Risk Measures with Generalized Secant Hyperbolic Dependence

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Abstract

In this paper we propose to model the dependence of multiple time series returns with a multivariate extension of the generalized secant hyperbolic distribution (GSH) using the NORTA (NORmal-to-Anything) approach and the Koehler and Symanowski copula function. The two methodologies permit to generate random vectors with marginals distributed as a GSH distribution and given correlation matrix, which can be used to measure the risk of a portfolio using the Monte Carlo method.

1 Motivation

It is well known that distributions of many financial quantities have heavy tails, are skewed and have other non-Gaussian characteristics. A multivariate distribution often used in risk management is the asymmetric Student-t distribution which seems to be the distribution which better captures the non-normality features of financial data. A drawback of the Student-t distribution is represented by the technical difficulties due to the evaluation of moments of the marginals, especially in the case of fractional degrees of freedom.

In alternative, we propose to model the dependence of multiple time series returns with a multivariate extension of the generalized secant hyperbolic distribution (GSH) using the NORTA (NORmal-To-Anything) approach and the Koehler and Symanowski copula function. The two methodologies permit to generate random vectors with marginals distributed as a GSH distribution and given correlation matrix, which can be used to measure the risk of a portfolio using the Monte Carlo method. The main advantage of the GSH distribution over the Student-t distribution is that all the moments are finite for each value of the shape parameter.

The paper is organized as follows. After introducing the GSH distribution in next section, in section 3 we present the two methodologies, the NORTA (NORmal-To-Anything) approach and the Koehler and Symanowski copula function, to model the dependence of multiple time series returns with a multivariate extension of GSH distribution; then, in section 4 we show the results of an empirical experiment. Conclusions are left to the last section.

2 The GSH Distribution

A random variable X is said to follow a generalized secant hyperbolic distribution having parameters μ, σ, λ, with $\mu \in \mathbb{R}$, $\sigma > 0$, and $\lambda > -\pi$, in symbols $X \sim \mathrm{GSH}(\mu, \sigma, \lambda)$, if its density function, for $x \in \mathbb{R}$, is

$$f_{\mathrm{GSH}}(x; \mu, \sigma, \lambda) = \sigma^{-1} f_{\mathrm{GSH}}\left(\frac{x-\mu}{\sigma}; 0, 1, \lambda\right),$$

where

$$f_{\mathrm{GSH}}(z; 0, 1, \lambda) = \frac{c_1}{2(a + \cosh(c_2 z))}$$

and

$$\begin{array}{llll} a = \cos(\lambda) & c_2 = \sqrt{\dfrac{\pi^2 - \lambda^2}{3}} & c_1 = \dfrac{\sin(\lambda)}{\lambda} c_2 & \text{for } -\pi < \lambda < 0 \\ a = 1 & c_2 = \sqrt{\dfrac{\pi^2}{3}} & c_1 = c_2 & \text{for } \lambda = 0 \\ a = \cosh(\lambda) & c_2 = \sqrt{\dfrac{\pi^2 + \lambda^2}{3}} & c_1 = \dfrac{\sinh(\lambda)}{\lambda} c_2 & \text{for } \lambda > 0 \end{array}$$

X is symmetric around μ because $f_{\mathrm{GSH}}(x) = f_{\mathrm{GSH}}(-x)$ for any value of x. μ and σ are the mean and the standard deviation of X, while λ decides the kurtosis of the distribution. In other terms, if $X \sim \mathrm{GSH}(\mu, \sigma, \lambda)$, then

$$Z = \frac{X - \mu}{\sigma}$$

is its standardized form, with distribution $\mathrm{GSH}(0, 1, \lambda)$. For $\lambda = 0$, X has a logistic distribution.

The cumulative distribution function of Z is

$$F_{\text{GSH}}(z) = \begin{cases} \frac{1}{2} + \frac{1}{\lambda}\tan^{-1}\left[\tan\left(\frac{\lambda}{2}\right)\tanh\left(\frac{c_2}{2}z\right)\right] & \lambda \in (-\pi, 0) \\ \frac{1}{2}\left[1 + \tanh\left(\frac{c_2}{2}z\right)\right] & \lambda = 0 \\ \frac{1}{2} + \frac{1}{\lambda}\tanh^{-1}\left[\tanh\left(\frac{\lambda}{2}\right)\tanh\left(\frac{c_2}{2}z\right)\right] & \lambda > 0 \end{cases}$$

The quantile function for $0 < p < 1$ is given by

$$F^{-1}_{\text{GSH}}(p) = \begin{cases} \frac{2}{c_2}\tanh^{-1}\left[\cot\left(\frac{\lambda}{2}\right)\tan\left(\frac{\lambda}{2}(2p-1)\right)\right] & \lambda \in (-\pi, 0) \\ \frac{2}{c_2}\tanh^{-1}(2p-1) & \lambda = 0 \\ \frac{2}{c_2}\tanh^{-1}\left[\coth\left(\frac{\lambda}{2}\right)\tanh\left(\frac{\lambda}{2}(2p-1)\right)\right] & \lambda > 0 \end{cases}$$

X has kurtosis given by

$$Ku = \begin{cases} \dfrac{21\pi^2 - 9\lambda^2}{5(\pi^2 - \lambda^2)} & \lambda \in (-\pi, 0] \\ \dfrac{21\pi^2 + 9\lambda^2}{5(\pi^2 + \lambda^2)} & \lambda > 0 \end{cases}$$

Note that Ku decreases as $\lambda \to \infty$ and $1.8 < Ku < \infty$. In particular, when $\lambda = \pi$, $Ku = 3$, which is the kurtosis of the normal distribution.

2.1 Skew GSH Distribution

In literature, many methods can be found in order to transform a symmetric distribution in a skewed one. In the present context, we apply the procedure used by Fernandéz and Steel (1998) to design a skew-t distribution to the density of GSH.

A random variable X is said to follow a skew generalized secant hyperbolic distribution having parameters $\mu, \sigma, \gamma, \lambda$, with $\mu \in \mathbb{R}$, $\sigma > 0$, $\gamma > 0$, and $\lambda > -\pi$, in symbols $X \sim \text{SGSH}(\mu, \sigma, \gamma, \lambda)$, if its density function, for $x \in \mathbb{R}$, is

$$f_{\text{SGSH}}(x; \mu, \sigma, \gamma, \lambda) = \sigma^{-1} f_{\text{SGSH}}\left(\frac{x-\mu}{\sigma}; 0, 1, \gamma, \lambda\right),$$

where

$$f_{\text{SGSH}}(z; 0, 1, \gamma, \lambda) = \frac{c_1}{(\gamma + \frac{1}{\gamma})(a + \cosh(c_2 \gamma^{-\text{sign}(z)} z))}.$$

The parameters of the distribution can be interpreted as follows:

- μ is a position parameter,
- σ is a scale parameter,
- γ decides the skewness of the distribution (the density is symmetric for $\gamma = 1$, right skewed for $\gamma > 1$ and left skewed for $0 < \gamma < 1$),
- λ decides the kurtosis.

The characteristics of this distribution can be found in Palmitesta and Provasi (2004).

3 Modelling the Dependence Structure

3.1 NORTA Method

Cario and Nelson (1997) described the "NORmal To Anything" (NORTA) method for generating iid replicates of random vectors with specified marginals and covariance structure. The NORTA method starts by generating a random vector $\boldsymbol{Z}$ with a multivariate normal distribution and transforms $\boldsymbol{Z}$ to obtain a random vector $\boldsymbol{X} = (X_1, \ldots, X_p)$ with the desired marginals and covariance matrix.

Let F_i be the distribution function of X_i, for $i = 1, \ldots, p$. The NORTA method generate i.i.d. replicates of $\mathbf{X}$ by the following procedure.

1. Generate an $\mathbb{R}^p$ valued standard normal random vector $\boldsymbol{Z} = (Z_1, \ldots, Z_p)$ with mean vector $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{Z}} = (\boldsymbol{\Sigma}_{\boldsymbol{Z}}(i,j) : 1 \leq i, j \leq p)$, where $\boldsymbol{\Sigma}_{\boldsymbol{Z}}(i,i) = 1$ for $i = 1, \ldots, p$.
2. Compute the vector $\boldsymbol{X} = (X_1, \ldots, X_p)$ via
$$X_i = F^{-1}(\Phi(Z_i))$$
for $i = 1, \ldots, p$, where Φ is the distribution function of a standard normal random variable.

A vector X generated by this procedure will have the prescripted marginal distribution. To see this, note that each Z_i has a standard normal distribution, so that $\Phi(Z_i)$ is uniformly distribuited on $(0, 1)$ and $F^{-1}(\Phi(Z_i))$ will have the required marginal distribution. The covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{Z}}$ should be chosen so that it induces the required correlation structure on $\boldsymbol{X}$.

3.2 A Copula Approach

3.2.1 The Multivariate Uniform Distribution

Koehler and Symanowski (1995) introduce a multivariate distribution that can be viewed as a generalization of the Cook-Johnson family of distributions. Consider the p-dimensional random variable $\boldsymbol{U} = (U_1, \ldots, U_p)'$ with support on the unit hypercube $(0,1]^p$ and cumulative distribution function (cdf)

$$F(u_1, \ldots, u_p) = \prod_{i=1}^{p} u_i \prod_{j=i+1}^{p} K_{ij}^{-\alpha_{ij}},$$

where

$$K_{ij} = u_i^{1/\alpha_{i+}} + u_j^{1/\alpha_{j+}} - u_i^{1/\alpha_{i+}} u_j^{1/\alpha_{j+}},$$

with $\alpha_{ij} = \alpha_{ji} \geq 0$ for all (i,j) and $\alpha_{i+} = \alpha_{i1} + \cdots + \alpha_{ip} > 0$ for all $i = 1, \ldots, p$.

Deriving the cdf with respect to $u_1, \ldots, u_p$, we obtain the probability density function (pdf) of $\boldsymbol{U}$:

$$\begin{aligned} f(u_1, \ldots, u_p) = & \prod_{i=1}^{p} \left(D_i \prod_{j=i+1}^{p} K^{-\alpha_{ij}} \right) \\ & \cdot \left[1 + \sum_{i=1}^{p} \sum_{j=i+1}^{p} \left(\frac{\alpha_{ij}}{\alpha_{i+}\alpha_{j+}} D_i^{-1} D_j^{-1} K_{ij}^{-2} u_i^{1/\alpha_{i+}} u_j^{1/\alpha_{j+}} \right) \right], \end{aligned}$$

where

$$D_i = \alpha_{i+}^{-1} \left[\alpha_{ii} + \sum_{k \neq i}^{p} (\alpha_{ik} u_k^{1/\alpha_{k+}} K_{ik}^{-1}) \right] \quad \text{and} \quad K_{ij} = K_{ji}.$$

It is possible to obtain a scheme to generate $\boldsymbol{U}$ using gamma distributions. Let $Y_1, \ldots, Y_p$ be i.i.d. $Gamma(1,1)$ and, independently, $G_{11}, G_{12}, \ldots, G_{pp}$ be $Gamma(\alpha_{ij}, 1)$ with $G_{i+} = \sum_{j=1}^{p} G_{ij}$. Then, the joint pdf of

$$U_i = \left(1 + \frac{Y_i}{G_{i+}} \right)^{-\alpha_{i+}},$$

for $i = 1, \ldots, p$, has the cdf of $\boldsymbol{U}$. $\boldsymbol{U}$ is positively associated. Variations in the standard form that also take into account negative association can be obtained by applying the transformation $V_i = 1 - U_i$ to some, but not all, variables.

3.2.2 The Copula Function

Recall that a copula C is a p-dimensional distribution function defined on $[0,1]^p$ with uniform marginal distributions. From the Sklar's theorem we have that any cdf F with marginals $F_1, \dots, F_p$ can be written as

$$F(x_1, \dots, x_p) = C(F_1(x_1), \dots, F_p(x_p))$$

for any copula function C which is uniquely determined on $[0,1]^p$ for F distributions with marginals absolutely continuous. Vice versa, any copula function C can be used to join any set of cdf $F_1, \dots, F_p$ to build a multivariate cdf F with marginals $F_1, \dots, F_p$. The copula function C of their joint distribution can be obtained computing

$$C(u_1, \dots, u_p) = F(F_1^{-1}(u_1), \dots, F_p^{-1}(u_p))$$

where the F_i^{-1} are the quantile function of the marginals.

A KS copula is given by

$$F(x_1, \dots, x_p) = \prod_{i=1}^{p} F_i(x_i) \prod_{j=i+1}^{p} K_{ij}^{-\alpha_{ij}}$$

The pdf is

$$\begin{aligned} f(x_1, \dots, x_p) = &\prod_{i=1}^{p} \left(f_i(x_i) D_i \prod_{j=i+1}^{p} K^{-\alpha_{ij}} \right) \\ &\cdot \left[1 + \sum_{i=1}^{p} \sum_{j=i+1}^{p} \left(\frac{\alpha_{ij}}{\alpha_{i+}\alpha_{j+}} D_i^{-1} D_j^{-1} K_{ij}^{-2} F_i(x_i)^{1/\alpha_{i+}} F_j(x_j)^{1/\alpha_{j+}} \right) \right], \end{aligned}$$

where

$$\begin{aligned} D_i &= \alpha_{i+}^{-1} \left[\alpha_{ii} + \sum_{k \neq i}^{p} (\alpha_{ik} F_k(x_k)^{1/\alpha_{k+}} C_{ik}^{-1}) \right], \\ K_{ij} &= F_i(x_i)^{1/\alpha_{i+}} + F_j(x_j)^{1/\alpha_{j+}} - F_i(x_i)^{1/\alpha_{i+}} F_j(x_j)^{1/\alpha_{j+}}. \end{aligned}$$

It is immediately verified that, to simulate the joint distribution of $X_1, \dots, X_p$ with the Monte Carlo method, it is sufficient to generate some variates from $\boldsymbol{U}$ and then apply the inverse transformation to each marginal.

3.2.3 Measures of association

The conditional means of C are not linear functions of the values of the conditioning variables. Consequently, it is more reasonable to measure dependence between variables using concordance coefficients as the Kendall's tau or the Spearman's rho than using the linear correlation coefficient. The level of association of (X_1, X_2) depends on the level of the parameters α_{ij}.

The cdf of (X_1, X_2) approaches the upper Fréchet bound as $\alpha_{12} \to 0$ when both $\alpha_{11}/\alpha_{1+} \to 0$ and $\alpha_{22}/\alpha_{2+} \to 0$ provided that α_{11} and α_{22} decrease to zero faster than α_{12}. It approximates $(X_1 X_2)$, paired independence, when

1. either $\alpha_{12} \to 0$ and both α_{1+} and α_{2+} are finitely nonzero or
2. both $\alpha_{1+} \to \infty$ and $\alpha_{2+} \to \infty$.

In order to stress these properties, we quote the values of Kendall's τ and Spearman's ρ coefficients of the KS copula function for some values of α_{12} when $\alpha_{11} = \alpha_{22} = 0$ and X_1 and X_2 are uniform on zero and 1. It must be emphasized that τ and ρ depend only on the copula function C.

Association parameter	Kendall's coefficient	Spearman's coefficient
0.1	0.8333	0.9581
0.5	0.5000	0.6822
1.0	0.3333	0.4784
1.5	0.2500	0.3654
2.0	0.2000	0.2949
2.5	0.1667	0.2470
3.0	0.1429	0.2124
4.0	0.1111	0.1657
5.0	0.0909	0.1358
10.0	0.0476	0.0714
∞	0.0000	0.0000

3.3 Tail dependence coefficients

The tail-dependence coefficients provide asymptotic measures of the dependence in the tails of the bivariate distribution for (X_1, X_2). The upper tail-dependence coefficient for X_1 and X_2 is

$$\lim_{q \to 1} P(X_2 > F_2^{-1} | X_1 > X_1 > F_1^{-1}) = \lambda_{\mathcal{U}},$$

provided that a limit $\lambda_{\mathcal{U}} \in [0, 1]$ exists. The lower tail dependence coefficient is

$$\lim_{q \to 0} P(X_2 > F_2^{-1} | X_1 > X_1 > F_1^{-1}) = \lambda_{\mathcal{L}},$$

Association parameter	$\lambda_{\mathcal{L}}$
0.1	0.9330
0.5	0.7071
1.0	0.5000
1.5	0.3536
2.0	0.2500
2.5	0.1768
3.0	0.1250
4.0	0.0884
5.0	0.0625
10.0	0.0313

Table 1: Coefficients of the lower tail dependence for some values of α_{12} when $\alpha_{11} = \alpha_{22} = 0$.

provided that a limit $\lambda_{\mathcal{L}} \in [0, 1]$ exists.

Also these measures depend only on the copula function C of (X_1, X_2) and can be obtained easily by means of the expressions used by Joe (1997), which are given by

$$\lambda_{\mathcal{U}} = \lim_{q \to 1^-} \frac{\bar{C}(q,q)}{1-q}, \qquad \lambda_{\mathcal{L}} = \lim_{q \to 0^+} \frac{C(q,q)}{q},$$

where $\bar{C}(u,u) = 1 - 2u + C(u,u)$ is known as the survivor function of the copula.

On the basis of this expression, it is immediate to verify that, for the KS copula function $\lambda_{\mathcal{U}} = 0$, while $\lambda_{\mathcal{L}} \to 1$ when the cdf of (X_1, X_2) approaches the upper Fréchet bound and $\lambda_{\mathcal{L}} \to 0$ when X_1 and X_2 approach the indipendence. Coefficients of the lower tail dependence for the KS copula are given in Table 2 for some values of α_{12} when $\alpha_{11} = \alpha_{22} = 0$.

3.3.1 Minimum Distance

We want to obtain the parameters of the KS copula which permit to generate, as with NORTA, multivariate random samples with known marginals and given correlation matrix. With this aim, we can minimize the distance (MD) between the association matrix $\boldsymbol{P}$ obtained with the Kendall's tau or Spearman's rho coefficients computed for each bivariate marginal of the copula and the corresponding copula-implied association matrix $\boldsymbol{R}(\boldsymbol{\alpha})$.

The MD minimizes respect to the $p(p+1)/2$ elements of $\boldsymbol{\alpha}$

$$(\boldsymbol{P}^{\dagger} - h(\boldsymbol{\alpha}))'\boldsymbol{W}^{-1}(\boldsymbol{P}^{\dagger} - h(\boldsymbol{\alpha}))$$

where $\boldsymbol{P}^{\dagger}$ and $h(\boldsymbol{\alpha})$ indicates vectors $p(p-1)/2 \times 1$ which contain, respectively, the only elements of the given correlation measures and theoretical association matrices and $\boldsymbol{W}^{-1}$ is a weight matrix.

4 An Application

In this section we show the results of an experiment in order to verify if the approach is correct, both in terms of comparison between the skew Student-t and the skew GSH distributions and in terms of accuracy of the copula approach.

4.1 The data

The raw data used in this paper are weekly prices of four market indices:

- the S&P 500 Composite index (S&PCOMP),
- the NASDAQ Composite index (NASCOMP),
- the NIKKEI 500 index (JAPA500) and
- the MSCI AC World index (MSACWFL).

The observations were obtained by Datastream for the period 1/1/1988 to 12/31/2003. We compute returns as the first differences of the natural logarithms of each series, $r_t = \ln I_t - \ln I_{t-1}$, where I_t indicates the price at time t. This gives a sample of $T = 854$ returns.

If we express the dependence structure of the four indices with the Kendall's tau coefficient, we obtain the following matrix:

$$\boldsymbol{P}_{\tau} = \begin{bmatrix} 1.0000 & 0.6347 & 0.2499 & 0.6807 \\ 0.6347 & 1.0000 & 0.2421 & 0.5517 \\ 0.2499 & 0.2421 & 1.0000 & 0.4578 \\ 0.6807 & 0.5517 & 0.4578 & 1.0000 \end{bmatrix}$$

while if we use the Spearman's rho coefficient we obtain:

$$\boldsymbol{P}_{S} = \begin{bmatrix} 1.0000 & 0.8245 & 0.3642 & 0.8577 \\ 0.8245 & 1.0000 & 0.3524 & 0.7423 \\ 0.3642 & 0.3524 & 1.0000 & 0.6337 \\ 0.8577 & 0.7423 & 0.6337 & 1.0000 \end{bmatrix}.$$

4.2 Fitting the marginals

We have considered as marginals of the KS copula the skew Student-t and the skew generalized secant hyperbolic distributions. The two following tables present the results of the maximum likelihood estimation of the parameters of the two distributions for the four series of returns. In the table we also present the values of the averaged loglikelihood function at the maximum (shown in italics). On the basis of these values, note that the two distributions show a good fitting to the data.

ML Estimation of the Skew Student-t Distribution

	S&PCOMP	NASCOMP	JAPA500	MSACWFL
Location	0.0062	0.0096	0.0013	0.0055
Scale	0.0222	0.0357	0.0300	0.0196
Skewness	0.8664	0.8506	0.9593	0.8593
Kurtosis	5.4329	3.4153	4.5528	4.8423
LogLik	*2.4179*	*2.0313*	*2.1460*	*2.5503*

ML Estimation of the Skew Generalized Secant Hyperbolic Distribution

	S&PCOMP	NASCOMP	JAPA500	MSACWFL
Location	0.0061	0.0010	0.0011	0.0055
Scale	0.0220	0.0334	0.0294	0.0193
Skewness	0.8738	0.8440	0.9645	0.8599
Kurtosis	-1.4014	-2.1752	-1.7483	-1.6308
LogLik	*2.4160*	*2.0360*	*2.1477*	*2.5497*

4.3 Fitting the KS copula

The estimated parameters of the KS copula function for the four series of returns are shown in the following table. The values refer to the minimum distance estimates with association matrices based on the Kendall's tau (MDE$_\tau$) and the Spearman's rho (MDE$_\rho$). The last row of the table shows the distance value between the empirical and theoretical association matrices.

MD Estimation of the KS Copula Function
($\boldsymbol{W} = \boldsymbol{I}$)

Parameters	MDE_τ	MDE_ρ
α_{11}	0.0072	0.0013
α_{12}	0.0640	0.0608
α_{13}	0.0208	0.0181
α_{14}	0.0574	0.0478
α_{22}	0.0181	0.0157
α_{23}	0.0239	0.0227
α_{24}	0.0437	0.0523
α_{33}	0.0517	0.0457
α_{34}	0.0684	0.0712
α_{44}	0.0110	0.0094
Distances	0.0000	0.0000

The structure of $\boldsymbol{\alpha}$ obtained with the minimum distance method proves quite similar (it is equivalent at the second decimal digit) depending on measuring the association between variables using the Kendall's tau or the Spearman's rho;

The distance of $\boldsymbol{P}_\tau$ and $\boldsymbol{P}_\rho$ from $\tilde{\boldsymbol{\alpha}}$ is virtually zero and it denotes the skill of the KS copula function to model complex dependence structures among subsets of marginals.

5 Conclusions

In this paper we have proposed to model the dependence of multiple time series returns with a multivariate extension of the generalized secant hyperbolic distribution (GSH) using the NORTA (NORmal-to-Anything) approach and the Koehler and Symanowski copula function. Fitting the distribution to the returns of four market indices, the S&P 500 Composite index (S&PCOMP), the NASDAQ Composite index (NASCOMP), the NIKKEI 500 index (JAPA500) and the MSCI AC World index (MSACWFL), with the skew Student-t and the skew generalized secant hyperbolic, we find that this copula function succeeds in properly interpreting the dependence structure of data, apart from the marginals. The two methodologies permit to generate random vectors with marginals distributed as a GSH distribution and given correlation matrix, which can be used to measure the risk of a portfolio using the Monte Carlo method. It is in progress the implementation in Mathematica of the NORTA method and of the KS copula to measure the risk of a portfolio with the Monte Carlo method.

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