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Gleb Koshevoy
Ernesto Savaglio

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Gleb Koshevoy* and Ernesto Savaglio†

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Abstract

We analyse the *rationality* of a Decision Maker (DM) who chooses from lists of sets of alternatives. We show that the DM's choice behaviour, represented by a choice function f defined on such lists, can be 'improved' by a two-step procedure. We first construct the *list-envelop* of f that is the union of all choice sets induced by f on every possible partition (lists) of a set of alternatives. Then, we take among the list-envelop choice functions those that are Plott functions (see [4]) and characterize such a family by using a suitable property, allowing a DM to disregard alternatives that are worthless for the choice process. Then, using a rule that breaks the ties that could occur in every choice set, we characterize the set of Plott functions that has a prescribed *shelling* (see (5) below) of the choice sets. Finally, we study the 'relational system' that satisfies some compelling properties and *rationalizes* the class of choice functions studied in the present work.

JEL Classification $D01$

Keywords Envelop Choice Function, Plott function, Sequential Rationality, Path-independency, Shuffle of Linear Orders.

*CEMI, Russian Academy of Science of Moscow, *e-mail: koshevoyga@gmail.com*

†DEc, University of Pescara and DEPS, University of Siena, *e-mail: ernesto@unich.it*

1 Introduction

The present work is devoted to the problem of studying the choice of a Decision Maker (DM) who *sequentially* selects alternatives from *lists of sets* of alternatives. In particular, we want to analyse the *rationality* of finding solutions to a complex choice process by breaking that difficult problem into parts that are easier to conceive and understand. Such a practical routine is naturally rational as long as it improves the ability of the DM to make good selection.
For instance, in cognitive sciences, Newell and Simon [13] have shown as a mental strategy to find a solution to a complicated problem is usually to divide it in a number of parts so as to minimize the dependencies among those parts, and to maximize the possibility to get the best solution. As a matter of fact, the mental technique of breaking a complex problem into smaller parts in order to gain a better understanding of it has been applied in the study of mathematics and logic as well as in any decision process since before Aristotle.
Now, it seems to be naturally legitimate to ask that the solution-set of all sub-problems competes (or better coincides) with the solution-set of the complex problem they derive from. We also require that solving a difficult problem, (characterised by a high level of complexity), dividing the latter in more elementary problems (with a greater degree of simplicity in resolution), must not depend on the *ordering* in which the DM faces and solves those sub-problems. In other words, if the choice set is the outcome of a process in which the elements from lists of sets of alternatives are compared, then the requirement of routewise invariance may turn out to be somewhat very natural and compelling. Such a kind of *rationality* property of a process of choice is called *pseudorationality* by Moulin [12], who refers it to the work of Plott [14] on the so-called *path-independent choice functions*. Indeed, Plott [14] studied the class of choice functions which requires the same choice set for every ordered list of alternatives, and then moreover that this invariant result be identical with the choice from the set from which the ordered lists come from, so that the sequential choice processes appear as in stable pursuit of that choice.[1]
In what follows, we suppose that, for each set of alternatives A, a DM considers all possible sub-set partitions of A, which are *lists* of sets of alternatives. For a

[1]For the sake of truth, we acknowledge here that was in fact Afriat [1] the first to propose 'path independency' as a suitable rationality property of a choice function.

choice function defined on those lists and representing the DM's behaviour, we construct another choice function, that we call *list-envelop*, that takes the union of all the choice sets from the different possible (partitions) lists of A.
The list-envelop choice function represents an *improvement upon* the choice process as long as it is *rational* to split a complex problem in a finite set of sub-problems easier to solve. Namely, when making a decision is overwhelming because there are many potential outcomes and risks that could result from the wrong choice while dividing a complicated decision problem in parts lead to an increased opportunity of getting a good choice and avoiding regret.
We consider the behaviour of a DM as *rational* whenever her choice is not sensitive to any particular order of presentation of the different lists of sets of alternatives. Therefore, we regard a choice function, representing such a DM's attitude, as *rational* if it belongs to the class of the so-called *Plott functions* (see [5] and below for a formal definition).
However, the set of list-envelop choice functions is larger than the class of Plott functions. Some of the choice functions of the former class cannot be *rationally improved* by any of the latter (see Examples 1 and 2 below). Since a list-envelop choice function f represents a procedure to improve the DM's choice from a 'complex' set, we then consider the maximal Plott function g that is less or equal to f.[2] Such a selection procedure, named *Plottization* by Koshevoy and Danilov [4], helps the DM to *further correct and improve* the choice by removing the so-called 'non-rational' choices, that could eventually emerge from the selection, from the choice sets. Proposition 1 shows that such an *improvement* fails in the case of revealed preferences $\preceq_f$ defined on the set of alternatives X which are cyclic.[3]
The next issue, discussed in Section 3, consists in the axiomatic characterization of the class of Plott functions that are among the fixed points of the mapping transforming a choice function into a list-envelop choice function (see Theorem 1). The only axiom used here to characterize such a class of choice functions is motivated by the study on the provision of stable outcomes to the extension of the *Gale-Shapley algorithm* due to Roth [17]in the many-to-many matching setup (see [7]). In words, this axiom says that, when a complex problem is solved by splitting it in a finite number of more elementary sub-problems easier to solve,

[2]We remind here that, in general, $g \leq f$ whenever, for any set of alternatives A, $g(A) \subseteq f(A)$.

[3]Here $x \preceq_f y$ means that for any $x, y \in X$ and a choice function f, x is at least as good as y, if $x \in f(x, y))$ and cyclic means that $x \preceq_f y \preceq_f z \preceq_f x$ for any $x, y, z \in X$.

the set of alternatives that at each step, *sequentially*, are disregarded because worthless will never been selected as possible solutions to the problem faced by the DM: any solution-set that was rejected at the previous steps of the sequential choice procedure from lists of sets of alternatives will be also rejected at any future step.
Since a list-envelop choice function f fails to specify a single solution to a problem when the DM is indifferent between alternatives, a tie-breaking rule, consistent with our notion of rationality, is necessary. In Section 4, we explore one tie-breaking rule among the many arbitrary complex that are available. That rule allows us to characterize (see Theorem 2) the Plott functions f on a directed partition $\mathscr{X} = (X_1 \cup X_2, \cup, \cdot \cup X_r)$ of the set of all alternatives X, for which the following sequence of choices, $X_1 = f(X)$, $X_2 = f(X \setminus X_1)$, ..., $f(X \setminus \cup_{j=1,\ldots,k} X_j) = X_{k+1}$, $k = 2, \ldots, r-1$ holds true: such a partition is a *shelling* of X. The characterization proposed can also be seen as a generalization of Proposition 2 in Rubinstein and Salant [18].
Finally, we look for a heuristic criterion that helps the DM to decide which solution-set among the many available to solve her complex problem could be better. Answering to such a question (in Section 5) allows us to find a *rational* to the class of choice functions analysed in the present work. Indeed, we find an easy to implement relation that establishes that a solution-set will be disregarded whenever another one arises to be better and therefore preferred to the former (see formula (7) below). We show that such a *relational system*, defined on any possible list of sets of alternatives in which a set (problem) can be partitioned, satisfies several compelling properties that are necessary and sufficient to characterize (see Theorem 3) the rationality behind the class of choice functions studied here.
A long, but not exhaustive, list of prominent examples (in Section 6), showing the relevance and connection of our approach with some significant works from the current economic literature, concludes.

2 Notation, definitions and preliminary results

Let X be a finite set of alternatives and 2^X be the set of all possible subsets of X. A *choice function* $f : 2^X \to 2^X$ is a contraction operator, i.e. it is a mapping such that, for any $A \in 2^X$, $f(A) \subseteq A$. The set of all choice functions is denoted with $\mathbf{CF}(X)$. We allow the DM to carry out the empty choice, that is, for some set A, it might be the case that $f(A) = \emptyset$. We also define a natural order $\leq$ on $\mathbf{CF}(X)$ by establishing that for any $f, g \in \mathbf{CF}(X)$ and any $A \in 2^X$, $f \leq g$ whenever $f(A) \subseteq g(A)$.

Rubinstein and Salant [18] have analysed a choice model in which the DM doesn't choose from *sets* of alternatives, but she faces a problem of selecting from a list $L = \{x_1, ..., x_K\}$ of alternatives, with $x_i \in X$ for any $i = 1, ..., K$. We generalize their notion by considering, for each set $A \in 2^X$, a *list* that is an ordered partition of A, namely an (ordered) collection of non-intersecting sets, $\mathcal{A} = (A_1, \ldots, A_s)$, where $s := l(\mathcal{A})$ is the length of the partition and $A_i \cap A_j = \emptyset$, $i \neq j$, $A = A_1 \cup \ldots \cup A_s$.

Denote by $\mathbf{Par}(A)$ the set of all lists (or possible partitions) of A, and by $\mathbf{Par}(2^X) = \bigcup_{A \in 2^X} \mathbf{Par}(A)$ the set of all possible lists for any $A \in 2^X$. Then, a choice function, defined on $\mathbf{Par}(2^X)$, is a function,

$$\alpha : \mathbf{Par}(2^X) \to 2^X,$$

that assigns to any list $\mathcal{A} \in \mathbf{Par}(A)$, a subset of A, namely $\alpha(\mathcal{A}) \subseteq A$. We simply call α the *choice function from sets of lists* (or **CSL**).

One of the problems we deal with in the present paper concerns the *rationality* of the class of **CSL**. If the list from which a DM chooses is not observed or a DM is interested to analyse, for all $A \in 2^X$, the options that are selected for all the ordered collections of sets in which A can be partitioned, then, for a **CSL** α, we define the choice correspondence:

$$f_\alpha(A) = \bigcup_{\mathcal{A} \in \mathbf{Par}(A)} \alpha(\mathcal{A}), \quad A \in 2^X.$$

that we call the *list-derived choice function.*

We say that the DM's behaviour will be *rational* whenever her choice does not depend on any particular order of presentation (of the lists) of the alternatives. More specifically, we consider a choice function, representing such a DM's attitude, as *rational*, if it belongs to the class of the so-called *Plott functions* (see [4]). Indeed, Plott [14] proposed to consider a choice as *rational* if it is not dependent on the way we divide a set of alternatives from which we make a choice. It means that if a set A is divided into two subsets B and C, making a choice first from B and then from C and finally making a choice from the union of these two choice sets, it must be the same as to make the choice from A. Namely, for a choice function f and for any $A = B \cup C$,

$$f(A) = f(f(B) \cup f(C)). \tag{1}$$

A choice function which satisfies (1) is called Plott function in [4] and more often *path-independent*.[4] We denote the set of all Plott functions with $\mathbf{Pl}(X)$ and we observe that $\mathbf{Pl}(X)$ is stable under the union, i.e. if f_1 and f_2 are two Plott functions, the function $f_1 \cup f_2$, defined as $(f_1 \cup f_2)(A) = f_1(A) \cup f_2(A)$, with $A \in 2^X$, is a Plott function, (see [4, 14]).

In the present work, we say that a choice function α, defined on the set of all possible lists from a set of alternatives, is *rational* if the list-derived choice function f_α is a Plott function.

In order to study such a rationality, we proceed in two steps.

Step 1. For a choice function f, we first define a choice function $\hat{f}$, that we call the *list-envelop* of f, by the rule:

$$\hat{f}(A) = \bigcup_{A = A_1 \cup \ldots \cup A_s} f(A_s \cup (f(A_{s-1} \cup (\ldots \cup f(A_2 \cup f(A_1)))))). \tag{2}$$

The list-envelop $\hat{f}$ is a kind of *improvement* upon the choice function f, namely if f is 'rational' on a 'simple' choice set A_i, it is expected that $\hat{f}$ might improve the choice on a more 'complex' choice set A.
However, we observe that the set of fixed points of the mapping $f \to \hat{f}$, that we call the *list-enveloped choice*, contains the set of Plott functions as a proper subset. Indeed,

Example 1. Let f be a choice function defined on a three elements set $X = \{1, 2, 3\}$ and specified as $f(i) = \{i\}$, $f(1, 2) = f(1, 3) = \{1\}$, $f(2, 3) = \{2, 3\}$, $f(1, 2, 3) = \{2\}$. The function f does not satisfy path-independence, since $f(1, 2, 3) = \{2\}$ and $f(f(1, 2) \cup 3) = \{1\}$.
We have $\hat{f}(A) = f(A)$ for $|A| \leq 2$, and $\hat{f}(1, 2, 3) = \{1, 2\}$. Therefore, also $\hat{f}$ is not path-independent. Moreover, we have that $\hat{\hat{f}} = \hat{f}$. $\square$

Step 2. Since a list-envelop mapping does not necessarily yields a Plott function, we consider the so-called *Plottization* procedure of a choice function, as

[4]The difference between these two classes of choice functions is that for the Plott functions the non-emptyness of the choice sets is not required.

defined by Danilov and Koshevoy [4]. Namely, for a choice function $f \in \mathbf{CF}(X)$, the function

$$\tilde{f} = \bigcup_{h \in \mathbf{Pl}(X),\, h \leq f} h \tag{3}$$

is the *Plottization* of f, that is the maximal Plott function that is less or equal to f. Such a function always exists, since the choice function $\mathbf{0}$ is a Plott function. Going back to Example 1, we observe that the Plottization of f is $\mathbf{0}$, while the Plottization of $\hat{f}$ is the function $\tilde{f}$ specified as $\tilde{f}(i) = \{i\}$, $\tilde{f}(1,2) = \tilde{f}(1,3) = \{1\}$, $\tilde{f}(2,3) = \{2,3\}$, $\tilde{f}(1,2,3) = \{1\}$. This function is *rationalized* by two linear orders $1 > 2 > 3$ and $1 > 3 > 2$. Then, thanks to this procedure, we detect that $\{2\} \in f(1,2,3)$ in Example 1 was a 'non-rational' choice, which might be "corrected and improved upon", (see Section 3 for more details).
Hence, in what follows, we study the foregoing two-step rational improvement of a choice function f by considering the *Plottization* of the list-envelop $\hat{f}$.[5]
Before starting our analysis, it is worth noticing here that some choice functions do not allow to be improved upon. Indeed,

Example 2. Let f be a non-empty-valued choice function on $X = \{1,2,3\}$ specified by $f(1,2) = \{1\}$, $f(2,3) = \{2\}$, $f(1,3) = \{3\}$. Then $\hat{f}(X) = X$ and $\tilde{\hat{f}} = \mathbf{0}$. □

The reason for such a failure in the 'improvement'of f to be a 'more rational' function is that the revealed preferences on X corresponding to f, and defined as $x \preceq_f y$ if $x \in f(x,y)$, are *cyclic.*
So, in general, we state that:

Proposition 1 *For a non-empty valued choice function f, such that the revealed preferences $\preceq_f$ are acyclic, the two-step improvement $\tilde{\hat{f}}$ is a non-empty valued Plott function.*

Proof. The set of linear orders which break ties of $\preceq_f$ define the desired non-empty valued Plott function.[6] □

[5]Notice that for any choice function f, the list-envelop $\hat{f}$ coincides with f at all two-element sets and singletons.

[6]Compare with Section 3 of [18], and Section 4 below.

Remark 1. We notice that the Plottization of f is non-empty if and only if f satisfies the property that if, for any $A \in 2^X$, there exists $a \in f(A)$ such that $\emptyset \neq B \subset A$, and $a \in B$, then $a \in f(B)$. As a consequence of this, the choice function f in Proposition 1 also satisfies this property. □

3 Sequentially consistent choice procedure and Plottization as a choice improvement process

In what follows, we construct a choice set from lists of sets of alternatives as induced by an 'usual'choice function $f : 2^X \longrightarrow 2^X$. Then, we study the corresponding list-envelop $\hat{\hat{f}}$ of f, when it occurs that $\hat{\hat{f}}$ belongs to the class of Plott functions.

In order to do thta, we first observe that, within a choice context, there exists some irrelevant information that is detrimental or useless to the choice process. Identifying crucial information presented in a problem and then being able to correctly recognize its usefulness is essential. We expect that a rational DM (or simply the representation of her behaviour via a choice function), that improves her choice at any stage of the selection procedure, disregards alternatives that are useless to solve her choice problem and those alternatives will never be chosen an any future step of the decisional process. Such a suitable criterion represents the principle on which our procedure to improve the choice relies and it can be formalized as follows.

For any $A \in 2^X$, let $\mathcal{A} = A_1 \coprod A_2 \coprod \cdots \coprod A_k$ be a k- sets partition of A (namely, lists of sets of alternatives) with (eventually) some $A_i = \emptyset$, $A_i \cap A_j = \emptyset$ for all $i \neq j$ and $A = A_1 \cup A_2 \cup \ldots \cup A_k$. Then, for a choice function $f : 2^X \to 2^X$, define the following sequence of choice sets:

$$C_0 = A,\, C_1 = f(A_1),\, C_2 = f(C_1 \cup A_2) = f(f(A_1) \cup A_2) \cdots, C_k(\mathcal{A}, f) = f(C_{k-1} \cup A_k).$$

We call $C_{l(\mathcal{A})}(\mathcal{A}, f) := C_k(\mathcal{A}, f)$, with $k = 1, ..., l(\mathcal{A})$, is the *choice from the list* of $\mathcal{A}$, induced by f. Then, we define the corresponding *list-envelop* of f as:

$$\hat{f}(A) = \bigcup_{\mathcal{A}} C_{l(\mathcal{A})}(\mathcal{A}, f),$$

where the union is taken over all directed partitions of A. We observe that this map coincides with (2).

Finally, we say that a choice function f is a *sequentially consistent choice procedure*, namely a selection mechanism that, at each stage, disregards all the alternatives that a DM has not chosen at the previous stages, if f satisfies the following:

Sequential Consistency (**CON**) For any $A \in 2^X$, any directed partition $\mathcal{A} = A_1 \cup A_2 \cup ... \cup A_k$ and any $B \subset A \setminus C_k$, there holds

$$f(B \cup C_k) \cap B = \emptyset. \tag{4}$$

The class of choice functions f that satisfies **CON** is named *sequentially consistent* (**SC**). We show that the class **SC** coincides with the class of Plott functions $\mathbf{Pl}(X)$, namely the DM, that is not order-dependent in her choices, never chooses what she disregards as detrimental or worhtless. In order to show that, let us introduce two well-known properties of the choice functions:

Heritage (**H**) For any A, $B \in P(X)$, if $A \subseteq B$ then $f(B) \cap A \subseteq f(A)$.

Heritage condition has been first introduced by Chernoff (postulate 4 in [3]) and it means that if an alternativey a is chosen from a set B, then it will be also chosen from the smaller set $A \subseteq B$ including a.[7]

Outcast (**O**) For any $A, B \in P(X)$, if $f(A) \subset B \subset A$, then $f(B) = f(A)$

Chernoff (postulate 5 in [3]) proposed property **O**. It is also called Aizerman property, in honour of Mark Aizerman who first made an extensive use of it (see e.g. [2]). It says that removing from a set the alternatives that are not chosen by a DM does not affect the worth of the set.

Now, for a choice function f to satisfy both **H** and **O** axioms is tantamount to satisfy Plott's path-independence property (see, for example, Lemma 6 in [12]). Then, we establish that:

[7]There are some other reformulation of this axiom. For instance, in the literature on stable matchings (Roth and Sotomayor, [17], Definiton 6.2 p.173), the Heritage axiom is the *Substitutability property*: if a worker a is hired by a firm from a list B, she will be also hired from any shorter list ($A \subseteq B$), containing it.

Theorem 1 *The following statements are equivalent:*

1. *f is **SC**;*
2. *f is* **Pl**.

Proof. 2) implies 1). This follows from the multilinear representation of Plott's path-independence, namely $f\left(\bigcup_{i=1}^{k} f(A_1)\right) = f\left(\bigcup_{i=1}^{k-1} f(A_i) \cup f(A_k)\right)$. Indeed, posit wlog $A = B \cup A \backslash B$, then, by (1) , we have:

$$f(A) = f(f(B) \cup f(A \backslash B)) = f(f(B) \cup A \backslash B).$$

where $f(f(B) \cup A \backslash B) = f(C_1 \cup A \backslash B) = C_2$. Then, for any $D \subset A \backslash C_2$, $f(C_2 \cup D) \cap D \subset f(A) \cap D = \emptyset$.

1) implies 2). In order to prove such a statement, we observe that if f is **SC**, then, obviously, it is a *2-sequentially consistent* choice function, namely it satisfies **CON** for any partition $\mathcal{A} = A_1 \cup A_2$. Therefore, without loss of generality, we show that if f is *2-sequentially consistent*, then f is a Plott function.
Let us first establish the following:

Claim 1. A choice function f that is 2-sequentially consistent is idempotent, i.e. for any $A \in 2^X$, $f(f(A)) = f(A)$.

Suppose on the contrary that, for some $A \in 2^X$, $f(f(A)) \neq f(A)$ that means that there is some $x \in f(A)$, which is not in $f(f(A))$. Then, consider the partition $A = f(A) \cup (A \backslash f(A))$. By 2-*sequentially consistency* $C_2 := f(f^2(A) \cup (A - f(A))$, that means there is some $x \in f(A)$ which is not in C_2, that contradicts the fact that f is 2-*sequentially consistent* since $f(A) = f(C_2 \cup (A \backslash C_2)$.

Claim 2. For any $A, B \in 2^X$ and a *2-sequentially consistent choice function f, if $B \subset f(A)$, then $f(B) = B$.*

Indeed, for any $A \in 2^X$, consider a partition $A = B \cup (A \backslash B)$ and suppose by contradiction that $B \backslash f(B) \neq \emptyset$. Now, since $B \backslash f(B) \not\subset f(f(B) \cup (A \backslash B)) = C_2$, the inclusion $B \backslash f(B) \subset f(A) = f(C_2 \cup (A - C_2))$ contradicts the 2-sequentially consistency property.

Let us check now that a *2-sequentially consistent choice function f satisfies property* **O** by supposing by contradiction that it does not hold, i.e. for any $A, B \in 2^X$ and a *2-sequentially consistent* choice function f, $f(A) \subset B \subset A$ implies $f(B) \neq f(A)$ and, in particular, that $f(B) \subset f(A)$.

Now, $f(B) \subset f(A)$ implies $f(B) \cup (A \backslash B) \subset f(A) \cup (A \backslash B)$, that is $f(f(B) \cup (A \backslash B)) \subset f(f(A) \cup (A \backslash B)) \subset f(B \cup (A \backslash B)) = f(A) \subset f(B)$ that implies that $C_2 \subset$

$f(B)$ that is impossible. Indeed, if $A\backslash B$ does not contribute to the choice (a DM doesn't choose any alternative from $A\backslash B$), then $f(f(B)\cup(A\backslash B))$ reduces to $f(f(B))$ that by Idempotence implies $f(B)\subset f(B)$ that is impossible. On the contrary, if $A\backslash B$ contains alternatives that could be selected by a DM, then we get a contradiction because $f(f(B)\cup(A\backslash B))$ cannot be a subset of $f(B)$.

By analogy, we treat the case that $f(B)\supset f(A)$. Hence $f(B)=f(A)$ as required.

In order to prove that *2-sequentially consistent choice function f satisfies property* **H**, consider, for $A\in 2^X$, the partition $A=B\cup(A\backslash B)$ and suppose that for $B\subset A$, $f(A)\cap B\nsubseteq f(B)$. Then, we know that $C_2=f(f(B)\cup(A\backslash B))$, $D=(f(A)\cap B)\backslash f(B)$ is nonempty and also $A\backslash D$ must be nonempty. Now, $D\subset C_2$ and also $A\backslash D\subset A\backslash C_2$, that implies that $f(A\backslash D)\subset f(A\backslash C_2)\subset f(A)$. By **O**, $f(A\backslash C_2)=f(A\backslash D)$. But, by the 2-sequentially consistency property $f(A\backslash C_2)$ must be empty, so it will be $f(A\backslash D)$, hence a contradiction that implies that a 2-sequentially consistent choice function must satisfy **H**. Hence, since a function that satisfies **H** and **O** also satisfies Plott's path-independency (1), we get that 1) implies 2) and complete the proof. □

Remark 2. It is straightforward to observe that **O** implies 1-sequential consistency, while the opposite is not true. Indeed, let $A=\{x,y,z,w\}$ and $f(\{x,y,z,w\})=\{x,y\}$, $f(B)=\{x\in B : x\succ y \text{ for any } y\in B\}$, i.e. $f(B)$ selects the best element w.r.t. $x\succ y\succ z\succ w$ for all strict subsets B of A.

On the contrary, 1-sequential consistency implies a weaker form of **O**, namely, for any $A, B\in\mathcal{P}(X)$, if $f(A)\subset B\subset A$, then $f(B)\subset f(A)$. While 1-sequential consistency and Idempotence of f implies Outcast. □

We have shown that the rationality of a DM, whose choice is not influenced by the particular position of an alternative in a list, consists in breaking a complex problem in more simple ones that can be solved sequentially, disregarding at each step the alternatives that do not concur to the solution of the problem solved before. Such a kind of rationality is easily implementable applying the iterated procedure in (2) and corresponds to the rationality of a DM who maximize transitive and complete preference relations, as the following result shows:

Corollary 1 *If $\succ_1,\dots\succ_n$ is a sequence of orderings and $f\in$ **SC**, then for all*

$A \in 2^X$

$$f(A) = \bigcup_{i=1}^{n} \max_{A} \succ_i$$

i.e. f *can be written as the union of all choice functions* $l_i(A) := \max_A \succ_i$, $i = 1, \ldots, n$.

Finally, it is worth considering that Theorem 1 is a first 'attempt' to connect the present work on rationality of choices from lists of sets of alternatives and the now quite large literature on *sequentially rational choice* (see [8] and [9] and references therein), but the exploration of such a topic is best left for future research.

4 Tie-breaking and priority indicator

The selection from smaller subsets is simpler than the choice from a single large choice problem. However, the elements of a choice set may not be distinct. In fact, any selection procedure is often specified in a way that allows for multiple winners or ties to occur, without making it clear how these ties are to be broken. An instance of such a type could be a voting rule like the 'plurality voting'where multiple alternatives may have the highest score, another one is the 'successive choice rule' (see [6], [19], [22]), where in each round it eliminates the alternatives with the lowest preference, and ties may occur.
However, in many cases, what is desired is a single winner or just one alternative. In these situations, a tie-breaking rule is required. We could have many types of tie-breaking rules, such as random or deterministic, lexical or arbitrary. Here, we study the family of linear-ordered tie-breaking rules as a suitable and natural rule to break ties.[8]

In order to do that, we suppose to have a partition of X into non-intersecting sets X_i, such that $\mathcal{X} = X_1 \coprod X_2 \coprod \cdots \coprod X_r$. For such a collection $\mathcal{X}$ of disjoint sets $X_1, \ldots, X_r$, we consider the set $\mathcal{P}(\mathcal{X})$ of the Plott functions[9] for which:

$$f(X) = X_1, \text{ and } f(X \setminus \cup_{j=1,\ldots,k} X_j) = X_{k+1}, \ k = 1, \ldots, r-1, \tag{5}$$

[8]It is worth noticing that our approach to tie-breaking also generalizes Rubinstein and Salant's (Section 3, [18]) characterization of single valued path-independent choice functions using the so-called *priority indicators* to the case of choice functions that are not necessarily single valued.

[9]Since $\mathcal{X}$ is a partition of X, the class of Plott functions here coincides with the class of the path-independent choice functions.

holds true. We notice that the set $\mathcal{P}(\mathcal{X})$ is stable under intersection and union of functions of it.
We characterize the set $\mathcal{P}(\mathcal{X})$ following Danilov and Koshevoy [4], providing a rule to break ties when they occur during the process of rational improvement of the choice.

Before proceeding, let us recall the notion of *shuffle* of linear orders on sets X and Y (see [4] for further details).
A linear order $x_{i_1} > x_{i_2} > \cdots$ on X can be seen as a word $x_{i_1}x_{i_2}\cdots$, then a shuffle of two words $v = v_1v_2...v_n$ and $w = w_1w_2...w_n$ is a word of the form $w_1v_1w_2v_2...w_nv_n$, with some of the subwords w_i, v_j that might eventually be empty. In other terms, for shuffling two words we take the initial piece of one of the two words and we append the initial piece of the other word to the former. Then, we return to the first word and we take the next piece starting from the interruption place and again we switch to the second word, and so on. In order to illustrate this procedure, suppose $w = xyz$ and $v = abcd$ are two words. Then, the word $xabyczd$ is one of the (possible) shuffles of w and v. A shuffle of three or more words is defined similarly.

By analogy, the shuffle of two linear orders $>_1$ and $>_2$ is obtained as the *simplification* of the shuffle of the corresponding words. Namely, we take the shuffle of the words $x_{i_1}x_{i_2}\cdots$ and $x_{j_1}x_{j_2}\cdots$ corresponding to $>_1$ and $>_2$ and cancel each second appearance of a letter. Again, an example illustrates the procedure . Let $X = \{x_1, x_2, x_3, x_4\}$ be the set of alternatives and $>_1 := x_1 >_1 x_4 >_1 x_3 >_1 x_2$ and $>_2 := x_3 >_2 x_4 >_2 x_2 >_2 x_1$ be two linear orders on X. A shuffle of the corresponding words be $x_1\, x_3x_4\, x_4x_3x_2\, x_2x_1$. Then, the corresponding linear order $x_1 > x_3 > x_4 > x_2$, being the simplification of $x_1\, x_3x_4\, x_4x_3x_2\, x_2x_1$, is a shuffle of $>_1$ and $>_2$, and is an element of the set $>_1 \underline{Ш} >_2$, where $\underline{Ш}$ denotes the shuffle of two linear orders.

Now, for the collection $\mathcal{X}$ of disjoint sets $X_1, \dots, X_r$, let $\mathcal{L}^+(\mathcal{X})$ and $\mathcal{L}^-(\mathcal{X})$ be two collections of linear orders on X. We construct $\mathcal{L}^+(\mathcal{X})$ in r steps.

Step 0. Denote by $\mathcal{L}_r$ the set of linear orders on the set X_r.

Step 1. Define the set of linear order on $X_{r-1} \cup X_r$ as a subset of shuffles of $\mathcal{L}_{r-1}$ and $\mathcal{L}_r$, such that the top element of a linear order of $\mathcal{L}_{r-1}\underline{Ш}\mathcal{L}_r$ belongs to X_{r-1}. Denote this subset of shuffles by $\mathcal{L}_{r-1}Ш\mathcal{L}_r$.[10]

[10]Notice that, since the sets X_k and X_{k-1} are disjoint, we do not need the simplification operation on shuffles of the corresponding words for shuffling $\mathcal{L}_k$ and $\mathcal{L}_{k-1}$.

Step $r-k$. Consider the set

$$\mathcal{L}_{r-k} Ш (\mathcal{L}_{r-k+1} Ш (\mathcal{L}_{r-k+2} Ш (\cdots Ш (\mathcal{L}_{r-1} Ш \mathcal{L}_r).$$

and denote by $\mathcal{L}(\mathcal{X})^+$ the output set of the above construction of linear orders at the step $r-1$.

The collection $\mathcal{L}(\mathcal{X})^-$ is the subset of $\mathcal{L}(\mathcal{X})^+$ of linear orders such that all elements of X_k, with $k = 1, \dots, r-1$, dominate all elements of X_{k+1}.

Then, for any $A \subseteq X$, $f^+_{\mathcal{X}}(A)$, with $f^+_{\mathcal{X}} \in \mathcal{P}(\mathcal{X})$, is the union of maximal elements of the linear orders of $\mathcal{L}(\mathcal{X})^+$ being restricted to A, therefore it is the maximal element of $\mathcal{P}(\mathcal{X})$. The function $f^-_{\mathcal{X}}$ is defined, for each $A \subseteq X$, as the union of maximal elements of the linear orders of $\mathcal{L}(\mathcal{X})^-$ restricted to A and it is the minimal element of $\mathcal{P}(\mathcal{X})$.

We further need to introduce the notion of *basement* of a choice function f (see [4]).

A simple word $w = x_1 x_2 ... x_n$ is a word without repeated letters. It defines the following linear order $w = (x_1 \succ_w x_2 \succ_w ... \succ_w x_n)$ on the subset $\underline{w} \subset X$, where $\underline{w}$ denotes the subset of X constituted by the letters of w. This ordering defines the *linear* choice function l_w on X, namely, for $A \subseteq X$, $l_w(A) = \emptyset$ if $A \cap \underline{w} = \emptyset$ and $l_w(A)$ equals the maximal element of $A \cup \underline{w}$ if the intersection is not empty. The choice function l_w is a Plott function, called *linear* Plott function in [4]. The main peculiarity of l_w is that it contains at most one element.

Danilov and Koshevoy [4] define the *basement* of a choice function f as the set $Bas(f)$ of linear Plott functions l such that $l \leq f$ and establish that any Plott function f is equal to the join of all linear Plott functions of the basement of f, namely $f = \vee(Bas(f))$ (see Theorem 1 in [4]) . We further observe that, since there is a bijection between the set of simple words and linear Plott functions, $Bas(f)$ can be regarded as the collection of simple words w such that $l_w \leq f$. Hence, we get:

Theorem 2 *Let f be a Plott function. Then $f \in \mathcal{P}(\mathcal{X})$ if and only if the basement of f, $Bas(f)$, is in-between the sets $\mathcal{L}^-(\mathcal{X})$ and $\mathcal{L}^+(\mathcal{X})$ of linear orders on X, i.e.,*

$$\mathcal{L}^-(\mathcal{X}) \subseteq Bas(f) \subseteq \mathcal{L}^+(\mathcal{X}) \tag{6}$$

It is also possible to check that $Bas(f^+_{\mathcal{X}}) = \mathcal{L}^+(\mathcal{X})$ and $Bas(f^-_{\mathcal{X}}) = \mathcal{L}^-(\mathcal{X})$, and therefore, we can also regard (6) as:

$$Bas(f^-_{\mathcal{X}}) \subseteq Bas(f) \subseteq Bas(f^+_{\mathcal{X}}).$$

Proof. Since $X_1 = f(X)$, any linear Plott function of $Bas(f)$ has 'top' elements of X_1, that is for any $l \in Bas(f)$, $l(X) \in X_1$ and for any $a \in X_1$, there exists an $l \in Bas(f)$, such that $a = l(X)$. For the restriction of an $l \in Bas(f)$ to $X \setminus X_1$, we get that the $l(X \setminus X_1) \in X_2 = f(X \setminus X_1)$, and again for any $b \in X_2$, there is an $l' \in Bas(f)$ such that $l(X \setminus X_1) = b$. For level k with X_k, we have the same properties. This shows that $Bas(f) \subset \mathcal{L}^+(\mathcal{X})$.

The inclusion $\mathcal{L}^-(\mathcal{X}) \subseteq Bas(f)$ follows since the Plott choice function $f^-_{\mathcal{X}}$ has $\mathcal{L}^-(\mathcal{X})$ as basement. $\square$

Theorem 2 establishes that any Plott function, satisfying (5), is the join of linear Plott functions inducing a linear order on the elements of X. The methodology presented here for breaking the ties is an efficient and clear-cut tie-break rule, that also generalizes the priority indicators tie-breaking rule studied by Rubinstein and Salant ([18]).
We further observe that for a Heritage choice function g, $\mathcal{Y} = Y_1 \coprod \ldots \coprod Y_s$ is the corresponding partition of X, where

$$g(X) = Y_1, \text{ and } g(X \setminus \cup_{j=1,\ldots,k} Y_j) = Y_{k+1},\ k = 1, \ldots, s-1.$$

This partition is correctly defined, (see [5], Section 4). Then, we also have the following characterization of the Heritage choice functions with a given $\mathcal{Y}$:

$$\mathcal{L}^-(\mathcal{Y}) \subseteq Bas(g).$$

5 Rationalizability and comparison of sets

The comparisons is a critical and fundamental pathway to choose, in particular when you need to compare solutions to complex problems. However, little basis, a priori, are known about how to make a choice among different solutions and to understand their 'goodness'We then need to find some plausible criterion that helps us to decide which procedures or set of solutions could be better than another to solve some difficult problem. In general, if the members of sets, we need to compare, are not mutually exclusive as it is the case, for instance, of matching, voting, coalition formation and sequentially rationalizable choices just to mention a few, one can have the necessity to take preferences over sets as a primitive.
In other words, we wonder if it is possible to find a preference relation that (a)

be consistent with our class of choice functions (**Pl** or equivalently **SC**), and that (b) sequentially compare solution-sets from the sub-problems in which our difficult problem is divided and that (c), at each step, do not choose what was discarded in a previous step. Namely, for any given partition $A = A_1, ..., A_n$, if the DM's choice follows (2), then she needs to make comparisons among the elements of a sub-set and the elements of a sub-set that is next in the list in order to find the best alternatives (solution sets).
In the present context, a plausible criterion that a DM could adopt consists in ranking a set A_k at least as good as set A_{k+1} if for any (solution) element a of A_{k+1} there exists some (solution) element b of A_k that is preferred to a and therefore is chosen. The foregoing criterion defines a binary relation $\preccurlyeq$ induced by the choice that the DM pursues. Formally,

Definition 2. For any $A \in 2^X$, let $\mathcal{A} = A_1 \coprod A_2 \coprod \cdots \coprod A_k$ be a k- sets partition of A with (eventually) some $A_i = \emptyset$, $A_i \cap A_j = \emptyset$ for all $i \neq j$ and $A = A_1 \cup A_2 \cup \ldots \cup A_k$. Then, for a choice function $f : 2^X \to 2^X$

$$A_{k+1} \preccurlyeq A_k \quad \text{if, for every} \quad a \in A_{k+1}, \quad f(a \cup A_k) \subseteq A_k \tag{7}$$

It is possible to observe that the binary relation $\preccurlyeq$ in (7) satisfies several interesting properties.

Desiderability (D). For any $\{a\} \in X, \emptyset \prec \{a\}$.

that means there are not (absolutely) undesirable alternatives, there are not simple problems in which a complex one is divided which do not have solution.

Monotonicity (MON) . For all $A_k \, and \, A_{k+1}$, $A_{k+1} \subseteq A_k$ implies $A_{k+1} \preccurlyeq A_k$.

MON property is very compelling assumption in the literature on ranking sets of alternatives in terms of the freedom of choice they provide. Here, following Manzini and Mariotti (2012), we imagine that a complex problem is divided in many simpler sub-problems easier to be solved. Of course, the larger is the set of possible solutions to a given complex problem the more tools we will have to solve it satisfactorily.

If a binary relation $\preccurlyeq$ is transitive and satisfies **MON**, then it satisfies the following two convenient properties:

Contraction (CON). For any A_{k+2}, A_{k+1}, A_k, if $A_k \subseteq A_{k+2} \preccurlyeq A_{k+1}$, then $A_k \preccurlyeq A_{k+1}$.

Expansion (EXP). For any A_{k+2}, A_{k+1}, A_k, if $A_k \preccurlyeq A_{k+2} \subseteq A_{k+1}$, then $A_k \preccurlyeq A_{k+1}$.

The first axiom highlights the fact that if what we choose from a set is somehow better of what we choose from another, then, *a fortiori*, this holds true for any subset of the latter.
On another hand, if a set offers more 'suitable solutions' to a complex problem than another, then the set containing the former as its subsets will certainly provide more suitable tools than the latter for solving the difficult problem we face.
It is easy to agree that, going out for having a dinner, if we discard two restaurants A_k and A_{k+1} in favour of another A_{k+2}, it should be not rational to prefer a restaurant collecting all the dishes of the A_k and A_{k+1} menus to A_{k+2}. Formally,

Coherence (CH). For all A_{k+2}, A_{k+1}, A_k, if $[A_{k+2} \preccurlyeq A_{k+1} \textit{ and } A_k \preccurlyeq A_{k+1}]$, then $A_{k+2} \cup A_k \preccurlyeq A_{k+1}$.

Axiom **CON** above says that what is never chosen, will never be chosen and therefore can be discarded without any problem. This means that just some *essential* elements of a set matter. This intuition is analytically captured by the following axiom proposes by Puppe [15]:

Independence of Non-Essential Alternatives (IND). For every $A \in 2^X$, let $f : 2^x \longrightarrow 2^x$ be a choice function, then

$$A \preccurlyeq f(A) \tag{8}$$

We are now ready to state what is the connection between the class of sequentially rational choice functions (**SC**) introduced above (and of course the equivalent class **Pl**, see Theorem 1) and the relational structure (of Definition 2) that compares sets in terms of the desirability of the solution set for the DM.

Theorem 3 *Let $f : 2^X \to 2^X$ be a choice function and $\preccurlyeq$ a binary relation on 2^X, then the following statements are equivalent:*
*1. f is a **SC**;*
*2. $\preccurlyeq$ satisfies **D**, **MON**, **CON**, **EXP**, **CH**, and **IND**.*

Proof. (1. $\Rightarrow$ 2.) In order to prove such a statement, we again observe that if f is **SC**, then it is a *2-sequentially consistent choice function* (see the proof of Theorem 1). Therefore, without loss of generality, we have to show that if f is a *2-sequentially consistent choice function*, then the binary relation under Definition 2 satisfies all the properties defined above.
So, take a set $A \in 2^X$ such that $A = A_1 \cup A_2$ and suppose for some $x \in X$ that $\preccurlyeq$ is such that $x \preccurlyeq \emptyset$ that contradicts the fact that $\preccurlyeq$ satisfies **D**. By definition of 2-sequentially consistency of f we can have that $f(A_1) = \emptyset$ and, supposing that $A_2 = \{x\}$, that $f(\emptyset \cup x) = \emptyset$ that contradicts the fact that f must be non-empty valued. In order to check **MON**, consider that if $A \subseteq B$, then $\forall a \in A$, $f(a \cup B) = f(B) \subseteq B$ by definition of choice function, that means that all what matters is in B and therefore $A \preccurlyeq B$, that is what **MON** requires. If $A \preccurlyeq B$ whenever, for any $a \in A$, $f(a \cup B) \subseteq B$, then, if $A' \subseteq A$, by set-inclusion we have that $f(a' \cup B) \subseteq B$ that means that $A' \preccurlyeq B$ as per **CON** request. In order to check **EXP** consider $a \preccurlyeq A$ and that $A \subseteq B$. From $a \preccurlyeq A$ we know that $f(a \cup A) \subseteq A$, then, since $A \subseteq B$ we need to check that $f(a \cup B) \subseteq B$ that entails that $a \preccurlyeq B$. Let us assume that $a \notin B$ otherwise, by inclusion, we obviously get the result and suppose $f(a \cup B) \nsubseteq B$. Then, let $a \in f(a \cup B)$, since f satisfies Heritage, $a \in f(a \cup A) \subseteq A$, therefore $a \in A \subseteq B$ that contradicts the fact that $a \notin B$. To prove **IND**, we need to check that for any $a \in A$, $f(a \cup f(a)) \subseteq f(A)$, that implies $A \preccurlyeq f(A)$. By property **O** of f, we have that $f(A) \subseteq a \cup f(A) \subseteq A$. The fact that f is idempotent applied to $f(f(A)) \subseteq f(a \cup f(A)) \subseteq f(A)$ entails the desired result.
(2. $\Rightarrow$ 1.) Given a binary relation $\preccurlyeq$ endowed with the properties **D**, **MON**, **CON**, **EXP**, **CH**, and **IND** and the triple (a, A, B) such that $A \subseteq B$ and $a \in (f(B) \cap A)$, we want to prove that $a \in f(A)$. Take a choice function $f_{\preccurlyeq}(C) = \{c \in C : c \npreccurlyeq C \setminus c\}$, where $\preccurlyeq$ evidently satisfies **IND** and suppose, on the contrary, that $a \notin f(A)$, i.e. $a \preccurlyeq A \setminus a$. By $A \subseteq B$ and **EXP**, we have $a \preccurlyeq B \setminus a$, that means that $a \notin f_{\preccurlyeq}(B)$, that contradicts the assumption that $a \in (f(B) \cap A)$ and prove that **H** holds true.
Let us check that f also satisfies the Outcast property by supposing that $f(A) \subseteq$

$B \subseteq A$. By **IND**, we have $B \subseteq A \preccurlyeq f(A)$, that implies $B \preccurlyeq f(A) \subseteq B$. Now, $B \preccurlyeq f(A)$ entails that $f(B) \subseteq f(A)$. Indeed, if it is not, then there exists some $b \in f(B)$ that is not in $f(A)$ and therefore $f(A) \subseteq B \setminus b$. The latter with $f(B) \subseteq f(A)$ imply that, by **EXP**, $B \preccurlyeq B \setminus b$, that contradicts the fact that $b \in f(B)$ and states that $f(B) \subseteq f(A)$ holds true. The reverse inclusion, i.e. $f(A) \subseteq f(B)$, is a trivial consequence of the fact tha f satisfies (see above) **H**. Indeed, from $f(A) \subseteq B \subseteq A$, consider the second part of the inclusion, namely $B \subseteq A$, by **H**, $f(A)$
$bigcapB \subseteq f(B)$ and by the first part of the inclsuion, anmely $f(A) \subseteq B$, we have the desired result , i.e. $f(A) \subseteq f(b)$. The double inclusion gives the result that also **O** is satisfied, hence f is a 2-sequentially rational choice function. □

Theorem 3 establishes a full description of the logical relationship between the notion of rationality analysed in this paper and the preferences of a DM, which *rationalize* the class of choice functions analysed in the present work.

6 Some prominent examples of 'rational' choice from lists

We conclude by checking through some well-known prominent examples if the rationality of a DM, who chooses sequentially from lists of alternatives, is exactly (i.e. coincides with) that one provided by a Plott function.

Example 3. (Rational choice) Given a complete, asymmetric and transitive binary relation $\prec$ over a finite set X, the DM chooses the maximal element of each A_i of $A = \{A_1, ..., A_n\}$ according to $\prec$. It is easy to check that the corresponding f is a Plott function.

Example 4. (Satisficing ([20])) Given a complete, asymmetric and transitive binary relation $\prec$ over a finite set X, the DM chooses the elements of each A_i in $A = \{A_1, ..., A_n\}$ that lie above a given bliss point x^*, if there is not any one she chooses the last element of A_i. Applying our framework to the foregoing sequential choice, we observe that, for any A, $\alpha(A) = A$ and the corresponding list derived choice function $f_\alpha(A) = A$ if x^* is not in A or $f_\alpha(A) = \{x \in A : x^* \prec x\}$ if x^* is in A. Notice that $f_\alpha(A)$ could also be empty-valued. Now, it is straightforward to observe that for $A = B \cup C$ could be the case that $f_\alpha(A) \neq f_\alpha(f_\alpha(B) \cup C)$ and therefore f_α is not a Plott function.

Example 5. (Place-dependent rationality) The goods are divided on the shelves of a supermarket on the basis of the seller's own interest in buying the product. Supermarket-marked goods are to the center while at the bottom or top of the shelves, more hidden, the other goods. The DM has a binary preference relation $\prec$ over the set $X \times \{1, 2, ..., K\}$ where the pair (x, j) denotes the good x in the $j - th$ place on the shelve. A *rational* DM, whose consideration of the goods depends on their relative positions on the shelves, chooses the good x_j for which $(x_i, i) \prec (x_j, j)$ for all $1 \leq i \leq K$, where each good-position comparison occurs in every one of the $n - partition$ sets in which all the goods X of the supermarket are divided. In the present example, $f_\alpha(A) = \emptyset$ if $|A| > K$ and also if $|A| = K$ otherwise $f_\alpha(A) = \{\ x_k | (x_k, k) \succ (x_l, l)\}$. Therefore, x will be chosen, namely $x \in f_\alpha(A)$, if and only if there exist a permutation $\sigma : A \to A$ such that $(\sigma(a), |\sigma(a)|) \succ (\sigma(b), |\sigma(b)|)$ for any b such that $\sigma(b) \prec \sigma(a)$. We remark that for $A = B \cup B$, if $|A \cup B| > K$, $f_\alpha(A)$ could be equal to the empty set, while $f_\alpha(A) \neq f_\alpha(f_\alpha(B) \cup C)$ is different from $\emptyset$. Hence, again, the foregoing sequential choice procedure is not compatible (is not consistent) with a Plott function.

Example 6. (The first elements establishes the ordering) For every element $a \in A$, there exists a corresponding ordering $\prec_a$ that works as a reference point respect to which the DM chooses as in e.g. [21]. In particular, for $A = (A_1 \cup A_2 \cup, ..., \cup A_n)$ and a choice function f, if $x_1 = f(A_1)$, then consider x_1 as a given target and rank all the alternatives in X according to their desiderability (as expressed by a utility function or a suitable defined distance) with respect to x_1, in order to have the corresponding ordering $\prec_{x_1}$. Then, $f_\alpha(A) = \{\prec_{x_1}, a \in A\}$, but for $A = B \cup C$, $f_\alpha(B \cup C) \supset f_\alpha(f_\alpha(B) \cup C)$ and again f_α is not Plott.

Example 7. (Successive Choice, see [22], [11], [19]) A binary relation $\prec$ on X could be understood as a dominance relation, so $x \prec y$ means that y dominates x according to some criterion. From a list of items $\{x_1 ...., x_k\}$, a DM picks x_1 at time 0. Then, at the next step, she continues to store x_1 or replaces it with another item x_{1+k}, if $x_1 \prec x_{1+k}$. In the present case,

$f_\alpha = \{x \in A : \nexists y \in A \, with \, x \prec y\}$ (see [5]). Now, if $\prec$ is an order (namely an asymmetric and transitive binary relation, see again [5]), then f_α is a Plott function, but in general it is not.

Example 8. (The DM stops when the value of her choice decreases) A DM, with an ordering on X, chooses an item x_k, where k is the maximal integer such that $x_1 \prec x_2 \prec \cdots \prec x_{k-1} \prec x_k$, namely she stops the choice when the value of the next item in the list is smaller than the previous one, i.e. $x_{k+1} \prec x_k$. In the foregoing example, $f_\alpha(A) = \{a \in A : x \, is \, not \, minimal \, in \, A\}$. Now, if $A = B \cup C$, it is possible to observe that $f_\alpha(A) \supset f_\alpha(f_\alpha(B) \cup C)$, and therefore f_α is not Plott.

Example 9. (Good and bad company) The elements of a list can be classified in 'good'and 'bad'. The fact that there are many bad elements around a good one makes the latter even better, namely $x_l \prec x_k$ if $|x_i, closer \, to \, x_k| > |x_j, closer \, to \, x_l|$. The DM chooses the element that is more surrounded by the largest number of consecutive bad elements. Therefore, $f_\alpha(A) = \{a \in A : a \, is \, good\}$. It is interesting to observe that f_α is Plott but it is not path-independent.

Example 10. (Knockout tournament) Let $\succ$ be a complete and asymmetric binary relation on a finite set X. A DM chooses from a set A the alternative x_{2k-1} that dominates the alternative x_{2k} for $1 \leq k \leq |\frac{1}{2}K|$ according to $\succ$. She continues until the list of alternatives in A is terminated and just one alternative remains. In the present situation $f_\alpha(A) = \{x \in A : x \, beats \, \log_2 |A| \, elements \, of \, A\}$. Then, it is easily checked, by considering for instance a set $A = \{x_1, x_2, x_3\}$ with $x_1 \succ x_2 \succ x_3 \succ x_1$, that f_α is not a Plott function.

Example 11. (Almost-random choice function) Let $(X, \prec)$ a binary relational system, where X is a finite set of possible alternatives and $\prec$ is a complete and transitive binary relation, $A = \{x_1, ..., x_K\} \in 2^X$ and $n_K = |log_2 K|$. For $1 \leq i \leq n_k$, let us define $\beta_i = 1$ if $x_{i+1} \prec x_i$ and $\beta_i = 0$ if $x_i \prec x_{i+1}$. The DM chooses the alternative x_k, where k is the $n_k - digit$ binary number $\beta_1 \beta_2 \cdots \beta_{n_k}$. In other words, the DM counts $\beta_1 + 2^{\beta_2} + 4^{\beta_3} + 8^{\beta_4} + \ldots + (2^{n_k - 1})^{\beta}_{n_k}$, obtaining a number between 1 and K and the corresponding k is her choice f_α. The choice function f_α is a Plott function.

Example 12. (Limited attention, [10], [6]) A DM is looking for something on the web but she has a limited amount of time to check all the possible alternatives that a search engine shows. So, she actually consider only n alternatives out of the N provided by the web. Thus, she has an 'attention ordering' of A, which establishes the n alternatives on which she focuses and a preference ordering that entails the best-alternative among the elements selected by the 'attention ordering'. In the present case, $f_\alpha(A) = \{x \in A : x \in \Gamma(A)\, and\, for\, all\, y \in \Gamma(A)\, , x \leq y\, implies\, y = x\}$ where Γ is the attention filter induced by the 'attention ordering'(see e.g. [6], [10]) and $\leq$ is a linear ordering on Γ. We observe that f_α is not Plott. Indeed, let $A = \{x_1, x_2, x_3, x_4, x_5\}$, with an attention ordering that dictates to be focused only on the first two elements of the set of alternatives. If we split A into two set $B = \{x_1, x_4, x_5\}$ and $C = \{x_2, x_3\}$, we have that $\Gamma(B) = \{x_1, x_4\}$ and $\Gamma(C) = \{x_2, x_3\}$. Now, if $x_4 \leq x_2 \leq x_1 \leq x_3$, we obtain that $f_\alpha(f_\alpha(B) \cup C) = \{x_3\}$ while $f_\alpha(A) = \{x_1\}$. So f_α is not a Plott function.

Some of the foregoing examples propose a choice that is not consistent with the behaviour of a DM that selects the set of her best alternatives through a sequentially rationalizable choice procedure that is order-independent. This means, that the rational choice they represent cannot be improved upon by the general choice mechanism we have studied in the present work.

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