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A consistent representation of Keynes's
long-term expectation in financial markets

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Abstract

This paper advances an intuitive representation of Keynes's notion of long-term expectation. We introduce the epsilon-contamination approach and represent the conventional judgment by the Steiner point of agents' common probability set. We anticipate a change in conventional judgment by updating the Steiner point.

Keywords: Keynes, long-term expectation, epsilon contamination, uncertainty, multiple priors.

JEL classification: D81.

1 Introduction

In 9 august 2007 French bank BNP Paribas suspended three of its funds as problems in the U.S. subprime mortgage sector. It was the ant fact of the subprime crises that started in 2008 with the Lehman and Brothers failure and the beginning of the Great Recession. In November 2008, during a briefing by academics at the London School of Economics the Queen asked: "Why did nobody notice it?" In a three-page letter, Tim Besley, an external member of the Bank of England's monetary policy committee, and Peter Hennessy, political historian, summarized the conclusions of the seminar at the British Academy in June 2009. There were conducive conditions: imbalance in the global economy, low unemployment, low inflation, cheap consumer goods, ready credit, lower borrowing costs etc. Moreover, the authors put in evidence the existence of a general feel good factor that originated the psychology of denial that caused "the failure of the collective imagination of many bright people, both in this country and internationally, to understand the risk to the system as a whole". Since then, economists and academics have been examining the 'never again' question widely. We think that the crucial question is how people form their long-term expectation that determine their actions in real and financial markets. In this perspective, we think that J.M. Keynes represents a lighthouse in foggy current misinterpretations. Keynes thinks that professional investors and speculators in the stock exchange are forced to predict the mass psychology of the market, that is to inform and foresee "changes in the conventional basis of valuation a short time ahead of the general public" (Keynes 1936, p. 134). Keynes observes that "knowledge of the factors which will govern the yield of an investment some years hence is very slight and often negligible" (p. 134). Different from heroic times, when, according to Keynes, investment "was partly a lottery, though with the ultimate result largely governed by whether the abilities and character of the managers were above or below the average" (p. 134), if the separation between ownership and management prevails, then "certain classes of investment are governed by the average expectation of those who deal on the Stock Exchange as revealed in the price of shares, rather than by the genuine expectations of the professional entrepreneur" (Keynes 1936, p. 136).

Keynes condenses the process that induces to anticipate the change of convention in the famous metaphor of financial markets as a newspaper beauty contest¹. Keynes maintains that an investor does not have to anticipate what

¹explains the activity of professional investors that are forced to anticipate the change of conventional valuation by the following methaphor: "Professional investment may be likened to those newspaper competitions in which the competitors have to pick out the six prettiest faces from a hundred photographs, the prize being awarded to the competitor whose choice most nearly corresponds to the average preferences of the competitors as a whole; so that each competitor has to pick, not those faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of the other competitors, all of whom are looking at the problem from the same point of view. It is not a case of choosing those which, to the best of one's judgment, are really the prettiest, nor even those which average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I

will be the fundamental value of a firm in the future, but rather should estimate other investors' valuation. The individual assessed value is different from "the outcome of a weighted average of quantitative benefits multiplied by quantitative probabilities" (p. 145). In fact, as Keynes argues, to make an investment decision, "we are assuming, in effect, that the existing market valuation, however arrived at, is uniquely correct in relation to our existing knowledge of the facts which will influence the yield of the investment, and that it will only change in proportion to changes in this knowledge; though, philosophically speaking it cannot be uniquely correct, since our existing knowledge does not provide a sufficient basis for a calculated mathematical expectation. In point of fact, all sorts of considerations enter into the market valuation and are in no way relevant to the prospective yield" (p. 137).

We define a general functional to represent Keynes's long-term expectation and following a recent paper (Basili and Chateauneuf, 2016) we also set up the way to represent how a speculator anticipates changes of conventional judgment. Section 2 defines long-term expectation in an epsilon contamination approach incorporating the decision maker's attitude about insufficient and vague information. Section 3 sets up an aggregation scheme of opinions expressed through different probability distributions. Facing the set of all probability distributions attached by agents to possible events, the speculator is assumed to consider the weighted probability distribution of agents' cores, that is the weighted probability distribution of the intersection of all the investors' probability distribution consistent with the market. Such a weighted probability distribution is the Steiner point of the convex capacity that emerges from the aggregation of agents' opinions that represents the conventional judgment. Gajdos et al. (2008) show that in the case of a finite state space, the Steiner point always exists and can be valued through the Shapley value. On the contrary, in the case of an infinite countable state space, since the Steiner point is defined with respect to the outer angle or curvature, the Steiner point has no continuous extension to all convex bodies in infinite dimensional Hilbert space (i.e. Vitale 1985, p. 247). Section 3 approximates the Steiner point at the limit. The idea is very intuitive and straightforward: it is assumed that each agent has an interval of probabilities on each state and that intervals are distributed as a Fisher-Tippett distribution, that is a general distribution for extremes that includes Weibull, Gumbel, and Frechet distributions. By attaching an extreme distribution to intervals, convergence holds: the more extreme are events, the lower are the probabilities and the closer is the interval. Section 4 defines the professional investor long-term expectation that is the result of 'the average expectation of those who deal on the Stock Exchange as revealed in the price of shares' and the competence "to anticipate what average opinion expects the average opinion to be" (Keynes 1936, p. 139). Section 5 concludes.

believe, who practise the fourth, fifth and higher degrees." (1936, p. 140)

2 Uncertainty, multiple-priors and epsilon-contamination approach

Evaluation of an asset depends on expectations of prospective yields but this long-term expectation, as Keynes claims, is based "partly on existing facts which we can assume to be known more or less for certain, and partially on future events which can only be forecasted with more or less confidence" (p.133). Crucially relevant facts at the base of individual expectation are often *very uncertain*, even if Keynes makes clear that "by very uncertain I do not mean the same thing as improbable" and in so doing he establishes a direct relation between the notion of confidence in the General Theory and the weight of arguments in the Treatise on Probability (Note 1, p. 133). Keynes clarifies that "the state of long-term expectation, upon which our decisions are based, does not solely depend, on the most probable forecast we can make. It also depends on the confidence with which we make this forecast - on how highly we rate the likelihood of our best forecast turning out quite wrong" (p. 133). In this perspective of uncertainty², we shall assume that each investor does not have a unique prior on states of the World, but rather a finite set of probability distributions (multiple priors), none of which is considered sufficiently reliable. To represent the individual state of confidence, that depends on "the actual observation of the markets and business psychology" (p. 134), we assume that each agent's preferences can be represented by the epsilon contamination (ε - *contamination*, henceforth) of some probability measure³: Eichberger and Kelsey, 1999; Nishimura and Ozaki, 2002; Asano, 2008; Gajdos et al., 2008; Kopylov, 2009; Cerreia et al., 2013.

2.1 Framework

Let $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ be the set of states of the World, $\mathcal{P}(\Omega)$ the sigma-algebra of all the subsets of Ω and P the set of probability measures, such that $P = \{p : p \text{ is a probability measure on } \Omega\}$.

Let F be a mixture space and $f, g \in F$ are act (simple and compound lotteries or assets), such that $f : \mathcal{P}(\Omega) \rightarrow R$. Consider $C \subset P$ and $\epsilon \in [0, 1]$. Then

$$f = \epsilon f(p(\omega)) + (1 - \epsilon) f(q(\omega)), \quad (2.1)$$

(Huber 1981, Berger 1984, Berger and Berliner 1986).⁴

²Because of continuity with Keynes terminology we call *uncertainty* what in current decision theory is named *ambiguity*.

³The ε - *contamination* emerges as a robust Bayesian method to quantify, in terms of a class of possible distributions, how partial and incomplete is the subjective information encompassed in a single prior distribution. In fact, "quantification of prior beliefs can never be done without error, and hence that one is left at the end of the elicitation process with a set Γ of prior distributions which reflect true prior belief; i.e., π_T is an unknown element of Γ " (Berger 1984, p. 73). Details are in Moreno and Cano (1991).

⁴This representation of agent's beliefs by ε - *contamination* of a given prior has been applied in economics (Epstein and Wang, 1994; Carlier et al., 2003; Nishimura and Ozaki,

Let $E = \{s_1, \dots, s_k, \dots\}$ be a finite or countable set of agents. Suppose every agent has got an opinion, formally an opinion of an agent s_i is a convex set C_i , contained in P . Under no-arbitrage condition, in frictionless and complete financial market, assets are valued by a linear function of their payoffs (mathematical expectation), that is the price of any asset can be computed by its expected value with respect to a unique probability. If there are incompleteness or trade frictions but arbitrage-free condition holds, an asset price can be evaluated by the Choquet integral with respect to a non-additive probability of its payoffs. If the nonadditive probability is concave, then the pricing rule is sublinear. The core of the concave capacity is close and compact and is the set of all the agent's opinions consistent with the market⁵. Following Chateauneuf et al (1996), Jouini (2000), Jouini and Kallal (2001), Castagnoli et al. (2002), Araujo et al. (2012; 2018), for any asset $f \in F$ there exists a financial pricing rule $D : R^\Omega \rightarrow R$, that is a function over future payoffs contingency to state space $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$. Such a pricing rule D is *subadditive, arbitrage free, positive homogeneous, monotonic and constant additive*⁶. Araujo et al. (2012) point out (Theorem 2) that for a given pricing rule $D : R^\Omega \rightarrow R$, there exists a unique closed and convex set $K \subset P$ of probability measures, where at least one element is strictly positive, such that for any asset f : $D(f) = \max_{k \in K} E_k(f)$, where $E_k(\cdot)$ is the standard expectation with respect to k .

2.2 Individual long-term expectation

The ε -contamination approach allows to consider the agent's asset evaluation as the combination of \bar{D} , the asset price observed in the market and the confidence in his most reliable forecast. Because of uncertainty every agent forms his long-term expectation by distorting asset price with his confidence and combining it with his own most reliable, or any other motivated probability distribution such as the probability distribution that induces minimum expected utility that solves the Ellsberg Paradox, evaluation of that asset.

Then agent's long-term expectation can be summarized by the following criterion

Criterion 1 *Agent's long term expectation can formally be defined by*

$$\gamma_i(f) = [\varepsilon \bar{D}(f) + (1 - \varepsilon) D_i(f)] \quad (2.2)$$

where $p_i \in C_i \subset P$, $\varepsilon \in [0, 1]$ and $D_i(f)$ is the expectation of f with respect to p_i .

Agent's long-term expectation reveals that he is $\varepsilon \times 100\%$ confident that the uncertainty he faces is summarized by the market price, but at the same time,

2006; Wolitzky, 2016). It is well known that ε can be considered as the individual confidence on a model of financial market or probability distribution that represents it.

⁵Chateauneuf et al (1996) first studied and characterized the sub-additive Choquet pricing rule and showed that if the non-additive probability is a concave capacity, the set of the agent's probability distributions consistent with the market is unique and coincides with the core.

⁶Details are in Araujo et al. 2012.

he is aware that with $(1 - \varepsilon) \times 100\%$ uncertainty could be better represented by another probability distribution in the set C_i of all reasonable evaluations. In sum, the ε -contamination interpretation of agent's long-term expectation allows describing imprecision of knowledge and behavioral effects of its awareness.

3 Main motivation

As noted before, Keynes assumes a convention influences investment decisions and such a general evaluation is "the outcome of the mass psychology of a large number of ignorant individuals" (1936, p. 138). A way to define the mass psychology is by aggregating agents' opinions, expressed by probability distributions on future states of the World⁷. An aggregation of agents' opinions is that of choosing a particular set of agents E , at most countable, each one giving a range of probability distributions; every such an agent s_i , as Ω is the space of states, has a family of probabilities C_i on it, which he considers reasonable. An adequate way to do all this is, for every such an individual, to associate with him a convex subset C_i of the probabilities on Ω . Let K be the family of all convex sets in P . An opinion multifunction is every $\mathcal{O}_{E_i} : E_i \rightarrow \mathcal{K}$, $i \rightarrow C_i$. Finally, given an opinion \mathcal{O}_{E_i} , the prevailing opinion \mathcal{O}_E is defined as: $\bigcap_i C_i$. In the finite dimensional case, i.d. if Ω is finite and under the hypothesis that $\bigcap_i C_i$ is not the empty set⁸, the idea is that the properly balanced opinion has got to be the Steiner point of $\bigcap_i C_i$: the conventional judgement⁹. If the number of the events is not finite yet countable, some difficulties occur: so in Section 3 we define a suitable aggregation of agents' opinion¹⁰.

3.1 Preliminaries

Let (X, d) be a metric space. In what follows $B(x; r) \subset X$ is the usual ball centered on x and with radius r . If $X = R^d$ with its usual Euclidean norm, we set S^{d-1} the unit hypersphere centered on the origin. If \mathcal{H} is a Hilbert space, we denote with $\langle \cdot, \cdot \rangle$ its standard inner product and $\|\cdot\|$ the induced Hilbert norm.

Definition 1 *Let (X, d) be a metric space. For every couple C_1, C_2 of bounded closed subsets of X we define their Hausdorff distance as:*

⁷Opinion as a distribution is a usual assumption, e.g. de Finetti.

⁸ $\bigcap_i C_i$ can be considered the agents common information set (opinions), that is their subjectively elaborated and evaluated information about market asset evaluation.

⁹The Steiner point or curvature centroid of smooth convex bodies is additive, uniformly continuous and satisfies an invariance property with respect to isometries.

¹⁰The Bayesian axiomatic approach to consensus distribution would not appear satisfying, not even in the sophisticated versions (copula models) and elicitation based on behavioral combination methods (e.g., DeGroot and Montera, 1991). If investors' opinions are not all independent and equally likely, each investor has to cope with ambiguity and stochastically dependent evaluations. As a consequence, each investor could calibrate the aggregation of investors' opinions through her confidence or degree of belief by pooling methods based on Dempster's rule of combination or theory of evidence, combination rules based on possibility distributions and fuzzy measures, or aggregation based on multiple priors or capacity.

$$d_{H,X}(C_1, C_2) = \{\inf \rho > 0 : C_2 \subset C_1 + B(0, \rho), C_1 \subset C_2 + B(0, \rho)\}.$$

Let now X be a Banach space. We denote with $\mathcal{C}(\mathcal{H})$ the family of all the closed sets of \mathcal{H} and let $\mathcal{K}(X)$ be the family of its compact and convex subsets. Let also $\mathcal{C}_F(\mathcal{H})$ be the family of all the finite-dimensional elements of $\mathcal{C}(\mathcal{H})$, that is the family of those contained in some finite-dimensional affine subspace of \mathcal{H} and let $\mathcal{K}_F(\mathcal{H}) := \mathcal{C}_F(\mathcal{H}) \cap \mathcal{K}(X)$ be the set of finite-dimensional compact and convex sets of X . Finally, for a $C \in \mathcal{C}_F(\mathcal{H})$ define $\dim(C) := \min\{\dim(L) : C \subset L, L \text{ a finite-dimensional affine subspace of } \mathcal{H}\}$.

It is a well-known result (Castaing and Valadier 1977, Theorem II-14, p. 47) that:

Proposition 1 $(\mathcal{K}(X), d_{H,X})$ is a complete metric space.

Next, we recall the definition of the classical Steiner point for a d -dimensional convex body (see e.g. Schneider 1993). Let \mathcal{H} be a Hilbert space.

Definition 2 Let $C \in \mathcal{K}(\mathcal{H})$ and $C \subset L$ where L is a d -dimensional linear subspace of \mathcal{H} . Then its Steiner point $s(C)$ is defined as

$$s(C) := d \int_{S_{\mathcal{H}} \cap L} u h_C(u) d\sigma(u),$$

where $h_C(u) := \sup\{\langle u, x \rangle : x \in C\}$ is the support function of C , $S_{\mathcal{H}}$ denotes the unit hypersphere in \mathcal{H} centered on the origin and σ is the normalized Lebesgue measure on $S_{\mathcal{H}} \cap L$.

The Steiner point is independent of the choice of the finite-dimensional Euclidean subspace L containing C , so that the previous definition makes sense; it only depends on the inner product.

Let first analyze the case of $\Omega = \{w_1, \dots, w_n\}$ a finite set of states of the World; so to set our ideas in a simpler situation. We shall treat the countable case further.

Given the sigma-algebra $\mathcal{P}(\Omega)$ of all the subsets of Ω , we identify isometrically the convex set $P(\Omega)$ with $\{\pi : \pi \text{ is a probability measure on } \Omega\} = \left\{ \pi : \Omega \rightarrow [0, 1] \text{ such that } \sum_{j=1}^n \pi(j) = 1 \right\}$, with $[0, 1]^n \cap \left\{ \sum_{j=1}^n \pi(j) = 1 \right\}$, with its Hilbert space natural structure.

A reasonable way to investigate these events is simply to choose a certain number of agents E to give an opinion, or a range of opinions as follows: every such an agent s_i is asked to give to Ω a probability or, in more uncertain situations, a possible set of probabilities C_i . Thus, with the previous identifications, for every agent i , C_i is contained in a linear set of dimension less or equal to $n - 1$, and the common opinion is a convex set contained in a linear set of dimension less or equal to $n - 1$.

For every agent chain $\{s_1, \dots, s_k, \dots\}$, and so for $E = \{s_1, \dots, s_k, \dots\}$ a reasonable way to have an aggregation of agents opinions, as remarked in a recent paper Basili and Chateauneuf (2016) is to choose the Steiner point of the common

opinion $\mathcal{O}_{\cap E}$. In that paper an opinion was chosen this way: for every agent i a range of possible values for every admissible value $\pi(j)$ is chosen such that $a_{ij}(j) \leq \pi(j) \leq b_{ij}$, $1 \leq j \leq n$. So, $C_i = ([a_{i1}, b_{i1}] \times \dots \times [a_{in}, b_{in}]) \cap \left\{ \sum_{j=1}^n \pi(j) = 1 \right\}$.

The following important result is an Hilbert space adapted situation of classical results (See Shvartsman 2004, Theorem 1.2, with the Lipschitz constant asymptotic evaluation due to Vitale (1985, Appendix)).

Proposition 2 *Let \mathcal{H} be a Hilbert space. Then the mapping $s: (\mathcal{K}_F(\mathcal{H}), d_{H,\mathcal{H}}) \rightarrow \mathcal{H}$ which associates to every element of $\mathcal{K}_F(\mathcal{H})$ its Steiner point is such that, for every $C_1, C_2 \in \mathcal{K}_F(\mathcal{H})$, and setting $d = \dim(C_1 \cup C_2)$,*

$$\|s(C_2) - s(C_1)\| \leq l(d)d_{H,\mathcal{H}}(C_2, C_1),$$

where

$$l(d) = \frac{\Gamma(d/2 + 1)}{\sqrt{\pi}\Gamma(d/2 + 1/2)} \sim \sqrt{d/2 + 1},$$

as $d \rightarrow +\infty$, with Γ the standard Euler Gamma function.

Furthermore, $l(d)$ is the minimal possible constant fulfilling the previous inequality.

We finally remark that if the opinions are chosen as in Basili and Chateauneuf (2016), then we actually restrict our Steiner selector to the set of compact convex sets of $P(\Omega)$ contained in a linear space whose dimension does not exceeds $n - 1$, having thus the possibility of a unique Lipschitz constant. The stability with respect to the Hausdorff metric is at its best in such a situation.

3.2 Steiner point with countable states of the World

Unfortunately, there is no way to define a suitable generalization of the notion of Steiner point to general convex bodies not contained in a finite dimensional subspace of a Banach or even a Hilbert space This is because, e.g. in the Hilbert case, the Lipschitz constant l in Proposition 2 increases as \sqrt{d} when the dimension d increases, not permitting, in general, any approximation argument by means of finite dimensional convex bodies ¹¹. So, in general, when the set Ω is a countable set there is no way to proceed. As a matter of fact, some reasonable possibility arises when there is a natural way to create an ordering of Ω , when the tail of Ω is considered constituted by extreme events, for example. In situations like this, it usually happens that the way the Lipschitz constant l behaves as d goes to $+\infty$ is compensated the right way by the distribution itself.

To approach this way of analysis, let now consider the more general case of a countable set of states of the World $\Omega = \{w_1, \dots, w_n, \dots\}$ and a finite number of agents $E = \{s_1, \dots, s_k\}$ (the case of a countable number of agents can be

¹¹See R. A. Vitale (1985) for more discussion and details.

analogously treated, with minor changes in notation and no difference in methods, even if it is not realistic in our economical analysis). This case we shall use the Hilbert space $l_2 = \left\{ (x_1, x_2, \dots) : \sum_{j=1}^{+\infty} x_j^2 < \infty \right\}$ and the Banach space $l_1 = \left\{ (x_1, x_2, \dots) : \sum_{j=1}^{+\infty} |x_j| < \infty \right\}$.

We recall that l_1 is the dual space of the separable Banach space $c = \{(x_1, x_2, \dots) : \lim_{j \rightarrow \infty} x_j \text{ exists and is finite}\}$. Thus the unit ball of l_1 centered in the null sequence is sequentially weakly star compact. Given the sigma-algebra $\mathcal{P}(\Omega)$ of all the subsets of Ω , we shall identify the set of all the probability measures $P(\Omega)$ on Ω with $\left\{ (x_j) : 0 \leq x_j \leq 1, \sum_{j=1}^{+\infty} x_j = 1 \right\} \subset l_1(\Omega) \subset l_2(\Omega)$.

Assume further $\Delta_i^n = [a_{i1}, b_{i1}] \times \dots \times [a_{in}, b_{in}]$ where, with a little abuse of notation, we also suppose the possibility that for some i and j , $a_{ij} = b_{ij}$, so that in this case we set $[a_{ij}, b_{ij}] := \{a_{ij}\} = \{b_{ij}\}$. Set $\Delta^n = \bigcap_{i=1}^k \Delta_i^n$, which is either the empty set or it is a possibly degenerate n -rectangle $[\alpha_{n1}, \beta_{n1}] \times \dots \times [\alpha_{nn}, \beta_{nn}]$, where also in this case with a little abuse of notation, we consider the possibility that, for some j , $\alpha_j = \beta_j$, with $[\alpha_i, \beta_j] := \{\alpha_j\} = \{\beta_j\}$. Finally, consider $O_E^n = \Delta_n \cap P(\Omega)$. Using the previously introduced notation, we now state the following

Hypotheses 1

i) there exists an event j_0 such that for every expert i , $a_{ij} = 0$ if $j \geq j_0$; suppose furthermore that $O_E^{j_0} \neq \emptyset$; there exists a sequence (ζ_i) of non-negative real numbers such that, for every expert i , for all possible event j , $b_{ij} \leq \zeta_i$; furthermore let $\sum_{j=1}^{+\infty} \zeta_j < +\infty$.

ii) Suppose $\sum_{j=1}^{+\infty} \zeta_j l(j) < +\infty$, with $l(j)$ as in Proposition 2.

Note that

i) should be interpreted as that there exists some elementary event after which any expert can reasonably give no lower bound for the probabilities: only (possibly very decreasing) upper bounds can be given for all elementary events.

ii) to be fulfilled, one should choose for (ζ_i) a suitable extreme events distribution, for example

Remark 1 if i) is fulfilled, the (O_E^n) is a sequence of closed compact convex sets which, for $n \geq j_0$ is not-decreasing and not identically equal to the empty set and $\beta_n \leq \zeta_n$. Furthermore, $\sum_{j=1}^{+\infty} \beta_j$ is convergent. If also ii) is fulfilled, then $\sum_{j=1}^{+\infty} \beta_j l(j)$ is convergent.

We next need the following

Lemma 1 If $n \geq j_0$, then $d_{H,l_2}(O_E^{n+1}, O_E^n) \leq \sqrt{2}\beta_{n+1}$.

Proof. Let $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n, \bar{x}_{n+1}) \in O_E^{n+1}$. Then, setting π_n the projection of the whole l_2 onto R^n , isometrically identified with its subspace having zero components after the n -th, we get: $\pi_n(\bar{x}) = (\bar{x}_1, \dots, \bar{x}_n) \in \Delta_n$, so that: $\alpha_j \leq$

$\bar{x}_j \leq \beta_j$, $j \leq n$ and, furthermore, $0 \leq \bar{x}_{n+1} \leq \beta_{n+1} \leq \zeta_{n+1}$, so that $\sum_1^n \bar{x}_j = 1 - \bar{x}_{n+1} \leq 1$. Next, remark that, because $\Delta_n \neq \emptyset$ we have $\sum_1^n \beta_n \geq 1$.

Set $\phi_{\bar{x}}: [\bar{x}_1, \beta_1] \times \dots \times [\bar{x}_n, \beta_n] \rightarrow R$, $(y_1, \dots, y_n) \mapsto \sum_1^n y_j$.

By the intermediate value theorem there exists a \bar{y} such that $\phi_{\bar{x}}(\bar{y}) = \sum_1^n \bar{y}_j = 1$. Then $\bar{y} \in O_E^n$. Notice that, for $j \leq n$, $\bar{y}_j \geq \bar{x}_j$ and $\sum_1^n |\bar{y}_j - \bar{x}_j| = \sum_1^n \bar{y}_j - \bar{x}_j = \bar{x}_{n+1} \leq \beta_{n+1}$. So $\|\bar{y} - \pi_n(\bar{x})\| \leq \beta_{n+1}$. Because $\|\pi_n(\bar{x}) - \bar{x}\| = \bar{x}_{n+1} \leq \beta_{n+1}$, by Pythagoras theorem and because $O_E^n \subset O_E^{n+1}$ we get the announced result. ■

Theorem 1 *Suppose Hypotheses 1 are fulfilled. Then*

- i) there exists a strongly compact convex set $O \subset l_2$ such that (O_E^n) converges to O in the Hausdorff metric d_{H,l_2} ;*
- ii) the sequence $(s(O_E^n))$ of the corresponding Steiner points converges to a point $\bar{s}(O) = (\bar{s}_j)$ strongly in l_2 and weakly star in l_1 . In particular $\sum_{j=1}^{+\infty} \bar{s}_j = 1$, with $\bar{s}_j \geq 0$ for all j , so that $\bar{s}(O) \in P(\Omega)$.*

Proof. By Lemma 1 and Remark 1 (O_E^n) is a Cauchy sequence in $(\mathcal{K}(l_2), d_{H,l_2})$; so, by Proposition 1, *i)* is proved.

In order to prove *ii)*, notice that by Proposition 2 and Remark 1 $(s(O_E^n))$ is a Cauchy sequence in l_2 such that $\|s(O_E^n)\|_{l_1} = \sum_{j=1}^{+\infty} s(O_E^n)_j = 1$, so it is strongly convergent in l_2 and, because the unit ball in l_1 is sequentially weakly star compact, it is weakly star convergent in l_1 ; finally, because the sequence (1) which is constantly equal to 1 is in c and every $s(O_E^n)_j \geq 0$, we get that $\sum_{j=1}^{+\infty} \bar{s}(O_E)_j = \|\bar{s}(O_E)\|_{l_1} = 1$. ■

4 Conventional judgement and confidence in market asset price: a professional investor's behavior

Keynes considers the stock exchange populated by professional investors and speculators who are forced to anticipate the mass psychology of the market. As a consequence, the behavior of professional investors and speculators is the result of two different components: "the average expectation of those who deal on the Stock Exchange as revealed in the price of shares" and the competence "to anticipate what average opinion expects the average opinion to be" (Keynes 1936, p. 140). Then we obtain our primary result:

Criterion 2 *The professional investor's or speculator's long-term expectation $I(f)$ can be formally defined by*

$$I(f) = [\mu \bar{D}(f) + (1 - \mu) \bar{s}(C(f))] \quad (4.1)$$

where $\mu \in [0, 1]$ is the confidence of a professional investor in the asset price and $\bar{s}(C(f))$ is the expected value with respect to the Steiner point, that is what he considers conventional judgement, times future returns.

Crucially $(1-\mu)$ is the weight attached to what Keynes considers *the average opinion expects the average opinion to be*.

Then (4.1) precisely summarized the competitor’s behavior in the newspaper beauty contest suggested by Keynes. Then, the previous expectation is the solution of the problem and accurately explains how skilled Keynesian individuals that are long term investors, or speculators should solve the newspaper beauty contest. Confronting (2.2) and (4.1) it is clear that speculators and professional investors differ from ordinary agents through the ability or superior knowledge in trying to estimate the conventional valuation.

4.1 Updating

When an uncertain event occurs, people may change their long-term expectation. In fact the "conventional valuation which is established as the outcome of the mass psychology of a vast number of ignorant individuals is liable to change violently as the result of a sudden fluctuation of opinion due to factors which do not really make much difference to the prospective yield" (Keynes 1936, p. 138).

The investor has to anticipate this change, but because of dynamic consistency, he can not update the Steiner point only, since it could induce an order that is not coherent with his preference. A very simple way to update multiple priors models¹² is to apply the Bayes rule for each probability distribution (prior-by-prior) in P and $C \subseteq K \subset P$ and then *re-evaluate* the Steiner point. This method can guarantee dynamic consistency but is strenuous. It is possible to reduce the number of the probability distributions that need to be updated to calculate the new Steiner point after a given non-null even Ξ occurred.

Araujo et al (2016) point out how news modify the asset price. Araujo et al. characterize a new approach to updating the pricing rule that satisfies above conditions and the property called Dynamic Consistency to Certainty¹³

It follows that $C^E(f)$ is the updated pricing rule such that $p^\Xi \in P$, and for any asset f and real number h , $C(f_h^\Xi) \geq h$ if and only if $C^\Xi(f) \geq h$. that is *if the unconditional price of f is at least equal to h , then its conditional price must also be at least equal to h .*

So doing the investor anticipates the change in the conventional judgment and includes this anticipation, so that

$$I(f)_\Xi = [\mu \bar{D}(f) + (1 - \mu) \bar{s}(C^\Xi(f))], \quad (4.2)$$

where $\mu \in [0, 1]$.

Interesting enough the conditional Steiner point is elicited by the simple full Bayesian updating rule and it represents an appropriate 'average opinion' that

¹²Different solutions are: *rectangularity, menu dependence, change of subjective perception* etc.

¹³Given an event $\Xi \subset \mathcal{P}(\otimes)$ and the pricing rule D , Ξ is relevant if $-D(-\Xi^*) > 0$, then $p(\Xi) > 0$, for all $p \in C$ and $k^\Xi : \{p(\Xi) \in P | k \in K\}$ is the set of conditional probabilities. The updated pricing rule D^Ξ is the unique pricing rule that satisfies the Full Bayes Rule (Araujo et al 2016).

can be considered as a preferred rule with respect to every non bayesian rule unconditionally (de Finetti 1954).

The long-term expectation [4.2] represents the solution of the "battle of wits to anticipate the basis of conventional valuation a few months" (Keynes 1936, p.139). In fact, as Keynes argues, "it happens, however, that the energies and skill of the professional investor and speculator are mainly occupied otherwise. For most of these persons are, in fact, largely concerned, not with making superior long-term forecasts of the probable yield of an investment over its whole life, but with foreseeing changes in the conventional basis of valuation a short time ahead of the general public. They are concerned, not with what an investment is really worth to a man who buys it "for keeps", but with what the market will value it at, under the influence of mass psychology, three months or a year hence" (Keynes 1936, p. 139).

5 Concluding remarks

This paper proposes a different interpretation of Keynes's theory of long-term expectation and agents' ambiguity based on the ε -contamination approach of probability distributions. The ε -contamination interpretation of Keynes's long-term expectation theory makes direct and explicit the relationship between his long-term expectation notion and contemporary decision theory originated by the Ellsberg Paradox. The paper introduces a new representation of conventional judgement based on the Steiner point of the set of common opinions among agents. This work can give a formal description of the process by which professional investors try to anticipate the change in conventional judgment. The new representation of long-term expectation is also coherent with the behavior of competitors in the Keynes's beauty contest. Remarkably, this new representation of long-term expectation sheds light on Keynes's view of stock exchanges like casino, where speculators make the market by anticipating the change of conventional judgment.

6 References

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