context to have models with unstable equilibria is not a clear sign that the model has to be rejected since that model shows that as long as parameters are kept constant the behaviour of the system will diverge. A model with unstable equilibria could be understood as a partial model needing either an endogenous change in the parameters or the introduction of "floors and ceilings". In the latter case we have new laws governing the behaviour of the model in the extreme regions.

What I have said will probably seem odd to most people since the name of Harrod is associated with the concept of knife edge which implies absolute instability.But this is not correct in my opinion depending on a superficial attention paid to Harrod's writings (as it is documented in the careful review of the literature undertaken by Miconi (1967)). In

this respect it is also incorrect to state that Solow in reconciling natural and warranted rate solved the Harrod problem. From what I said up to now one can immediately spot striking similarities between Harrod and Richard Goodwin as far as the vision of the capitalism is concerned. No wonder since it is well known that Harrod was Richard's supervisor exactly at the time Harrod was writing his first book. In his preface to the Italian edition of his Dynamic Essays Goodwin states that he was much influenced by Harrod's vision and always found the techniques used by him totally unsuitable to the purpose at hand though he himself recognized that he struggled through many years to find the appropriate ones. I think incidentally that Harrod's techniques were not so bad as appeared to Goodwin though for different reasons (see later on). In this attitude Goodwin was influenced by Tinbergen's review of Harrod's The Trade Cycle where Tinbergen states that Harrod's analysis is formally wrong since a first order differential equation cannot give rise to oscillations.

This is of course true but Harrod was elaborating (implicitely) a non linear model both in that book and in his celebrated essay on economic dynamics. He probably did not use a non linear model for the reasons Tinbergen thought it would have been appropriate but it has been clearly demonstrated both by Medio (1979) and Glombowski-Krüger (1982) that a careful translation of the model offered in the book leads to a Lienard differential equation which admits of (at least) one limit cycle.(For the other essay see infra).This of course does not imply that the mechanism trough which Harrod purports to explain the cycle is satisfactory:far from that and Goodwin himself developed the multiplier-accelerator model in a different, much more satisfactory way. Finally Goodwin's vision is more complex since he tried to integrate a disaggregated analysis of the capitalistic system with its overall dynamics paying also particular attention to the role of innovations and exogenous factors in the process of development, a clear influence of another of his teachers, namely Joseph Schumpeter.

Another important aspect of Harrod's methodology (which assumes importance however only in his later writings) was his effort to link medium term dynamics with long term dynamics precisely with the purpose of showing that the system is unstable only in the small. At the same time he retains the belief that the equilibrium and the actual

growth path are not completely separated concepts. I think that this approach can be a useful one though it needs elaboration beyond the one sketched by Harrod.
So the themes of my considerations are sketched and I will try to develop them more systematically in what follows.

2.I will now reconstruct and criticize, where appropriate, the nature and the applications of the three concepts and their interrelations in the context of a theory of cyclical growth.What it is to be stressed is that Harrod's primum mobile is (in a typical Keynesian fashion) the expectation of a growing demand : even in the "long run" demand and not supply or potential output plays the most important role. This is clearly demonstrated if we modify the original Harrod's model to introduce a component in the investment function which is in relation to income or constant in absolute value.In both cases the warranted rate of growth is lower than otherwise precisely because it is expectation of income changes that does matter.
In the usual Keynesian static model it is assumed that output is adjusted to current demand without directly facing the expectation problem. But the latter is really implicit since firms at the beginning of period t will plan a level of output equal to the demand they will expect to be forthcoming over the whole period. The situation can be imagined as one where changes in demand are held to be transitory so that trough this channel no changes in capacity are elicited. Indeed investment is dependent on a totally different mechanism.
Therefore - one can argue - we can have a process of growth if we expect that demand will grow in the future and take the appropriate action to meet this demand: in this way we have a totally endogenous factor as determinant of growth.
Let me come to the definition of the actual rate.Implicit in the ex post equality between savings and investments is a growth rate of the economy at each point in time. Let us write a superscript p for ex post variables. Therefore in any given situation, be that of equilibrium or not, $S^p = I^p$ and by dividing trough by Y (income) and defining C as the marginal actual capital/output ratio we find the following truism which gives us the actual growth rate, $s/C = G$. This is clearly unique.

By employing ex ante definitions of savings and investments we can define a warranted rate of growth i.e. that rate of growth which is consistent with continuous ex ante equality between savings and investments. This is not a theory but simply a definition of $G^w = s^d/C^r$ where now s^d means the ex ante rate of savings and C^r the required capital/output ratio. At this stage nothing is said about the determinants of both s^d and C^r and their possible changes over the cycle.
The difference with the usual Keynesian analysis is that here we do take into account that at every moment in time we have to create additional capacity to produce a higher level of output whereas in the static model we take into consideration only those variations that can be accomodated by changes in the degree of utilisation of the given capacity (indeed in the static model we assume that investments do materialize at the beginning of the next period). However the assumption that Harrod does about the uniqueness of G^w seems a bit doubtful since now there is in general no presumption that in every possible circumstance s^d and/or C^r are uniquely determined by the choices of the agents in the economic system, even if prices are constant as it will be maintained throughout.
Finally we have a third concept, the definition of the optimum rate of savings, i.e. that rate of savings, $s^n = G^n/C^r$ where G^n is the rate of growth of the labour force measured in efficiency units that is assumed to be governed by slowly changing economic factors so that it can be considered exogenous to the analysis. It is optimum in the sense that if $s^d = s^n$ full employment obtains, if initial conditions are appropriate.
It follows that if we assume that G^w is unique then we are pretty sure that any G different from G^w implies that ex ante savings are different from ex ante investment and therefore there is disequilibrium in the commodity market. In particular if, say, G is greater than G^w then either s is greater than s^d or C is lower than C^r or both. In the first case the actual saving is greater than desired saving so that people will tend to save less and this will push G upwards (and viceversa). Now this step of the argument requires a model that specifies how people form expectations about future income and demand and this calls for a model which is behind Harrod's mechanism.

3.Now to provide a model that reflects all Harrod's hypotheses (some of them being implicit) has proved to be rather a difficult enterprise and many, many economists spent their time in this effort. Probably none of the models that have been set up are satisfactory nor totally faithful to Harrod's arguments which are a bit obscure or involved. It seems to me that the idea of Gw requires some more discussion since once we recognized that a model is behind the simple argument by Harrod then results-as it is obvious- depend on the actual model employed.

Suppose we are in a typical Keynesian model and that in a particular period investments ex ante exceed savings ex ante (Harrod prefers to state that investment ex ante exceed investments ex post since for him savings plans are always realized so that savings ex ante are always equal to savings ex post) so that output and income rise. Now the argument put forward by Harrod can be recast in the following manner: if firms do not want to vary their degree of capacity utilisation (because say the latter equals the desired level) then how firms should react if they think that the same rate of growth in output will persist? How they can cope with maintaining an optimal rate of capacity utilization if they have expectations on the rate of growth of demand as well? Here is the problem which is solved by introducing the acceleration principle which in fact requires no capacity underutilisation.

We specify the following model (which is only slightly different from the one put forward by Miconi (1967)):

1) $S(t) = sY(t)$

2) $I(t) = C^rY^*(t) + \{C^r - K(t-1)/Y(t-1)\}\ Y(t-1)$

3) $S(t) = I(t)$

4) $Y^*(t) = Y^2(t-1) \ / \ Y(t-2)$

In the first equation consumers are not allowed to have expectations on the level of income to which they adjust savings. The idea is that savers by taking quick action can easily adapt themselves to changing levels of income. But nevertheless this totally passive behaviour of the consumers that accept whatever level of income is forthcoming is to be noticed.

The second equation states the normal accelerator function corrected to take into account that outside equilibrium the capital which is in use is not the required one and therefore firms want not only to cope with expected demand but also to rebalance the capital/output ratio: so they will invest more(less) if the appropriate capital stock at time t-1 is less (more) than the actual one at time t-1.

The other equation to comment on is the fourth where firms expect the same rate of growth in demand that happened in the previous period. This is a form of static expectations with respect to the rate of growth or a special kind of extrapolative expectations with respect to the level of income where the coefficient is no longer a constant but the ratio of the two levels of income respectively at time t and at time t-1.

If we study the system above which has been formulated by Miconi (1967) in what- to the best of my knowledge-is the clearest (and the most faithful to Harrod's thought) paper on the instability problem, one finds that there is a stationary solution i.e. a rate of growth of output which will persist over time if the system already finds itself on that path at $t = t_0$.The system is non linear in the rate of growth of output and in the capital/output ratio. It can be shown to be unstable: see the appendix (sec.1) where a different route from the one followed by Miconi is attempted.

These results have been very much criticized in the literature and indeed there are plenty of good reasons for doing so but there are also bad reasons: let us start from an example of the latter. The expectation function - it has been argued - is naïve since for example adaptive expectations seem to approximate better the formation of actual expectations. Granted, but it is easy to show (see the appendix, sec.2) that if we substitute adaptive expectations for the Harrodian ones there will be no growth or more precisely there will be no endogenous growth, i.e. growth generated by firms themselves and their expectations about future demand. I am not suggesting that external forces like autonomous investments and the like do not play any role in the growth process: but simply that Harrodian expectations are a sufficient condition for endogenous growth.

And what is more if one modifies the above model by inserting a correction for the expectations function such that when expected demand is lower than actual firms will revise the expected value

downwards a little bit, then the system has the same warranted growth rate and it is still unstable (see the appendix, sec.3).
So it seems to me that up to now the main result still holds even if we reformulate the model in a direction that should have stabilized it. Therefore one could conclude that an equilibrium growth path can be generated by endogenous forces and that the system is expected to move away from the equilibrium path.
However this is not the whole story since the investment equation is surely uncomplete. It is true that when we have fixed coefficients to produce more output we need more capital (and labour) but we can have investment expenses even if the level of output is not changing. From a purely logical point of view in a one good economy as the one implicitly assumed so far one cannot see what sort of thing this extra investment could be. But if we think of the real world then even the most strenuous supporter of the accelerator will admit that, apart from the problem of depreciation, there is some kind of investment which is autonomous, i.e. independent of income and of the rate of change of it. The first thing that comes to my mind is exports and public expenditure in goods and in general those expenses that may be related to a prospective long period growth (i.e. expenses related to structural changes in demand and methods of production: see Harrod (1939),par.12). In this case the warranted rate of growth depends on the level of income at time t.
This is what Harrod himself shew in his 1939 paper but he did not become aware of the consequences of that. First of all models with the same structural coefficients may have different warranted rates of growth depending on the level of income at time t. In other words if the absolute level of exogenous expenditure remains constant over time the warranted rate of growth is rising over time. So that the warranted rate obtained in the simplest model is also the highest of a whole family which is tending to it. Secondly it is clear that any warranted rate of growth depends on the relation between autonomous expenditure and the level of income. This ratio can be made constant if autonomous demand grows at some exogenous rate (like in Hicks's model) which now becomes the trend growth rate around which actual rate fluctuates. However it is interesting to note that apart from Hicks's solution this opens the way to an influence of the governement

(or the foreign markets) on the warranted rate. Indeed by manoeuvring the ratio of public expenditure to income the government can alter the warranted rate and try to lower it in time of overexpansion thus bringing out a reduction of the warranted in relation to the actual rate.
At the same time however this relativisation of the warranted rate may deprive it of most of its attractiveness, I think, since is not such a clear thing like the pole star that can be safely assumed in the arguments of the theorists. This by the way may be what Keynes perceived in his discussion with Harrod himself after the latter submitted to the editor of the "Economic Journal" his famous 1939 paper: Keynes felt that it was against common sense to argue that the warranted rate of growth was unique and stated that in any given situation there could be many of them.[2]

4.It is interesting to note that an analogous outcome arises in a different context. I have already hint at the superficiality of considering Solow's celebrated model as the "solution" to Harrod's problem. Solow's model cannot answer the problem of the divergence between warranted and actual rate since it has no independent investment equation. The latter circumstance being the logical result of having built a model not of a Keynesian type (i.e. pushed by demand) but of a neoclassical variety where the only "stability" problem is the one connecting natural and warranted rates. This by the way is how

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[2] At this point a few words commenting an interesting and perceptive essay by Kregel (1980) might be in order. I will limit myself to note only a point of partial disagreement. Kregel argues that in discussing Harrod's works two separate questions got mixed up. One issue concerns the stability of the warranted growth path (B-stability in Kregel's terminology). The other is the variability of the warranted growth path over the trade cycle (A-stability). Kregel maintains that much of the discussion concerning Harrod's works depends on the failure of realizing that there were two different issues at stake. Though I agree that the two questions are logically separate, nevertheless I maintain that they are connected to each other. Were not the warranted rate of growth unstable how could we start explaining the cycle ? If the multiplier-accelerator mechanism were stable how could we imagine a persistent trade cycle ? In other words instability is a sufficient reason for considering the variability of the propensity to save and of the capital/output ratio. Otherwise we can discuss the latter assuming already that the trade cycle exists, but this seems exactly to beg the question.

Harrod's model is understood generally. The warranted rate is very often interpreted as that growth rate consistent with the continuous maintenance over time of full capacity. Therefore it is regarded to be a companion to the natural rate which is in fact the rate of growth consistent with full employment. It is not interpreted as we endeavoured to show as the result of a (class of)model(s). In other words the behavioural nature of the machinery is overlooked and it is implicitely interpreted as if it were Domar's model where such an interpretation is really appropriate.

Therefore I asked myself the following question: how one should have allowed a functional relationship between the capital/output ratio and the rate of interest into a Harrod type model? It turned out not to be a simple answer. In other words is it possible to extend to the analysis of growth the Keynesian apparatus in which we take into account that typical neoclassical element? The answer would appear to be in the negative, but I think it would be interesting to perform such an exercise.

If we maintain the Keynesian " determination" of the rate of interest on the money market (as long as there are transaction costs demand for money depends on the rate of interest) we have that the model can determine the growth rate of income, given the rate of interest which depends on the money supply (which is exogenous) and on the level of income which is not solved by the model. This requires that the disequilibrium in the financial market is eliminated rather quickly and in any case within the period in which the analysis is split. This seems rather plausible since we have defined the period as the time necessary to the multiplier process to be completed. In other words once the authorities fix the money supply there will be a fast change in the interest rate so that the equilibrium between demand for and supply of money is reached (before income changes appreciably). So there will be a warranted rate of growth for each level of the rate of interest that is consistent with the equilibrium in the money market, i.e. for a given couple of Y and M.

In other words the warranted rate of growth becomes dependent on initial conditions. So that we can say that given the level of income at time t then it is up to the monetary authorities to choose the level of interest by choosing the level of money supply. Once this choice has

been made then a particular warranted growth rate is also implicit. Since the required capital/output ratio falls with a higher rate of interest the latter has the effect, ceteris paribus, of rising the warranted rate.

We have assumed that the monetary authorities control the stock of money. This hypothesis is simply made because we wish to see whether in this case the model can be made stable and not because we think that this is the best (or the unique) available policy to the government. Having said that let us assume that the Central Bank exerts its control over the money supply and that at time t chooses a level of the money stock and therefore (after the elimination of the disequilibrium in the money market) an interest rate. If (for simplicity) income starts growing at the warranted rate to maintain the rate of interest constant and therefore the rate of growth also constant a unique growth rate of the money supply is needed depending on the rate of growth of income (given of course the elasticity of the demand for money with respect to income). If the latter is one, then the growth rate of the money supply will have to be equal to the warranted rate.

Suppose now that for some reason the economy leaves the warranted path and that the actual rate of growth is higher than the warranted. Then let us suppose that the Central Bank goes on in its policy and keeps the money supply growth unchanged (say because it believes Friedman). The rate of interest will rise and the new warranted rate with it. This of course will act as a stabilizing force reducing the gap between actual and warranted rates. This argument I think gives the rationale to some interesting considerations put forward by Harrod in his latest book where he asserts that there are many special warranted paths and that they change over time as a consequence of the actual behaviour of the economic system. However in his explanations he is a bit vague on how this can happen and asserts a certain cyclicity in the savings ratio itself. However it is not clear in Harrod's argument how the turning points could be determined. In our example the whole argument depends on the behaviour of the Central Bank which thinks that the rate of interest it chooses is a normal or an optimal or a desired one. So that when the economy leaves the warranted rate of growth it does not care. In this case the tendency to instability can be

checked and the more so the longer the Central Bank sticks to its predetermined growth rate of money supply. In this case sooner or later the warranted rate of growth should become equal to the actual one and this would require now and only now an increase in the rate of growth of the money supply to sustain the new equality between warranted and actual rate.

At this point of the exercise one has only to pause for a moment to convince oneself that this sounds rather implausible in practical terms. If as it is the case the monetary authorities cannot (or do not want to) keep the rate of interest constant two different scenarios open up. In the first the rate of interest is allowed to rise: in this case the warranted also rise with respect to the actual rate so that a cumulative deflation follows; in the other case the rate of interest is allowed to fall and in this case a boom will ensue. Up to now we keep the real wage constant (and therefore prices which depend on costs). If in the upswing it rises according to a modified form of the Phillips curve then the instability can be less or more accentuated depending on the relative effects of the interest rate and real wage changes on the capital/output ratio.

5.How one can judge this approach in the face of the problem of the integration of growth and cycles? There are various ways trough which these elements can be combined and in Harrod we can see three of them. The first is trough floors and ceilings, touched upon by Harrod (1951), the second is trough variations in the savings ratio and the capital/output ratio, the third one is trough the dependence of the warranted rate from the actual rate. The first has been only hint at by Harrod and has been much more consistently elaborated by Sir John Hicks (1950) and needs no further elaboration.

The second one derives from Harrod (1936) and relies on large changes in the savings ratio which depends on the share of profits and on the level of income itself. It is assumed that a rise in the share of profits rises the rate of savings and that the share of profits rises in the upswing. The latter circumstance is due to assuming that an imperfect competition long run equilibrium prevails in the economy. In this case as income rises the ratio of marginal cost to average cost rises as well and so the share of profits. The latter effect can also be

reinforced if we are prepared to assume that as income rises also the ratio of price to marginal revenue (= cost) rises (see Harrod (1936a). If the rate of savings rise enough to slow down the accelerator-multiplier process then there will be the upper turning point. (Changes in the capital/output ratio work against stability but they can be overlooked for simplicity:if one is interested is referred to pages 93-5 of Harrod 1936). The trouble with this explanation (apart from its empirical validity: it implies that the share of profits is procyclical and that the upper turning point is not the result of a profit squeeze) is that it is perfectly understandable if the model shows oscillations around a stationary level but it needs reformulation when output trend does exist. It clearly involves a peculiar savings function like the one exemplified by Glombowski and Krüger (1982). In this formulation the postkeynesian flexibility of the savings ratio is no longer a way of achieving full employment but the explanation of cycles. So that in my opinion the third way of attacking the problem could be more appealing.

In general terms it states the idea which seems to me rather interesting that short term dynamics and medium term dynamics do interact with each other. One has to specify plausible mechanisms but the general principle of adaptivity seems to be of general validity. Harrod postulates some relation of the type that G^w changes with a delay (sic!) in accordance to the difference between actual and warranted rate of growth. What I said before with respect to the modified Solow's model can be considered a more developed exercise on the line sketched by Harrod. For nothing but an exemplification one can have a look at the diagram.

By way of a conclusion the arguments put forward so far on one hand enrich the utilisation of the warranted rate of growth and on the other restricts the validity of the conditions under which it is unique.

6.As far as the natural rate of growth is concerned there are many theoretical results and empirical studies that work against the independence of the natural rate from the warranted and the actual ones if we lay more emphasis on the behaviour of the productivity than on that of population. Indeed Lord Kaldor postulated in his models a positive relation between the productivity growth rate and the

accumulation rate in per capita terms or alternatively between the former and the rate of growth of investment in per capita terms. At the same time models developed on the basis of the learning by doing hypothesis (see Arrow (1962)) also show that the rate of productivity growth is linked to the accumulation rate. This case is different from the usual solution given to the Harrod problem via changes in the capital/output ratio and/or in the savings ratio since it works on the other side of the equality. There is the possibility that a steady solution exists but as the system evolves the productivity growth shifts upwards and then we can have instability undermining the plausibility of the whole old good theory of cycles à la Hicks. In general the idea that the ceiling does not move endogenously (or does very slowly) is an extremely diffused idea which if abandoned can shake the confidence in many ideas about the process of economic growth.

What I have been trying to emphasize in this paper is that sometimes dynamics, i.e. the analysis of cycles an growth is much more complicated than Harrod thought especially if we are prepared to insert the actions of the government and the monetary authorities into the picture. In particular we have shown that in our example some properties of the warranted growth path (e.g. its uniqueness) are gone if we complicate the model in order to take into account other phenomena and the same is true for the natural rate of growth. At the same time and at the theoretical level I tried to elaborate some hints by Harrod on the relationship between actual and warranted rate of growth, though in a slightly different framework. I showed that at least in in my example instability can be eliminated in a way apparently similar to the one followed by Solow but substantially different since in my case the result depends heavily on the actions of the Central Bank. These arguments have been advanced having in mind Harrod's conception of the three growth rates and could be perhaps unwarranted to claim general validity for them.

APPENDIX

1.From the equations in the text we get the following system:

$sG(t) - CrG(t-1) + v(t-1) = Cr - s$

$v(t) - CrG(t-1) + v(t)G(t) = Cr$

where $v(t) \equiv K(t)/Y(t)$ and $G(t) \equiv [Y(t) - Y(t-1)]/ Y(t-1)$

Getting v(t) from the first and substituting into the second we find the following expression in G(t):

$CrG^2(t) - sG(t+1)G(t) - sG(t+1) - sG(t) - CrG(t-1) - s = 0$

To find stationary solution we set $G(t)=G(t-1)=G(t+1)=G$

i.e. $G^2(Cr-s) + G(Cr-2s) - s = 0$

There are one negative and one positive root since Cr-2s is positive:the positive one is the warranted rate of growth,ie s/(Cr-s) and the negative one,that we discard, equals -1 (Miconi (1967) does not mention the latter.)

To examine stability let us find the auxiliary equation of the homogenous equation which is

$\mu^3 + [(s-Cr)/s]\ \mu^2 + [(s-2Cr)/s]\ \mu + Cr/s = 0$

We know that if it is stable then $-\Sigma_i a_i < 1$ where a_i are the coefficients of the auxiliary equation.In our case the condition becomes $2Cr < 3s$ which is practically impossible so that the model is unstable. I wish to note that Miconi(1967) gives a different proof based on induction that does not require any hypothesis on the values of the parameters.

We can have some more information about the signs of the roots by noting that their products is equal to -Cr/s and therefore we can have either three negative roots or one negative and two positive roots; however since the sum of the roots is positive we have two positive roots and one negative root.Finally if we give plausible numerical values to the parameters, e.g. s = .2 and Cr = 2.7 we find that the discriminant is negative so that roots are real (no proper oscillation is possible, only monotonic movement plus an improper oscillation as Gandolfo (1980) labels it.)

2.Let us rewrite the model in the text unaltered with the exception of the second term in the investment function (that we omit for simplicity) and of the expectations function which is now of the adaptive kind:

$Y^*(t+1) - Y^*(t) = hY(t) - hY^*(t)$ where $h > 0$.

After some manipulations we get the following equation in G(t):

$G(t) + (1+a)G(t-1) + G(t)G(t-1) + 1+a-b = 0$

where $a \equiv (Cr/s)(1-h)-(1-h)$, $b \equiv (1-h)^2(Cr/s)+h(1-h)(Cr/s)$

To find the stationary solution we set $G(t)=G(t-1)=G$ which gives:

$G^2 + (2+a)G + (1+a-b) = 0$

It can be shown that both roots are negative so that no positive growth rate can emerge.

3. Let us rewrite the model in the text unaltered with the exception of the second part of the investment function (that we omit for simplicity) and the expectation function which is of the following type:

$Y^*(t) = Y^2(t-1)/Y(t-2) - \sigma\,[Y^*(t-1)-Y(t-1)]$

After some manipulations we get the following equation in G(t), the growth rate of output:

$G(t) - (Cr/s)G^2(t-1) + G(t)G(t-1) + (1+a-2Cr/s)G(t-1) + +1+a+Cr(\sigma-1)/s = 0$

where $a \equiv (Cr+\sigma s-\sigma Cr)/s$; $b \equiv 1+a-[Cr(1-\sigma)/s]$

To find stationary solutions we set $G(t)=G(t-1)=G$ and end with the following expression:

$G^2 + \{[(1+\sigma)Cr-s(2+\sigma)]/(Cr-s)\}G - (s+\sigma s)/(Cr-s) = 0.$

With plausible values of the parameters we conclude that we have one negative and one positive root. It can be shown that the positive root equals the one we get in the original model i.e. $s/(Cr-s)$ whereas the negative is equal to $-(1+\sigma)$.

To study local stability we rewrite the equation in the following way:

$G(t)[1+G(t-1)] = (Cr/s)G^2(t-1) - cG(t-1) - d$

where $c \equiv -(s+as-2Cr)/s$; $d \equiv [Cr(1-\sigma)-as-s]/s$ and compute

$$\delta G(t)/\delta G(t-1) = \{(Cr/s)G^2(t-1) + (2Cr/s)G(t-1) + Cr(1+\sigma)/s\} \,/\, \{1+G^2(t-1)\}$$

The above derivative is positive as long as G(t) is positive and can be either positive or negative when G(t) becomes negative; so that monotonic movements occur and maybe improper oscillations. To see whether the equilibrium is locally stable we compute the derivative in the equilibrium point where $G = s/(Cr-s)$ and we find that in order to have the root less than 1 we need $\sigma < (Cr-C^2r)/(Cr-s)^2$ which is impossible since σ is positive (see the condition stated in Smithies (1942).

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