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**QUADERNI DEL DIPARTIMENTO
DI ECONOMIA POLITICA E STATISTICA**

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Investment behaviour and “bull & bear” dynamics:
Modelling real and stock market interactions

n. 800 – Gennaio 2019



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January, 2019

Abstract

We develop a simple behavioural macrodynamic model in continuous-time with the purpose of investigating the interaction of the real economy and the financial markets. Building on Westerhoff (2012), we improve the specification of aggregate demand by distinguishing between consumption and investment expenditure and assuming that the latter is determined by the flexible accelerator principle. We remove the *ad-hoc* nonlinearity in the fundamentalist behavioural rule and allow the composition of the population between chartists and fundamentalists to be endogenously determined. The resulting nonlinear dynamic systems are shown to generate various dynamic regimes, among which the coexistence of periodic attractors with interesting economic implications.

Keywords: Real-financial interaction, multiplier, nonlinear accelerator, heterogeneous speculators, complex dynamics.

JEL Classification: E12, E24, E32, E44.

1 Introduction

Over the last thirty years, the boundedly rational heterogeneous agent literature that started with Day and Huang (1990) and Chiarella (1992) has successfully shown that trading activity of heterogeneous interacting speculators accounts for a large part of the dynamics of financial markets. Ten years or so after the financial crisis, there has been a renewed interest in studying the role of financial actors and institutions in amplifying fluctuations not only in the financial side of the economy but also in the real sector.

Different sources of behaviour heterogeneity have been identified such as trend extrapolation, noise trading, overconfidence, overreaction, optimistic or pessimistic traders, upward- or downward-biased traders, and so on (for a review on some recent developments, see Lux, 2009; Hommes, 2013; Dieci and He, 2018). On the other hand, the real side of the economy has been traditionally introduced in a disequilibrium framework in which output adjusts to excess demand. Complex dynamics, close to well known macroeconomic and financial stylised

*A preliminary version of this paper was presented at the 9th Conference on Nonlinear Economic Dynamics, Tokyo, Japan. We thank all participants for their comments and suggestions which allowed us to greatly improve the quality of the paper. All remaining errors and shortcomings are entirely of our responsibility.

facts, are obtained as a result of the interplay of heterogeneous agents with financial and real variables.

Westerhoff (2012) is one of the first contributions to develop a model in discrete time where the goods market is connected with the stock market. In his model, nonlinear interactions between aggregate demand, chartists, and fundamentalists result in complex “bull and bear” dynamics. A similar exercise was performed by Naimzada and Pireddu (2014) who introduced an extra nonlinearity in aggregate demand and assumed that the speed of adjustment in the stock market limits to infinite. Fiscal policy considerations were brought to attention by Cavalli et al. (2017) in an extension of previous models that adopts the nonlinear accelerator as a theory of investment.

The aforementioned contributions divided agents in the financial markets between chartists and fundamentalist but did not allow for endogenous changes in their composition. Endogenous switches between alternative heuristics were formalised by Naimzada and Pireddu (2015) and Cavalli et al. (2018). A basic assumption they make is that all agents populating the stock market are fundamentalist but unable to observe the underlying fundamental so that their beliefs are biased either to optimism or pessimism.

Other models, in general formulated in continuous time, have preserved the Keynesian adjustment of aggregate demand and the chartists/fundamentalist approach. For example, making use of Lux (1995) formalisation of herd behaviour in speculative markets, Franke (2012) developed a macrodynamic model with a preliminary financial distress variable. His set up has been extended allowing for a link between real and financial sides of the economy through Tobin’s q theory of investment (e.g. Franke and Ghonghadze, 2014; Flaschel et al., 2018). A different strand of the literature, on the other hand, has preferred the Brock and Hommes (1997) approach for modelling changes in the share of agents that use different heuristics, though always maintaining the basic view on expectations under bounded rationality (for a discrete time example, see Proaño, 2011, 2013).

Other financial actors such as banks have been brought to attention by Chiarella et al. (2015). In fact, there exists a vast literature on the crucial role of credit as a factor leading both to the instability of the system and to a strengthening of real-financial linkages in the economy. This tradition *a la* Minsky has recently incorporated different levels of behaviour heterogeneity, enlightening different aspects of the interdependency between firm’s external financial structure and the state of the economy (see, for example, Lojak, 2018).

It must be noted, however, that a full analysis of the interaction of the real economy and the financial market is still missing and we are far from a consensus.¹ In an attempt to contribute to fill this gap, in this paper we develop a simple behavioural macrodynamic model formulated in continuous-time with the purpose of investigating possible channels of interaction between the real economy and the stock market. Our choice of a continuous time approach has a twofold motivation. Although individual economic decisions are generally made in discrete time intervals, it is difficult to believe that they are coordinated in such a way as to be perfectly synchronized (Gandolfo, 2009, pp. 568-573). Moreover, a specification in continuous time is particularly useful for the formulation of dynamic adjustment processes based on excess demand and it is interesting to note that the first contributions on the topic explicitly advocated the use of continuous time models (see, for example, Goodwin, 1948).

Our starting point is the model by Westerhoff (2012) who was able to generate complex

¹Westerhoff and Dieci (2006), Chiarella et al. (2007) and Schmitt and Westerhoff (2014), leaving aside any consideration on the real side of the economy, investigate the case of switching between different financial markets. An intermediary case in which stock and housing markets interact can be found in Dieci et al. (2018).

dynamics of both national income and stock price providing an explanation of the irregularity of economic time-series. This is not surprising, however, given that the dynamic system of his model turns out to be a first-order 2×2 system of difference equations with a crucial – somehow *ad-hoc* – nonlinearity. In what follows, we rewrite the model in continuous-time and show that, as expected, the dynamics it is able to generate are much simpler than in the discrete-time case and even non-persistent for economically meaningful values of the parameters.

We proceed by considering a more general version of the model which we obtain by enriching the specification of its real side in various directions. First, following a suggestion by Westerhoff himself, we improve the specification of aggregate demand by distinguishing between consumption and investment expenditure and assuming that the latter is determined by the flexible accelerator principle. Second, we remove the *ad-hoc* nonlinearity on the fundamentalist behavioural rule. Finally, we allow for endogenous switches between chartist and fundamentalist behaviour in line with Lux’s (1995) mutual contagion mechanism in speculative markets. The resulting nonlinear, first order 4×4 system is shown to generate various dynamic regimes with interesting economic implications.

The reminder of the paper is organized as follows. Section 2 briefly describes and discusses Westerhoff’s model of the interactions between the real economy and the stock market. Section 3 is concerned with the study of a continuous-time version of the same model. Section 4 modifies the model by introducing the flexible accelerator. We continue, in section 5, by removing the *ad-hoc* nonlinearity in the fundamentalist behavioural equation and allowing the share of chartists and fundamentalist to change endogenously. Section 6 concludes. Details about the lengthy computations are contained in the Appendix at the end of the paper.

2 Interacting real economy and stock market in a macro-dynamic model

In order to study the dynamic interaction of the real economy and the financial market, Westerhoff (2012) considers a simple closed economy model with three basic ingredients: an adjustment mechanism of production to excess demand of goods, an adjustment mechanism of the stock price to excess demand of stocks, and a specification of goods and stocks demand in terms of both national income and stock price. More specifically, the equations of his model are the following.

With regard to the adjustment in the goods market (see, e.g., Blanchard, 1981), it is assumed that:

$$Y_{t+1} - Y_t = \alpha (Z_t - Y_t), \alpha > 0 \quad (1)$$

where Y is production, Z aggregate demand and α the speed of adjustment. To keep matters as simple as possible, and following the original presentation, we set $\alpha = 1$.

As far as the adjustment mechanism in the stock market is concerned, it is assumed that the stock price (P) is decided by a market maker who adjusts it to excess demand in the market. In its turn, the latter is given by the sum of speculative demand (by two different types of speculators, ‘chartists’ and ‘fundamentalists’)² and non speculative demand minus the supply of stocks, so that:

$$P_{t+1} - P_t = \beta (D_t^C + D_t^F + D_t^{NS} - N), \beta > 0 \quad (2)$$

²The inspiring reference for this characterisation of the stock market is the article by Day and Huang (1990).

where D^C , D^F are the stock demand by chartists and fundamentalists, respectively, D^{NS} stands for non speculative demand, and N is the supply of stocks. Since β is a scaling parameter, Westerhoff set it equal to one.

Assuming, for simplicity, that non speculative demand of stock fully absorbs the supply, this equation implies that stock price variations are entirely determined by speculative demand:

$$P_{t+1} - P_t = D_t^C + D_t^F \quad (3)$$

i.e., the market maker increases the stock price when the speculative demand is positive and vice-versa.

In this model, the first link between the real economy and the stock market is obtained thanks to the assumption that Z depends both on income of households and firms and their financial situation, as measured by the stock price:

$$Z_t = C_t + I_t + G = a + bY_t + cP_t, \quad a > 0, \quad 0 < b, c < 1 \quad (4)$$

where a is the sum of all autonomous expenditure, and b and c are the marginal propensity to spend from current income and stock market wealth, respectively.³

The second link between the real economy and the stock market is created by Westerhoff by assuming that speculative demand of stocks by both chartists and fundamentalists depend on the gap between the stock market fundamental value (F) and the current stock price, although with an opposite sign.⁴ Chartists are trend followers and expect that “bull and bear” markets will persist so that their stock demand is positive when the stock price is above its (perceived) fundamental value and vice-versa:

$$D_t^C = e(P_t - F_t), \quad e > 0 \quad (5)$$

Fundamentalists, on the contrary, expect that the stock price will return to its fundamental value so that they increase their stock demand when F is greater than P and vice-versa:

$$D_t^F = f(F_t - P_t)^3, \quad f > 0 \quad (6)$$

Finally, F is assumed to be perceived by both types of speculators as being proportional to national income:

$$F_t = dY_t, \quad d > 0 \quad (7)$$

Beside what already stressed above, namely that Eq. (7) implies the important simplification that chartists and fundamentalists believe in the same fundamental value (see, on this point, the discussion in [Westerhoff, 2012](#)), two other aspects of this formalization are worth remarking. First, it assumes that speculators cannot switch between strategies. Second, the nonlinearity in (6) appears to be somehow *ad hoc* and should be discussed and modified taking account of what is done in other contributions in the field. The motivation for it which is given in the literature is that it is reasonable to expect that the aggressiveness of fundamentalists increases with the mispricing they perceive (see, for example, [Tramontana et al, 2009](#)). We initially maintain both assumptions in order to present Westerhoff’s results, which are

³It is worth stressing that in this very simplified formalisation of aggregate demand neither a proper consumption function nor a proper investment function are specified. We will come back on this point later in the paper.

⁴The references in this regard are, among others, Day and Huang (1990), Hommes et al. (2005), Boswijk et al. (2007) and Tramontana et al. (2009).

the starting point for our own elaboration of an alternative model of the real economy-stock market interaction.

Westerhoff's results (2012, p. 18), which refer to the complete model where the dynamics of Y and P are jointly determined by the following dynamic system

$$Y_{t+1} = a + bY_t + cP_t \quad (8)$$

$$P_{t+1} = P_t + e(P_t - dY_t) + f(dY_t - P_t)^3 \quad (9)$$

can be shortly presented as follows.

The system admits three equilibrium points:

$$E_1 = (\bar{Y}_1, \bar{P}_1) = \left(\frac{a}{1-b-cd}, \frac{ad}{1-b-cd} \right) \quad (10)$$

$$E_{2,3} = (\bar{Y}_{2,3}, \bar{P}_{2,3}) = \left(\bar{Y}_1 \pm \frac{c}{1-b-cd} \sqrt{\frac{e}{f}}, \bar{P}_1 \pm \frac{1-b}{1-b-cd} \sqrt{\frac{e}{f}} \right) \quad (11)$$

where, to assure that they are all economically meaningful, we must impose that $1-b-cd > 0$, $a > c\sqrt{e/f}$ and $ad > (1-b)\sqrt{e/f}$.

Local stability analysis shows that the equilibrium point E_1 is always unstable, whereas the other two are locally stable for $e < (1+b)/(1+b+cd)$ (ibid.: pp. 7 and 18-19). These analytical results are then corroborated by numerical simulations (ibid.: 9-16) which are performed using the following values for the parameters:

$$a = 3, b = 0.95, c = 0.02, d = 1, e = 1.63, f = 0.3 \quad (12)$$

such that $e = 1.63 > (1+b)/(1+b+cd) \approx 0.9898$, so that not only the internal equilibrium point $E_1 = (100, 100)$, but also the two external ones $E_2 = (98.446, 96.115)$ and $E_3 = (101.55, 103.88)$ are unstable. Numerical simulation shows that in this case the model admits a chaotic attractor. In the opposite case, in which, leaving all the other parameters unchanged, we take a value of $e < 0.9898$, the two external equilibrium points become locally stable.

To conclude, the dynamics generated by the complete model are very rich and confirm the point first made by Day and Huang (1990), namely that the nonlinear interaction between the real economy and the stock market may give rise to complex "bull and bear" market dynamics. In our opinion, however, in order to assess the economic relevance of this result, it is necessary to discuss the role and plausibility of two crucial assumptions which are made by Westerhoff, namely, the formulation of the basic dynamic equations (1) and (3) in discrete time and the specification of fundamentalists' behaviour in Eq. (6). While we postpone the discussion of the second assumption until Sect. 5, in the next section we investigate the role played by the first one.

3 A formulation of the model in continuous time

Our purpose in this section is to introduce, analyse and discuss a continuous-time version of Westerhoff's model. In order to do that, we first notice that Eq. (1) in continuous time becomes:⁵

$$\dot{Y}(t) = Z(t) - Y(t) \quad (13)$$

⁵As is well known (see, e.g. Allen 1967), in the special case in which the adjustment mechanism involves a simple exponential lag, Eq. (13) implies that production adjusts with a continuously distributed lag to aggregate demand.

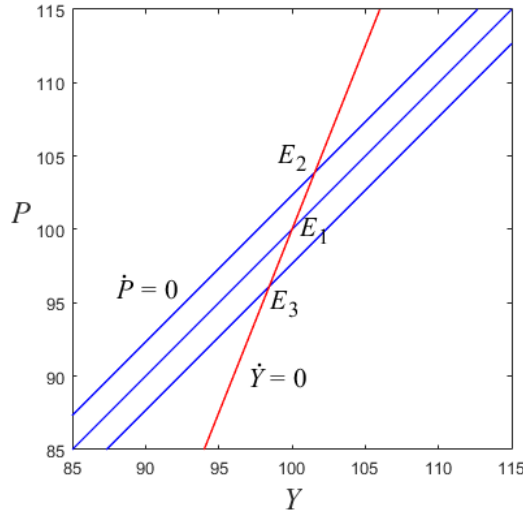


Figure 1: The two isoclines of the dynamical system with the parameter values given in (12).

Maintaining the specification of aggregate demand given in (4), we obtain the first equation of the dynamic system of our version of the model:

$$\dot{Y}(t) = a - (1 - b)Y(t) + cP(t) \quad (14)$$

Modifying in an analogous way Eq. (9), we obtain:

$$\dot{P}(t) = e[P(t) - dY(t)] + f[dY(t) - P(t)]^3 \quad (15)$$

Eqs. (14)-(15) form a continuous-time dynamic system in the two variables $Y(t)$ and $P(t)$, such that its equilibrium points (\bar{Y}, \bar{P}) are determined by the intersections of the two isoclines $\dot{Y}(t) = 0$ and $\dot{P}(t) = 0$ (see Fig. 1), i.e., by:

$$a - (1 - b)\bar{Y} + c\bar{P} = 0 \quad (16)$$

$$(\bar{P} - d\bar{Y}) \left[e - f(d\bar{Y} - \bar{P})^2 \right] = 0 \quad (17)$$

from which we obtain the same internal and external equilibrium points E_1 and $E_{2,3}$ as in Westerhoff's discrete-time version of the model.

In order to study the dynamics generated by the nonlinear system (14)-(15), we proceed by analysing the local stability of each of its equilibrium points.

3.1 Local stability analysis

The Jacobian matrix of the dynamic system (14)-(15) is given by:

$$\mathbf{J} = \begin{bmatrix} b - 1 & c \\ -ed + 3fd(dY - P)^2 & e - 3f(dY - P)^2 \end{bmatrix}$$

Thus, at the internal equilibrium point E_1

$$\mathbf{J}_1 = \begin{bmatrix} b - 1 & c \\ -ed & e \end{bmatrix}$$

such that:

$$\text{tr} \mathbf{J}_1 = b - 1 + e \quad (18)$$

$$\det \mathbf{J}_1 = -e(1 - b - cd) \quad (19)$$

and

$$\Delta_1 = (\text{tr} \mathbf{J}_1)^2 - 4 \det \mathbf{J}_1 \quad (20)$$

whereas, at $E_{2,3}$

$$\mathbf{J}_{2,3} = \begin{bmatrix} b-1 & c \\ 2ed & -2e \end{bmatrix}$$

such that

$$\text{tr} \mathbf{J}_{2,3} = b - 1 - 2e \quad (21)$$

$$\det \mathbf{J}_{2,3} = 2e(1 - b - cd) \quad (22)$$

and

$$\Delta_{2,3} = (\text{tr} \mathbf{J}_{2,3})^2 - 4 \det \mathbf{J}_{2,3} \quad (23)$$

Since, by assumption $0 < b < 1$, $1 - b - cd > 0$ and $e > 0$, we can conclude that:

$$\text{tr} \mathbf{J}_1 \geq 0, \quad \det \mathbf{J}_1 < 0, \quad \Delta_1 > 0$$

and

$$\text{tr} \mathbf{J}_{2,3} < 0, \quad \det \mathbf{J}_{2,3} > 0, \quad \Delta_{2,3} \geq 0$$

The implications of this analysis are summarised in the following Proposition.

Proposition 1 *The internal equilibrium point E_1 is a saddle point whereas the two external equilibrium points, E_2 and E_3 , are either both locally stable nodes when $\Delta_{2,3} \geq 0$ or locally stable foci when $\Delta_{2,3} < 0$.*

All trajectories will therefore converge to one or the other of the two external equilibrium points, with the exception of those departing from initial conditions on the stable manifold of the saddle point. With the set of parameter values (12), calculations show that the two external equilibrium points are locally stable nodes. The resulting dynamics are shown in Fig. 2, where the stable manifold of the saddle point E_1 is drawn in green and the unstable manifold in brown. Fourteen trajectories are also shown, seven of them converging to E_2 and the other seven to E_3 .⁶

The main conclusion of our analysis is that the possibility of persistent dynamics in this simple continuous-time version of Westerhoff's model must be excluded. In order to obtain a richer dynamics, an extension of the model is required.

⁶The figure clearly shows that the stable manifold of the saddle point E_1 separates the basins of attraction of the two external equilibrium points. In this figure, as in all other which follow, the attractive equilibrium points are marked by full dots (\bullet), the repulsive equilibrium points by open dots (\circ) and saddle points by squares (\square).

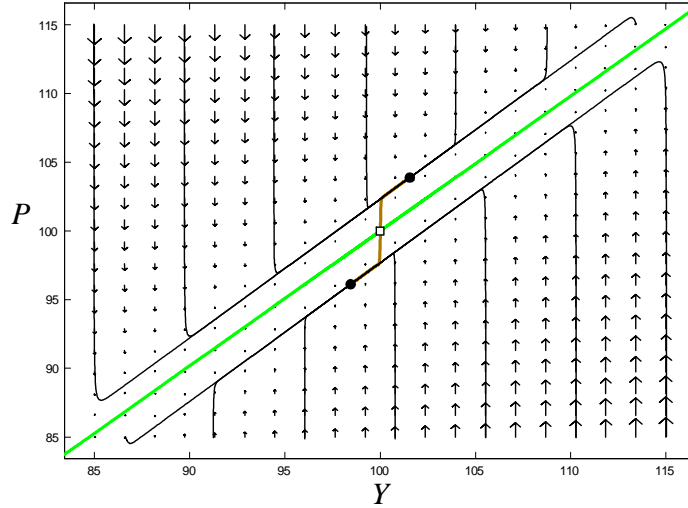


Figure 2: The dynamics of national income and stock price.

4 “Disaggregating” aggregate demand

To obtain a more satisfactory version of the continuous-time model – capable of generating persistent dynamics of the variables – we now make an important step forward, consisting in specifying in details a behavioural equation for each of the components of aggregate demand, namely, for investment in fixed capital and for consumption. In particular, our purpose is to do so by taking account of two basic dynamic facts, namely, that it takes time for investment in fixed capital to adjust to its desired level, which we take to be determined by the acceleration principle, and that a satisfactory representation of the latter require a nonlinear or piecewise-linear representation (see [Goodwin 1948, 1951](#), and the discussion in [Sordi, 2006](#), and [Sordi and Vercelli, 2006](#)).

First of all, we assume that consumption is determined by the following function:

$$C(t) = C_0 + bY(t) + cP(t), \quad 0 < b, c < 1 \quad (24)$$

which, in agreement with the empirical evidence and discussion given in a number of studies such as [Poterba \(2000\)](#), [Ludwig and Slok \(2002\)](#), and more recently [McMillan \(2013\)](#), states that consumption expenditure reacts positively to both national income and stock price.

Second, following a suggestion by Westerhoff himself ([2012, p. 16](#)), we assume that investment is determined by the acceleration principle. To keep the analysis as simple as possible, we disregard any consideration about depreciation and therefore there is no distinction between net and gross investment. However, to take account of the time-lags which are inevitably involved in the investment process, we introduce it in its “flexible form” ([Goodwin, 1948](#)). We do so by noticing that, although it is reasonable to assume that investment is a function of the rates of change of national income over a period of time, I is always lagging over such changes. In the extreme case, we can take investment as a function of the whole spectrum of past values of $v\dot{Y}$, where $v > 0$ is the accelerator coefficient, according to a so-called *continuously distributed lag* such that (cfr. [Allen, 1967](#)):

$$I(t) = \int_0^\infty w(\tau) \left[v\dot{Y}(t - \tau) \right] d\tau \quad (25)$$

where $w(\tau) \geq 0$ is the weight in fixing current investment which is attached to national income of τ periods ago and $\int_0^\infty w(\tau) d\tau = 1$.

In the special case in which the weights are of the exponential form:

$$w(\tau) = \gamma e^{-\gamma\tau} \quad (26)$$

we obtain from Eq. (25) a formulation for investment behaviour which has a straightforward economic interpretation. Indeed, inserting Eq. (26) into Eq. (25), we obtain:

$$I(t) = \gamma \int_0^\infty e^{-\gamma\tau} [v\dot{Y}(t-\tau)] d\tau$$

from which, through the change of variable $x = t - \tau$:

$$\frac{1}{\gamma} e^{\gamma t} I(t) = \int_{-\infty}^t e^{\gamma x} [v\dot{Y}(x)] dx$$

Then, differentiating both sides with respect to time and simplifying, we obtain the so-called *flexible accelerator* (Goodwin, 1948) :

$$\dot{I}(t) = \gamma [v\dot{Y}(t) - I(t)] \quad (27)$$

where, for simplicity, we take $\gamma = 1$. The fact that we have obtained such a formulation from the consideration of distributed lags in the relation between investment and changes in national income makes it more appropriate than the simple accelerator in a *macrodynamic* model.

Taking account of Eq. (27), the dynamics of the model is now generated by the following three-dimensional dynamic system:

$$\dot{Y}(t) = (b-1)Y(t) + I(t) + cP(t) + a \quad (28)$$

$$\dot{I}(t) = v(b-1)Y(t) + (v-1)I(t) + vcP(t) + va \quad (29)$$

$$\dot{P}(t) = e[P(t) - dY(t)] + f[dY(t) - P(t)]^3 \quad (30)$$

with equilibrium points analytically determined by solving:

$$(b-1)\bar{Y} + \bar{I} + c\bar{P} + a = 0 \quad (31)$$

$$v(b-1)\bar{Y} + (v-1)\bar{I} + vc\bar{P} + va = 0 \quad (32)$$

$$(\bar{P} - d\bar{Y}) [e - f(d\bar{Y} - \bar{P})^2] = 0 \quad (33)$$

From Eq. (32), taking account of Eq. (31), it follows that the only equilibrium value of investment is $\bar{I} = 0$. We can therefore conclude that the three equilibrium points of the present version of the model are given by:

$$E_1^* = (\bar{Y}_1, 0, \bar{P}_1)$$

and

$$E_{2,3}^* = (\bar{Y}_{2,3}, 0, \bar{P}_{2,3})$$

where \bar{Y}_i and \bar{P}_i , $i = 1, 2, 3$, are defined as in Eqs. (10) and (11).

4.1 Local stability analysis

In this section, we derive and discuss the local asymptotic stability of each of the three equilibrium points. The 3×3 Jacobian matrix of the dynamic system (28)-(30) is:

$$\mathbf{J} = \begin{bmatrix} b-1 & 1 & c \\ v(b-1) & v-1 & vc \\ -ed + 3df(d\bar{Y} - \bar{P})^2 & 0 & e - 3f(d\bar{Y} - \bar{P})^2 \end{bmatrix}$$

from which we obtain the following characteristic equation:

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$$

where

$$b_1 = -\text{tr}\mathbf{J}$$

$$b_2 = \text{sum of principal minors of } \mathbf{J}$$

$$b_3 = -\det \mathbf{J}$$

Thus, at E_1^* we have:

$$\mathbf{J}_1 = \begin{bmatrix} b-1 & 1 & c \\ v(b-1) & v-1 & vc \\ -ed & 0 & e \end{bmatrix}$$

such that:

$$\begin{aligned} b_1 &= 2 - b - v - e \gtrless 0 \\ b_2 &= \begin{vmatrix} v-1 & vc \\ 0 & e \end{vmatrix} + \begin{vmatrix} b-1 & c \\ -ed & e \end{vmatrix} + \begin{vmatrix} b-1 & 1 \\ v(b-1) & v-1 \end{vmatrix} \\ &= e(v-1) + e(b-1) + ced + (b-1)(v-1) - v(b-1) \\ &= 1 - b - e(2 - v - b - cd) \gtrless 0 \\ b_3 &= ed \begin{vmatrix} 1 & c \\ v-1 & vc \end{vmatrix} - e \begin{vmatrix} b-1 & 1 \\ v(b-1) & v-1 \end{vmatrix} \\ &= ed[vc - c(v-1)] - e[(b-1)(v-1) - v(b-1)] \\ &= -e(1 - b - cd) < 0 \end{aligned}$$

We can therefore state the following Proposition regarding the local stability of E_1^* .

Proposition 2 *Regardless of the signs of b_1 and b_2 , the internal equilibrium point E_1^* is unstable.*

Next, we find that the Jacobian matrix evaluated at $E_{2,3}^*$ is:

$$\mathbf{J}_{2,3} = \begin{bmatrix} b-1 & 1 & c \\ v(b-1) & v-1 & vc \\ 2ed & 0 & -2e \end{bmatrix}$$

such that:

$$\begin{aligned}
b_1 &= 2 - b - v + 2e \gtrless 0 \\
b_2 &= \begin{vmatrix} v-1 & cv \\ 0 & -2e \end{vmatrix} + \begin{vmatrix} b-1 & c \\ 2ed & -2e \end{vmatrix} + \begin{vmatrix} b-1 & 1 \\ v(b-1) & v-1 \end{vmatrix} \\
&= 2e(1-v) + 2e(1-b) - 2ced - (1-b)(v-1) + v(1-b) \\
&= 1 - b + 2e(1-v) + 2e(1-b-cd) \gtrless 0 \\
b_3 &= -2ed \begin{vmatrix} 1 & c \\ v-1 & cv \end{vmatrix} + 2e \begin{vmatrix} b-1 & 1 \\ v(b-1) & v-1 \end{vmatrix} \\
&= -2ed[vc - c(v-1)] + 2e[(b-1)(v-1) - v(b-1)] \\
&= 2e(1-b-cd) > 0
\end{aligned}$$

and therefore

$$\begin{aligned}
b_1 b_2 - b_3 &= (2 - b - v + 2e) [1 - b + 2e(1-v) + 2e(1-b-cd)] \\
&\quad - 2e(1-b-cd) \\
&= (2 - b - v) [1 - b + 2e(1-v) + 2e(1-b-cd)] \\
&\quad + 2e [2e(1-v) + 2e(1-b-cd) + cd] \gtrless 0
\end{aligned}$$

For $0 \leq v \leq 1$, all local stability conditions are satisfied:

$$\begin{aligned}
b_1 &> 0 \\
b_2 &> 0 \\
b_3 &> 0 \\
b_1 b_2 - b_3 &> 0
\end{aligned}$$

so that both external equilibrium points are locally stable. The implications of these results are summarised in the following Proposition.

Proposition 3 *For a sufficiently weak accelerator effect, such that the following three conditions are satisfied:*

$$\begin{aligned}
v-1 &< 1-b+2e \\
v-1 &< \frac{1-b}{2e} + 1-b-cd \\
A(v-1)^2 - B(v-1) + C &> 0
\end{aligned}$$

where

$$\begin{aligned}
A &= 2e > 0 \\
B &= 2e(1-b) + (1-b) + 2e(1-b-cd) + 4e^2 > 0 \\
C &= (1-b)[1-b+2e(1-b-cd)] + 4e^2(1-b-cd) + 2ecd > 0
\end{aligned}$$

the equilibrium points $E_{2,3}^*$ are locally asymptotically stable.

However, for certain values of v , it may happen that the last inequality is not satisfied. It may occur that the passage of the parameter through a critical value causes a qualitative change in the nature of the singular point and of the trajectories. In this case, the system in stable equilibrium can lose its stability, and may give rise to a limit cycle. Applying the existence part of the Hopf bifurcation theorem, we can state and prove the following Proposition.

Proposition 4 *For values of v in the neighbourhood of the critical value v_{hb} , such that:*

$$\begin{aligned} v_{HB} - 1 &< 1 - b - e \\ v_{HB} - 1 &< \frac{1 - b}{2e} + 1 - b - cd \\ A(v_{HB} - 1)^2 - B(v_{HB} - 1) + C &\approx 0 \end{aligned}$$

the equilibrium points $E_{2,3}^$ admit a family of periodic solutions.*

Proof. See Mathematical Appendix. ■

This result is in line with our aim of generating persistent fluctuations rooted in the interaction between real and financial markets. Periodic solutions might emerge as a result of an increase in the strength of the accelerator effect. We proceed examining the properties of the model using numerical simulations.

4.2 Numerical simulations

Convergence to equilibrium points $E_{2,3}^*$ is shown in Fig. 3, where, together with the parameter values listed in (12), we have used $v = 0.9$. The internal equilibrium point has eigenvalues equal $\lambda_1 \approx 1.600$, $\lambda_2 \approx -0.060 - 0.164i$ and $\lambda_3 \approx -0.060 + 0.164i$ so that it turns out to be a *saddle-focus* with a two-dimensional stable manifold whereas the two external points, with eigenvalues equal to $\lambda_1 = -3.2744$, $\lambda_2 = -0.0678 + 0.1590i$ and $\lambda_3 = -0.0678 - 0.1590i$, are locally *stable focus-nodes*. Two trajectories are shown in the figure, both starting from initial conditions very close to the saddle-focus, $(Y_1(0), I_1(0), P_1(0)) = (\bar{Y}_1, \bar{I}_1, \bar{P}_1 - 0.0003)$ and $(Y_2(0), I_2(0), P_2(0)) = (\bar{Y}_1, \bar{I}_1, \bar{P}_1 + 0.0003)$, but on different “sides” (one just above it, the other just below), one converging to the focus-node on the bottom-left of the space, the other to the focus-node on the top-right.

When $v > 1$, leaving unchanged all other parameters, it is possible to find a value of the coefficient of acceleration such that $b_1 b_2 - b_3 = 0$, i.e., such that the system undergoes a supercritical Hopf bifurcation at both external equilibrium points. Numerical calculation shows that this critical value of the coefficient of acceleration is $v_{HB} \approx 1.0362$ such that

$$\begin{aligned} b_1|_{v=v_{HB}} &> 0 \\ b_2|_{v=v_{HB}} &> 0 \\ b_3|_{v=v_{HB}} &> 0 \\ b_1 b_2 - b_3|_{v=v_{HB}} &\approx 0 \end{aligned}$$

For values of v greater, but close, to v_{HB} , the model may admit the coexistence of two stable limit cycles, each describing persistent and bounded fluctuations of the variables. An example of this is shown in Fig. 4, where we have taken $v = 1.039 > v_{HB}$ and drawn two trajectories

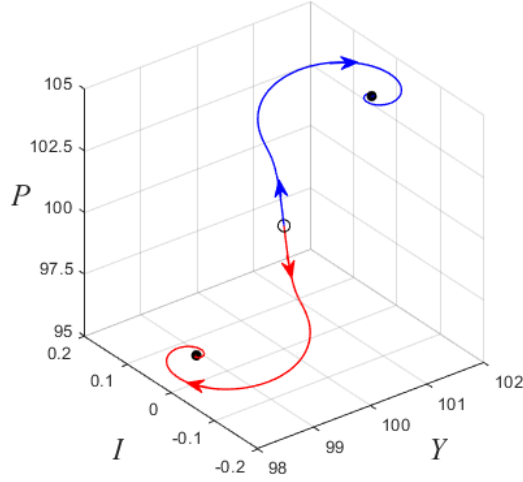


Figure 3: The dynamics of the three variables when $v = 0.9$.

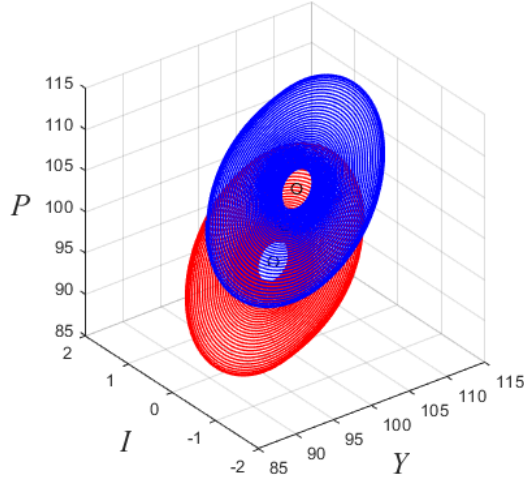


Figure 4: The dynamics of the three variables when $v = 1.039 > v_{HB}$.

with the same initial conditions as in the previous figure. The internal equilibrium point turns out to be *unstable focus-node* with eigenvalues equal to $\lambda_1 = 1.5977$, $\lambda_2 = 0.0107 + 0.1746i$ and $\lambda_3 = 0.0107 - 0.1746i$, whereas the two external equilibrium points, which have eigenvalues equal to $\lambda_1 = -3.2738$, $\lambda_2 = 0.0014 + 0.1728i$ and $\lambda_3 = 0.0014 - 0.1728i$, are *saddle-foci* with two-dimensional unstable manifolds.

When v is further increased, we observe another important change in the dynamics generated by the model as shown in Fig. 5 where we have taken $v = 1.0462$ and used the same two initial conditions as in the previous figures. The typology of equilibria remains the same as in Fig. 4, but now both trajectories, after an initial dynamics similar to the one shown in that figure, converge towards a unique attractor. Projections of this on the (Y, I) - and (Y, P) -planes are then shown in Fig. 6 and 7, respectively.

Real and financial markets interactions have produced significant boom-and-bust dynamics in the past. The importance of the accelerator effect over the business cycle justified the

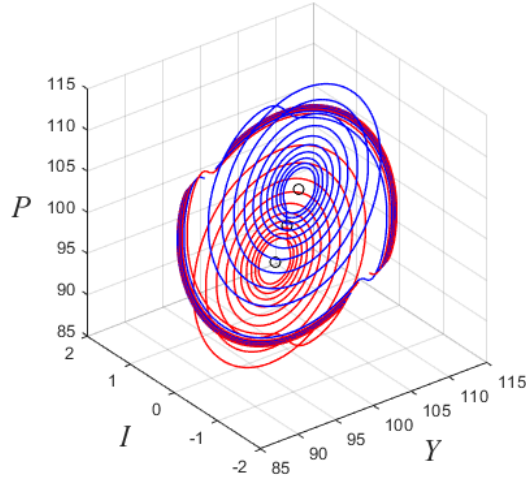


Figure 5: The dynamics of the three variables when $v = 1.0462$.

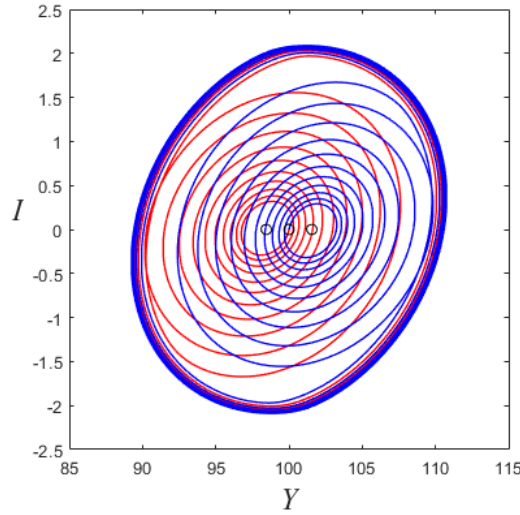


Figure 6: Projection on the (Y, I) -plane

modelling exercise presented in this section. It must be noted, however, that parameter values used in the calibration exercise are far from realistic. Moreover, our paper belongs to a literature that explains the behaviour of speculative markets taking into account that investors react to economic conditions but they are also influenced by the behaviour of other investors. It is our purpose in the next section to allow for a stronger accelerator effect as well as to extend the model in order to consider the possibility of switches between chartists and fundamentalist behaviour.

5 Endogenous sentiment dynamics

In an environment with strong asymmetric information, traders necessarily have to rely on what can be observed in the markets to take decisions concerning their actions. [Lux \(1995\)](#)

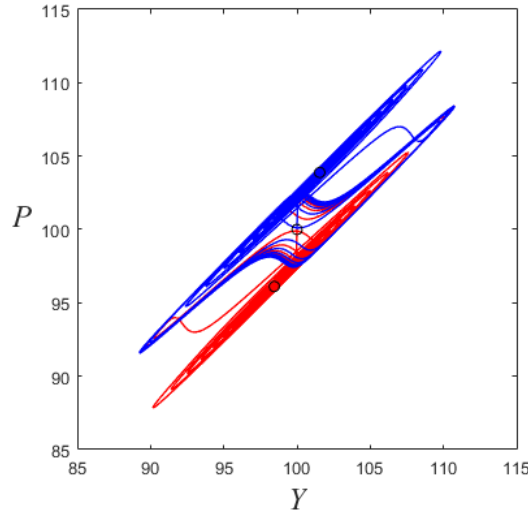


Figure 7: Projection on the (Y, P) -plane.

formalised a mechanism of mutual contagion in speculative markets that has been recently used to assess macroeconomic and stock market interactions (e.g. [Franke, 2012](#); [Flaschel et al., 2015, 2018](#)). Following this literature, we assume agents decide whether to take either a chartist or a fundamentalist stance depending on the current composition of the market. However, different from previous exercises, we leave aside the herd behaviour motivation and concentrate on a Minskyan mechanism in which agents update their expectations during good (bad) times and hence become more optimistic (pessimistic) about future economic prospects.⁷ Facing an increase of optimism (pessimism), the composition of the population will turn toward chartism (fundamentalism). This reflects investors' tendency to react more cautiously to larger market changes (see, for example, [Dieci et al., 2018](#)).

Suppose there is a fixed number of speculators trading in the financial market, N , that are divided between chartists, N^C , and fundamentalist, N^F , such that:

$$N = N^C(t) + N^F(t)$$

while the difference between these two groups, n , can be written as:

$$n(t) = N^C(t) - N^F(t)$$

Defining:

$$x(t) = \frac{n(t)}{N} \quad (34)$$

as the sentiment index, we have that $x \in [-1, 1]$ describes the average sentiment of speculators. At any given point in time, $x > 0$ indicates a dominance of chartists while $x < 0$ implies a majority of fundamentalists. An equal division of the population between these two groups gives $x = 0$.

Differentiating Eq. (34) with respect to time and making use of the definitions given above, we obtain:

$$\dot{x}(t) = \frac{\dot{n}(t)}{N} = \frac{\dot{N}^C(t) - \dot{N}^F(t)}{N} \quad (35)$$

⁷For a recent discussion and empirical evidence on herding behaviour in models with heterogenous agents, see [Franke and Westerhoff \(2016\)](#).

Changes in the sentiments index fundamentally depend on the difference between variations in the two groups that form the population.

Hence, we need to specify the dynamics of N^C and N^F taking into account that traders might change their own behaviour. Let $p^{F \rightarrow C}$ be the (transition) probability that a fundamentalist becomes a chartist, and $p^{C \rightarrow F}$ the probability that a chartist becomes a fundamentalist. In mathematical terms, we write:

$$\dot{N}^C(t) = N^F(t)p^{F \rightarrow C}(t) - N^C(t)p^{C \rightarrow F}(t) \quad (36)$$

$$\dot{N}^F(t) = N^C(t)p^{C \rightarrow F}(t) - N^F(t)p^{F \rightarrow C}(t) \quad (37)$$

that is, sentiments will change toward chartism if its share in the population multiplied by the probability of becoming a fundamentalist is lower than the share of fundamentalists multiplied by the probability of becoming a chartist and vice versa.

In line with [Flaschel et al. \(2018\)](#), the key behavioural assumption concerns the determinants of the transition probabilities, which are supposed to depend symmetrically on a switching index (s) capturing the expectations of traders on market performance. An increase in s raises the probability of a fundamentalist becoming a chartist, and decreases the probability of a chartist becoming a fundamentalist. However, differently from what was done in the above-mentioned contribution, we avoid the use of the standard exponential forms for the probabilities $p^{F \rightarrow C}$ and $p^{C \rightarrow F}$, and adopt the following simple linear specifications:

$$p^{F \rightarrow C}(t) = g + \mu s(t) \quad (38)$$

$$p^{C \rightarrow F}(t) = g - \mu s(t) \quad (39)$$

where μ is a measure of the sensitivity of the sentiments composition with regard to the switching index, and $g > \mu s$. It is well-known that the adoption of exponential probability functions opens the door to the existence of multiple equilibria values. Even though the use of these functions has a sounding theoretical motivation, it is our intention to show that our results do not rely on any particular (and to a certain point) *ad hoc* nonlinearity. This explanation justifies the use of Eqs. (38) and (39).

The switching index depends on the extent of disequilibrium in the goods market:

$$s(t) = h\dot{Y}(t) = h[Z(t) - Y(t)], \quad h > 0 \quad (40)$$

Agents tend to become more optimistic as economic activity expands. As their confidence increases, they estimate less carefully future economic values and become more trend-followers. This explains why s is increasing in the difference between Z and Y . It follows that the probability of fundamentalists becoming chartists increases during the expansion phase of the business cycle ($\dot{Y} > 0$) while agents are more likely to return to fundamentals during a recession ($\dot{Y} < 0$).

Making use of Eqs. (35)-(40), the dynamic equation for the endogenous composition of speculators is given by:

$$\dot{x}(t) = -2 \left[gx - \mu h \dot{Y}(t) \right] \quad (41)$$

Notice that, from Eq. (34), the share of chartists in the population is given by $N^C(t)/N = [1 + x(t)]/2$ while the share of fundamentalists by $N^F(t)/N = [1 - x(t)]/2$. In this way, the behavioural composition of speculators is endogenously determined as a function of their average sentiments.

Recalling the initial analysis by Westerhoff, stock prices are assumed to adjust to excess demand in the market. Taking into account that the number of agents with a chartist or fundamentalist attitude changes over time, we redefine Eq. (3) as:

$$\dot{P}(t) = \frac{N^C(t)}{N} D^C(t) + \frac{N^F(t)}{N} D^F(t) = \frac{1+x(t)}{2} D^C(t) + \frac{1-x(t)}{2} D^F(t) \quad (42)$$

so that variations in stock market prices depend on the sum of speculative demands weighted by the share of each type of agent in the population.

Chartists expect that “bull and bear” markets will persist so that their stock demand is positive when the stock price is above the perceived fundamental value and vice versa. Fundamentalists, on the contrary, expect that the stock price will return to its fundamental value and, therefore, they increase (decrease) their stock demand when F is greater (less) than P . While we maintain our continuous time formulation of Eq. (5) for chartists’ behaviour, we modify Eq. (6) to remove the previously discussed *ad hoc* nonlinearity in the response of fundamentalists:

$$D^F = f [F(t) - P(t)], \quad f > 0 \quad (43)$$

Therefore, substituting Eqs. (5) and (43) into (42), the dynamic equation for stock market prices becomes:

$$\dot{P}(t) = \left\{ f \left[\frac{1-x(t)}{2} \right] - e \left[\frac{1+x(t)}{2} \right] \right\} [F(t) - P(t)] \quad (44)$$

where to maintain the argument that fundamentalists respond stronger than chartists to price deviations, we impose $f > e$.

In the previous section, we specified behavioural expressions for each component of aggregate demand. By means of the acceleration principle, it was assumed that investment depends on the whole spectrum of past values of $v\dot{Y}$. It must be noted, however, that the values of parameter v required to obtain persistent dynamics were close to 1, which is quite unrealistic. More realistic values of v make the system locally unstable around the equilibrium points.

On the other hand, when describing investment behaviour, one needs to take into account the existence of resource constraints in the economy. Even when firms want to increase investment in response to higher demand, sooner or later the economy will achieve full capacity utilisation or face a shortage of inputs that constraints capital accumulation. There are also limits to the capacity of the firm to disinvest, even under strong aggregate demand contractions. Hence, to provide a proper specification of investment behaviour, we adopt the following piece-wise linear function (see Goodwin, 1951):

$$\phi(t) = \begin{cases} k_1, & \text{if } \dot{Y} > k_1/v \\ v\dot{Y}(t), & \text{if } -k_2/v < \dot{Y} < k_1/v \\ -k_2, & \text{if } \dot{Y} < -k_2/v \end{cases} \quad (45)$$

where $k_1, k_2 > 0$.

Taking into account these considerations, the dynamics of the model are now generated by the following four-dimensional dynamic system:

$$\dot{Y}(t) = (b-1)Y(t) + I(t) + cP(t) + a \quad (46)$$

$$\dot{I}(t) = \phi(t) - I(t) \quad (47)$$

$$\dot{P}(t) = \left\{ f \left[\frac{1-x(t)}{2} \right] - e \left[\frac{1+x(t)}{2} \right] \right\} [dY(t) - P(t)] \quad (48)$$

$$\dot{x}(t) = -2 [gx + h\dot{Y}(t)] \quad (49)$$

where to simplify notation and without loss of generality, we have assumed $\mu = 1$.

In steady-state $\dot{Y}(t) = \dot{I}(t) = \dot{P}(t) = \dot{x}(t) = 0$. This gives us the following equilibrium conditions:

$$(b-1)Y(t) + I(t) + cP(t) + a = 0 \quad (50)$$

$$\phi(t) - I(t) = 0 \quad (51)$$

$$\{f[1-x(t)] - e[1+x(t)]\} [dY(t) - P(t)] = 0 \quad (52)$$

$$gx + h\dot{Y}(t) = 0 \quad (53)$$

Given these equilibrium conditions, we can state and prove the following Proposition regarding the existence of a unique internal equilibrium point.

Proposition 5 *The dynamic system (46)-(49) admits a unique equilibrium solution, $(\bar{Y}, \bar{I}, \bar{P}, \bar{x})$, defined by:*

$$\begin{aligned} \bar{Y} &= \frac{a}{1-b-cd} \\ \bar{I} &= 0 \\ \bar{P} &= \frac{ad}{1-b-cd} \\ \bar{x} &= 0 \end{aligned}$$

Proof. In steady-state, $\dot{Y}(t) = 0$, so that, from Eq. (53), we have $\bar{x} = 0$. From Eq. (52), for $x = 0$ and because $f > e$, we obtain $dY(t) = P(t)$. Recall that for $\dot{Y} = 0$, we have $\phi(t) = 0$. Therefore, from Eq. (51), we have that $\bar{I} = 0$. Substituting $\bar{I} = 0$ and $dY(t) = P(t)$ into (50), we obtain $\bar{Y} = a/(1-b-cd)$. It immediately follows that $\bar{P} = ad/(1-b-cd)$. ■

Once we remove the cubic component in the fundamentalists' price adjustment equation, and adopt linear specification for the transition probabilities, the model is not capable any longer of generating multiple steady-state solutions. One should also notice that the values of Y and P that bring the goods market and stock market to equilibrium correspond to Westerhoff's internal equilibrium point. Furthermore, only an equal distribution between chartists and fundamentalists can stabilise sentiments in the stock market because it equalises speculative demand of both types of agents.

5.1 Local stability analysis

Even under this quite simple structure, the interaction between markets and agents can potentially generate more interesting dynamics. This is the result of the interaction of two destabilising forces. On the one hand, instability is an intrinsic part of the accelerator principle. Suppose, for example, that in a certain point of time the majority of traders are fundamentalist and the stock price is below (above) its fundamental value. As a response to $F > P$ ($F < P$), speculators start to buy (sell) stocks which initially makes P to increase (fall). An increase (reduction) in stock prices has a positive (negative) impact on consumption through the wealth effect channel. For realistic values of v , this will provoke a strong increase (reduction) in investment further increasing (reducing) output. An increase (a reduction) of Y actually increases (reduces) the fundamental value of the stock price so that the disequilibrium in the stock market is not corrected.

On the other hand, *ceteris paribus*, chartists tend to exert a destabilizing influence on the price of financial assets. For example, if in the limit there are no fundamentalists in the economy, i.e. $x = 1$, a positive shock in stock prices will induce two main effects. First, there is a positive and increasing spiral of P in the stock market because chartists are trend followers. Secondly, through the wealth effect on consumption, there is an increase in economic activity that reinforces chartism behaviour given that more optimistic agents are more likely to become trend-followers. Notice, nonetheless, that this last effect depends on the sign of the shock. A negative shock in stock prices will still generate a destabilising spiral of P in the stock market but the negative wealth effect that follows reduces economic activity and may act as a stabilising force. In any case, if the majority of financial players behave as chartists, there is an extra instability source on the system.

From the global point of view, our economy has one main stabiliser. The existence of resource constraints in the economy was modelled through a piece-wise investment function. As previously discussed, even when firms want to increase investment in response to a very high \dot{Y} , sooner or later the economy will reach full capacity or face a shortage of inputs that puts a constrain to capital accumulation. There are similar constraints to the capacity of the firm to disinvest that prevent investment to fall without limits. In what follows, we shall analyse the local stability of two subparts of the model separately. This exercise allows us to understand what are the sources of instability in the economy before exploring the complete model by means of numerical simulations.

5.1.1 Core investment-financial interactions

In section 4, we investigated the analytical properties of a 3-dimensional dynamic system that basically corresponds to a continuous-time version of Westerhoff's model with the inclusion of the flexible accelerator. Here, we will perform a similar exercise by taking the dynamic system (46)-(49) and assuming that sentiments are constant and such that $x = \bar{x} = 0$. Hence, the core system is formed by Eqs. (46)-(48). The main innovation lies in the absence of the cubic component in the fundamentalists' price adjustment equation. We proceed stating and proving the following Proposition regarding the stability of the unique internal equilibrium point.

Proposition 6 *For a sufficiently weak accelerator effect, such that the following three conditions are satisfied:*

$$v - 1 < 1 - b + \frac{f - e}{2} \quad (54)$$

$$v - 1 < 1 - b - cd + 2 \left(\frac{1 - b}{f - e} \right) \quad (55)$$

$$A(v - 1)^2 - B(v - 1) + C > 0 \quad (56)$$

where

$$A = (f - e)/2 > 0$$

$$B = (1 - b - cd)(f - e)/2 + (1 - b) + (1 - b)(f - e)/2 + (f - e)^2/4 > 0$$

$$C = [(1 - b) + (f - e)/2] (1 - b - cd)(f - e)/2 + (1 - b)^2 + cd(f - e)/2 > 0$$

the equilibrium point of the “core investment-financial” dynamic subsystem is locally asymptotically stable.

Proof. See Mathematical Appendix. ■

It is straightforward to see that for $v \leq 1$, conditions (54)-(56) are always satisfied. Moreover, although $v > 1$ does not necessarily imply that the equilibrium point is unstable, for realistic magnitudes of the accelerator effect the model is very much likely to be unstable. Deducting the properties of the precise bifurcation from a general mathematical analysis would be possible but not very illustrative. The general rule is that a sufficiently strong accelerator effect exerts a destabilising effect on the system making the equilibrium point unstable in Harrodian lines. A few numerical examples are sufficiently informative of the main mechanisms in motion and, hence, we will rely on numerical simulations to investigate if the ceiling and floor of the piece-wise investment function are capable of generating an attractor around equilibrium.

5.1.2 Core sentiment-financial interactions

As a next step, we introduce heterogeneous expectations and sentiment dynamics as in Eq. (49) while assuming that investment is constant and equal to its equilibrium value, i.e. $I = \bar{I} = 0$. Hence, the core system is now formed by Eqs. (46), (47), and (49). The introduction of heterogeneity in agents' expectations may play a destabilizing role in the economy mainly because chartists are trend-followers. Agents update their expectations during good times and become more confident turning toward chartism. However, it is also worth noting that when output is falling, confidence disappears and agents run back to fundamentals, which exerts a stabilising force. We proceed stating and proving the following Proposition regarding the stability of the unique internal equilibrium point.

Proposition 7 *The internal equilibrium point of the “core sentiment-financial” dynamic subsystem is always locally asymptotically stable.*

Proof. See Mathematical Appendix. ■

The investigation of local stability of each subsystem is not sufficient to give a complete picture of the dynamic behaviour of the full model. In order to understand its global properties, we now proceed to numerical simulations.

5.2 Numerical Simulations

This section examines the properties of the full 4D model by using numerical simulations. We first illustrate the main dynamics of each subsystem and only then, in a second step, we investigate the complete system, including the endogenous dynamics of aggregate sentiments. When choosing parameter values, we make reference to the same studies mentioned in the previous sections. They were also adjusted in order to provide outcomes with economic meaning. We would like to emphasise that, since we are not calibrating a real economy, the purpose of the exercise is to simply give an idea of magnitudes involved.

Our first subsystem corresponds to a core investment-financial interactions with sentiments taken as constant and equal to its equilibrium value. The calibration of the model is shown below:

$$\begin{aligned} a &= 3, \quad b = 0.95, \quad c = 0.02, \quad d = 1, \\ e &= 2, \quad f = 4, \quad k_1 = 1, \quad k_2 = 1, \end{aligned}$$

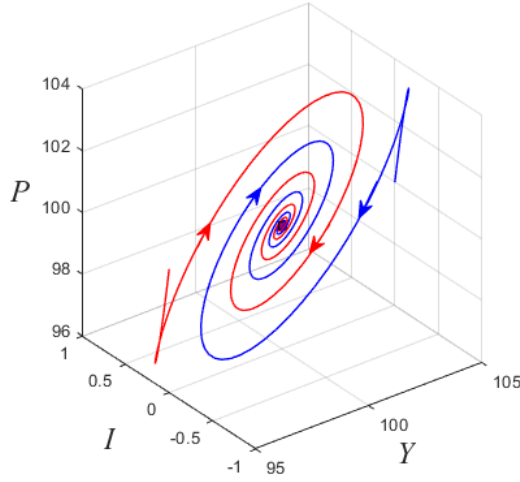


Figure 8: Cyclical convergence to equilibrium in the core investment-financial subsystem for $(Y_0, I_0, P_0) = (105, 0, 100)$ in blue and $(Y_0, I_0, P_0) = (95, 0, 100)$ in red, when $v = 1$.

As demonstrated in Proposition 3, for a sufficiently weak accelerator effect, the internal equilibrium point is locally asymptotically stable. Fig. 8 represents this case for $v = 1$ and two different initial conditions, $(Y_0, I_0, P_0) = (105, 0, 100)$ and $(Y_0, I_0, P_0) = (95, 0, 100)$ converging to $(\bar{Y}, \bar{I}, \bar{P}) = (100, 0, 100)$. When output is above (below) its equilibrium value, firms respond increasing (decreasing) investment which slightly increases (decreases) Y . Given that $f > e$, speculators start to buy (sell) stocks which initially makes P to grow (decrease) further elevating (reducing) output through the wealth effect on consumption. However, the instability introduced by the weak accelerator effect is not enough to destabilise the system. Once prices go above (below) the fundamental value, speculators start to sell (buy), which brings output down (up) through the wealth effect channel and is also amplified by the weak accelerator. This mechanism proceeds until equilibrium is reached.

For realistic values of the accelerator effect, the subsystem is very much likely to be unstable. As briefly discussed in the previous subsection, we use numerical simulations to investigate if the ceiling and floor of the piece-wise investment function are capable of generating an attractor around equilibrium. Fig. 9 shows the emergence of a limit cycle when we take, for example, $v = 2.5$. Maintaining the initial conditions as in the previous case, the main mechanism previously described now repeats itself indefinitely providing a representation of the intrinsic instability of the interaction between real and financial markets.

We now proceed by studying the second subsystem that corresponds to the core sentiment-financial interactions. Leaving aside for a moment any considerations about investment, we allow the composition of financial markets to change endogenously motivated by a mechanism that resembles Minskyan optimism waves. The calibration of the model is shown below:

$$\begin{aligned} a &= 3, \quad b = 0.95, \quad c = 0.02, \quad d = 1, \\ e &= 2, \quad f = 4, \quad g = 0.5, \quad h = 0.5 \end{aligned}$$

Fig. 10 shows the trajectories for two different initial conditions, $(Y_0, I_0, P_0) = (110, 100, 0.5)$ and $(Y_0, I_0, P_0) = (90, 100, -0.5)$ converging to $(\bar{Y}, \bar{P}, \bar{x}) = (100, 100, 0)$. In the analytical part of the paper, we proved that the internal equilibrium point of the system is asymptotically stable. Our numerical simulations confirm that this is indeed the case. When output is above

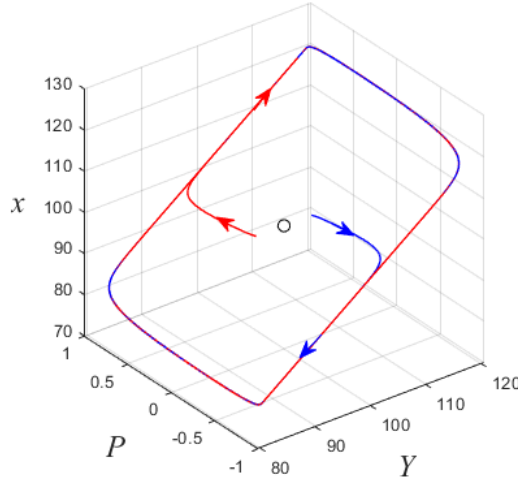


Figure 9: Convergence to a limit cycle in the core investment-financial subsystem for $(Y_0, I_0, P_0) = (105, 0, 100)$ in blue and $(Y_0, I_0, P_0) = (95, 0, 100)$ in red, when $v = 2.5$.

(below) its equilibrium value and most agents are chartists (fundamentalist), speculators start to sell (buy) which provokes prices in the stock market to fall (grow) leading to a reduction (increase) in output through the wealth effect on consumption. A reduction (increase) in output generates a wave of pessimism (optimism) so that agents turn to fundamentalism (chartist) behaviour. These dynamics continue until equilibrium is reached with an equal distribution between both types of speculators.

We are finally ready to look now at the full 4-dimensional system. Given the difficulties in deriving and interpreting the stability conditions for a system of such dimension, we chose to obtain some hints about the behaviour of the model through numerical simulations. The calibration used is shown below:

$$\begin{aligned} a &= 3, \quad b = 0.95, \quad c = 0.02, \quad d = 1, \quad e = 2, \\ f &= 4, \quad g = 0.05, \quad h = 0.05, \quad k_1 = 1.5, \quad k_2 = 0.1, \quad v = 2, \end{aligned}$$

Two main modifications were introduced with respect to the previous cases. First, we adopt an asymmetric piece-wise linear investment function to capture the idea that is more difficult for the firm to disinvest than to reach full capacity utilisation. Second, we reduce the magnitudes of parameters g and h to avoid prices falling below zero. Fig. 11 plots the time-series trajectories for two different initial conditions, $(Y_0, I_0, P_0, x_0) = (100, 0, 100, 0.5)$ and $(Y_0, I_0, P_0, x_0) = (90, 0, 100, -0.5)$, both of them converging to a limit cycle.

From this last figure, we can sketch a description of the dynamic interactions of the four variables over the cycle. During good times, when output is growing, the accelerator effect strongly increases output until capacity utilisation is fully used, further increasing Y . Agents update their expectations and become more optimistic about economic prospects. Hence, the composition of speculators turns toward chartism with $x > 0$. Chartists are basically trend followers and provoke an increase in stock market prices. At a certain point, output stops growing because the economy already reached full capacity, expectations are reverted, and the downward phase of the cycle begins. Speculators return to fundamentals, and investment strongly falls followed by stock prices. Once the majority of agents are fundamentalists and

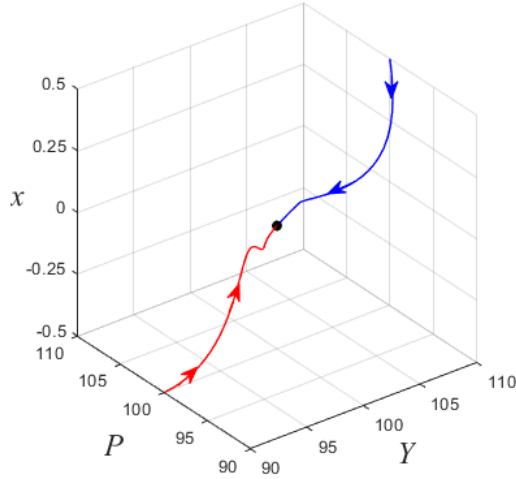


Figure 10: Convergence to equilibrium in the core sentiment-financial subsystem for $(Y_0, I_0, P_0) = (110, 100, 0.5)$ in blue and $(Y_0, I_0, P_0) = (90, 100, -0.5)$ in red.

prices fall below their fundamental value, speculators start buying stocks again which allows output and investment to recover. At this point the cycle restarts.

6 Conclusions

We have studied in this paper a stylized dynamic macroeconomic model of real-financial market interactions with endogenous aggregate sentiment dynamics. Building on the seminal paper by [Westerhoff \(2012\)](#), we rewrote the model in continuous time and show that the dynamics it is able to generate are much simpler than in the discrete-time case. We proceed by improving the specification of aggregate demand and distinguished between consumption and investment assuming that the latter is determined by the flexible accelerator principle.

Furthermore, we remove the *ad-hoc* nonlinearity on the fundamentalist behavioural rule. Finally, we allow for endogenous switches between chartists and fundamentalists in line with [Lux's \(1995\)](#) contagion mechanism in speculative markets. However, different from previous exercises, we leave aside the traditional herd behaviour motivation and concentrate on a Minskyan mechanism in which agents update their expectations during good times turning toward chartism while in bad times they run back to fundamentals.

We showed that the interaction between real and financial markets need not to be necessarily stable. For realistic values of the accelerator effect the core investment-financial subsystem was found to be unstable in Harrodian lines. Our numerical simulations indicate that the ceiling and floor of the piece-wise linear investment function are capable of generating an attractor around equilibrium. On the other hand, the core sentiment-financial subsystem was found to be always stable despite the positive feedback from production to sentiments.

Dynamics in the full 4-dimensional system were illustrated by means of numerical simulations and provide a more complete view of the story we are telling. The crucial nonlinearity in investment allows the emergence of a limit cycle with output, investment, stock prices and chartism increasing during the expansion phase of the business cycle while a downturn in sentiments is followed by the other endogenous variables of the model. Such persistent trajectories provide a representation of the intrinsic instability of the interaction between real and

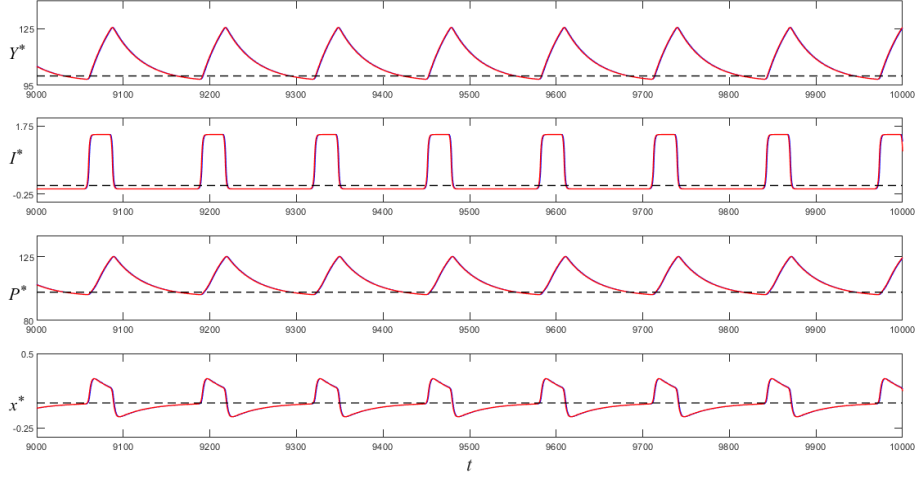


Figure 11: Trajectories of output, investment, stock prices, and sentiments in the full 4-dimension dynamic system.

financial markets.

A Mathematical Appendix

A.1 Proof of Proposition 4

To prove Proposition 3 using the (existence part of) the Hopf Bifurcation Theorem and using v as bifurcation parameter, we must first of all (HB1) show that the characteristic equation possesses a pair of complex conjugate eigenvalues $\theta(v) \pm i\omega(v)$ that become purely imaginary at the critical value v_{HB} of the parameter – i.e., $\theta(v_{HB}) = 0$ – and no other eigenvalues with zero real part exists at v_{HB} and then (HB2) check that the derivative of the real part of the complex eigenvalues with respect to the bifurcation parameter is different from zero at the critical value.

(HB1) Given the conditions required in order to $b_1 > 0$, $b_2 > 0$ and b_3 , in order that the characteristic equation has one negative real root and a pair of complex roots with zero real part we must have:

$$b_1 b_2 - b_3 = 0$$

a condition which, given the expression for $b_1 b_2 - b_3$, is satisfied for

$$A(v_{HB} - 1)^2 - B(v_{HB} - 1) + C = 0$$

(HB2) By using the so-called sensitivity analysis, it is then possible to show that the second requirement of the Hopf Bifurcation Theorem is also met. Substituting the elements of the Jacobian matrix into the respective coefficients of the characteristic equation:

$$\begin{aligned} b_1 &= 2 - b - v + 2e \\ b_2 &= 1 - b + 2e(1 - v) + 2e(1 - b - cd) \\ b_3 &= 2e(1 - b - cd) \end{aligned}$$

so that

$$\begin{aligned}\frac{\partial b_1}{\partial v} &= -1 \\ \frac{\partial b_2}{\partial v} &= -2e \\ \frac{\partial b_3}{\partial v} &= 0\end{aligned}$$

When $v - 1 < 1 - b - e$, $v - 1 < (1 - b) / 2e + 1 - b - cd$, and $A(v_{hb} - 1)^2 - B(v_{hb} - 1) + C = 0$, apart from $b_1 > 0$, $b_2 > 0$ and $b_3 > 0$ which is always true, one also has $b_1 b_2 - b_3 = 0$. In this case, one root of the characteristic equation is real negative (λ_1), whereas the other two are a pair of complex roots with zero real part ($\lambda_{2,3} = \theta \pm i\omega$, with $\theta = 0$). We thus have:

$$\begin{aligned}b_1 &= -(\lambda_1 + \lambda_2 + \lambda_3) \\ &= -(\lambda_1 + 2\theta) \\ b_2 &= \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 \\ &= 2\lambda_1 \theta + \theta^2 + \omega^2 \\ b_3 &= -\lambda_1 \lambda_2 \lambda_3 \\ &= -\lambda_1 (\theta^2 + \omega^2)\end{aligned}$$

such that:

$$\begin{aligned}\frac{\partial b_1}{\partial v} &= -\frac{\partial \lambda_1}{\partial v} - 2\frac{\partial \theta}{\partial v} = -1 \\ \frac{\partial b_2}{\partial v} &= 2\theta \frac{\partial \lambda_1}{\partial v} + 2(\lambda_1 + \theta) \frac{\partial \theta}{\partial v} + 2\omega \frac{\partial \omega}{\partial v} = -2e \\ \frac{\partial b_3}{\partial v} &= -(\theta^2 + \omega^2) \frac{\partial \lambda_1}{\partial v} - 2\lambda_1 \theta \frac{\partial \theta}{\partial v} - 2\lambda_1 \omega \frac{\partial \omega}{\partial v} = 0\end{aligned}$$

For $\theta = 0$, the system to be solved becomes:

$$\begin{aligned}-\frac{\partial \lambda_1}{\partial v} - 2\frac{\partial \theta}{\partial v} &= -1 \\ 2\lambda_1 \frac{\partial \theta}{\partial v} + 2\omega \frac{\partial \omega}{\partial v} &= -2e \\ -\omega^2 \frac{\partial \lambda_1}{\partial v} - 2\lambda_1 \omega \frac{\partial \omega}{\partial v} &= 0\end{aligned}$$

or

$$\begin{bmatrix} -1 & -2 & 0 \\ 0 & 2\lambda_1 & 2\omega \\ -\omega^2 & 0 & -2\lambda_1 \omega \end{bmatrix} \begin{bmatrix} \frac{\partial \lambda_1}{\partial v} \\ \frac{\partial \theta}{\partial v} \\ \frac{\partial \omega}{\partial v} \end{bmatrix} = \begin{bmatrix} -1 \\ -2e \\ 0 \end{bmatrix}$$

Thus:

$$\begin{aligned}\frac{\partial \theta}{\partial v} \Big|_{v=v_{HB}} &= \frac{\begin{vmatrix} -1 & -1 & 0 \\ 0 & -2e & 2\omega \\ -\omega^2 & 0 & -2\lambda_1 \omega \end{vmatrix}}{\begin{vmatrix} -1 & -2 & 0 \\ 0 & 2\lambda_1 & 2\omega \\ -\omega^2 & 0 & -2\lambda_1 \omega \end{vmatrix}} \\ &= \frac{-2\omega(2e\lambda_1 + \omega^2)}{4\omega(\lambda_1^2 + \omega^2)} = -\frac{1}{2} \left(\frac{2e\lambda_1 + \omega^2}{\lambda_1^2 + \omega^2} \right) < 0\end{aligned}$$

A.2 Proof of Proposition 6

The core investment-financial interactions subsystem is given by:

$$\begin{aligned}\dot{Y}(t) &= (b-1)Y(t) + I(t) + cP(t) + a \\ \dot{I}(t) &= v(b-1)Y(t) + (v-1)I(t) + vcP(t) + va \\ \dot{P}(t) &= \left\{ f \left[\frac{1-x(t)}{2} \right] - e \left[\frac{1+x(t)}{2} \right] \right\} [dY(t) - P(t)]\end{aligned}$$

with the respective Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} b-1 & 1 & c \\ v(b-1) & v-1 & vc \\ \frac{(f-e)d}{2} & 0 & -\frac{f-e}{2} \end{bmatrix}$$

The coefficients of the characteristic equation are given by:

$$b_1 = -\text{tr}\mathbf{J} = 1 - b + 1 - v + \frac{f-e}{2} \gtrless 0$$

$$\begin{aligned}b_2 &= \begin{vmatrix} v-1 & vc \\ 0 & -\frac{f-e}{2} \end{vmatrix} + \begin{vmatrix} b-1 & c \\ \frac{d(f-e)}{2} & -\frac{f-e}{2} \end{vmatrix} + \begin{vmatrix} b-1 & 1 \\ v(b-1) & v-1 \end{vmatrix} \\ &= \frac{[(1-v) + (1-b-cd)](f-e)}{2} + 1 - b \gtrless 0\end{aligned}$$

$$\begin{aligned}b_3 &= -\det \mathbf{J} = - \begin{vmatrix} b-1 & 1 & c \\ v(b-1) & v-1 & vc \\ \frac{d(f-e)}{2} & 0 & -\frac{f-e}{2} \end{vmatrix} \\ &= \frac{(1-b-cd)(f-e)}{2} > 0, \text{ always}\end{aligned}$$

Local stability requires $b_1, b_2, b_3 > 0$, and $b_1b_2 - b_3 > 0$. This last condition requires that:

$$\begin{aligned}b_1b_2 - b_3 &= \left[(1-b) + (1-v) + \frac{(f-e)}{2} \right] \left\{ [(1-v) + (1-b-cd)] \frac{(f-e)}{2} + (1-b) \right\} \\ &\quad - (1-b-cd) \frac{(f-e)}{2} \\ &= \left[(1-b) + \frac{(f-e)}{2} \right] \left[(1-v) \frac{(f-e)}{2} + (1-b-cd) \frac{(f-e)}{2} + (1-b) \right] \\ &\quad + (1-v) \left[(1-v) \frac{(f-e)}{2} + (1-b-cd) \frac{(f-e)}{2} + (1-b) \right] - (1-b-cd) \frac{(f-e)}{2} \\ &= \frac{(f-e)}{2} (v-1)^2 - \left[(1-b-cd) \frac{(f-e)}{2} + (1-b) + (1-b) \frac{(f-e)}{2} + \frac{(f-e)^2}{4} \right] (v-1) \\ &\quad + \left[(1-b) + \frac{(f-e)}{2} \right] (1-b-cd) \frac{(f-e)}{2} + (1-b)^2 + cd \frac{(f-e)}{2} \\ &= A(v-1)^2 - B(v-1) + C, \quad A > 0, \quad B > 0, \quad C > 0\end{aligned}$$

If $v \leq 1$, through direct substitution, it is easy to see that b_1 , b_2 , b_3 , and $b_1b_2 - b_3 > 0$. More generally, we have that:

$$\begin{aligned} b_1 > 0 & \text{ requires } v - 1 < 1 - b + \frac{f - e}{2} \\ b_2 > 0 & \text{ requires } v - 1 < (1 - b - cd + 2 \left(\frac{1 - b}{f - e} \right)) \end{aligned}$$

Notice that b_3 is always greater than zero. Hence, the final stability condition is:

$$A(v - 1)^2 - B(v - 1) + C > 0$$

A.3 Proof of Proposition 7

The core sentiment-financial interactions subsystem is given by:

$$\begin{aligned} \dot{Y}(t) &= (b - 1)Y(t) + \bar{I} + cP(t) + a \\ \dot{P}(t) &= \{f[1 - x(t)] - e[(1 + x(t))]\} [dY(t) - P(t)] \\ \dot{x}(t) &= -2[gx(t) - h\dot{Y}(t)] \end{aligned}$$

with the respective Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} (b - 1) & c & 0 \\ \frac{d(f - e)}{2} & -\frac{(f - e)}{2} & 0 \\ 2(b - 1)h & 2ch & -2g \end{bmatrix}$$

The coefficients of the characteristic equation are given by:

$$\begin{aligned} b_1 &= -\text{tr} \mathbf{J} = (1 - b) + \frac{(f - e)}{2} + 2g > 0 \\ b_2 &= \begin{vmatrix} -(f - e) & 0 \\ ch & -g \end{vmatrix} + \begin{vmatrix} (b - 1) & 0 \\ (b - 1)h & -g \end{vmatrix} + \begin{vmatrix} (b - 1) & c \\ d(f - e) & -(f - e) \end{vmatrix} \\ &= g[2(1 - b) + (f - e)] + (1 - b - cd)\frac{(f - e)}{2} > 0 \\ b_3 &= -\det \mathbf{J} = - \begin{vmatrix} (b - 1) & c & 0 \\ \frac{d(f - e)}{2} & -\frac{(f - e)}{2} & 0 \\ 2(b - 1)h & 2ch & -2g \end{vmatrix} \\ &= (1 - b - cd)(f - e)g > 0 \end{aligned}$$

Local stability requires $b_1, b_2, b_3 > 0$, and $b_1b_2 - b_3 > 0$. The crucial condition is:

$$\begin{aligned} b_1b_2 - b_3 &= \left[(1 - b) + \frac{(f - e)}{2} + 2g \right] \left\{ g[2(1 - b) + (f - e)] + (1 - b - cd)\frac{(f - e)}{2} \right\} \\ &\quad - (1 - b - cd)(f - e)g \\ &= \left[(1 - b) + \frac{(f - e)}{2} \right] \left\{ g[2(1 - b) + (f - e)] + (1 - b - cd)\frac{(f - e)}{2} \right\} \\ &\quad + 2g^2[2(1 - b) + (f - e)] \end{aligned}$$

which is always greater than zero. Therefore, the subsystem is locally stable.

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