UNIVERSITA' DEGLI STUDI DI SIENA

81

QUADERNI DEL DIPARTIMENTO
DI ECONOMIA POLITICA

Mauro Caminati

CYCLICAL GROWTH IN A LONG-PERIOD PERSPECTIVE

QUADERNI DEL DIPARTIMENTO DI ECONOMIA POLITICA

• Redazione: **Dipartimento** di **Economia Politica**, Piazza S. Francesco, 17 -53100 Siena- tel. 0577/298620

• La Redazione ottempera agli obblighi previsti dall'Art. 1 del D.L.L. 31.8.45 n. 660.

• Le richieste di copie della presente pubblicazione dovranno essere indirizzate alla Redazione

• I **Quaderni del Dipartimento di Economia Politica** dell'Università di Siena vengono pubblicati come servizio atto a favorire la tempestiva divulgazione di ricerche scientifiche originali, siano esse in forma provvisoria o definitiva. I **Quaderni** vengono regolarmente inviati a tutti gli istituti e dipartimenti italiani, a carattere economico, nonché a numerosi docenti e ricercatori universitari. Vengono altresì inviati ad enti e personalità italiane ed estere. L'accesso ai **Quaderni** è approvato dal Comitato Scientifico, sentito il parere di un referee.

• (Dal 1979 a tutto il 1987 la denominazione della presente Collana è stata: "**Quaderni dell'Istituto di Economia**". La numerazione progressiva rimane tuttavia la stessa).

Mauro Caminati

CYCLICAL GROWTH IN A LONG-PERIOD PERSPECTIVE

Siena, dicembre 1988

Mauro Caminati is "Ricercatore" in the Department of Economics of the University of Siena

Introduction.

In commenting on the low sensitivity of U. K. employment to cyclical downturns during the years 1955-66, Harrod argued[1] that the expectation of long-term full employment conditions had become structural in those years. The post-second-world-war growth phase provides a good historical example of optimistic long-term expectations quite unaffected by short-term disturbances. Cyclical recoveries could readily trigger cyclical booms in the prevailing economic climate of those years.

If we consider the recent phase of slow growth in the western economies following the 1973 oil shock, we observe that cyclical recoveries have not led to booms in the face of far less optimistic long-term expectations. One wonders whether cyclical fluctuations are convincingly explained by the mechanism of Harrodian instability during such periods of relative stagnation. Indeed, economic fluctuations during prosperity phases seem to differ from those prevailing in phases of stagnation not only on quantitative but also on qualitative grounds.

This remark seems to be at variance with standard trade-cycle theories in the "Keynesian" tradition, for the main part of investment behaviour is exclusively explained by current results in these theories. The main aim of this paper is to show how the qualitative difference referreed to above is better understood if the influence of long-term expectations on investment is explicitly introduced in trade-cycle theory. We suggest that this influence acts primarily through the effect of long-term expectations on the sensitivity of investment decisions to current results.

[1] R. Harrod, *Towards a New Economic Policy*, Manchester University Press, 1967.

1. Investment decisions in the real models of the business cycle.

1.1.

The term real models of the business cycle is used here to indicate the dynamic theories of output anticipated by M. Kalecki's *Essay on the Theory of the Business Cycle* published in Polish in 1933; these theories abounded after the publication of Keynes's *General Theory* up to the mid fifties. The coupling of some version of the consumption multiplier with the idea that investment expenditure is positively related to the current activity level and/or growth rate plays an essential role in all such theories; age and simplicity notwithstanding, they are still in the focus of modern economists engaged in macrodynamic theory along Keynesian lines.

In the present paper this group of models will be examined from a particular viewpoint. We shall overlook the sharp and obvious differences existing among them and concentrate on a single common feature relating to their attitude towards investment decisions and the consequences of this attitude. This is not to say, of course, that the models concerned have similar investment functions: for example some of them, such as Harrod's and Hicks's incorporate the acceleration principle, while others do not.

The common feature stressed here is that investment activity is divided into two parts that are governed by two different sets of expectations. One set is relatively independent of current activity levels and is either considered as exogenous, or as a slowly changing function of time, which may be influenced by the past history of the system. The other set of expectations is exclusively related to currently observed results concerning profits and/or output, given available capacity.

The above dichotomy, which is reflected by the distinction between autonomous and induced investment, is open to objections, some of which were clearly recognized at the time when the first conclusions from the fast growing literature on the real trade cycle models were being drawn. Harrod, in particular, was inclined to stress that the distinction was more a matter of degree than of kind, in the sense that

investment expenditures, that could be safely deemed as independent of current income, would not presumably be unaffected by a long depression. He insisted that though *"in a sufficiently short period all investment is autonomous"*[2], no investment expenditure is independent of output when the latter is measured over a sufficiently long time interval[3].

Harrod's remarks, as well other remarks made in the same period, aimed more at qualifying the notion of autonomous investment than at casting doubts on the adequacy of the twin-notion of induced investment; still, important doubts can be raised, and were raised e. g. by Keynes[4], on the ground that the high sensitivity of large components of investment expenditure to current results is exaggerated to the point of explaining such expenditures *exclusively* in terms of these results. If we confine our attention to output expectations (which are most relevant in the case of induced investment[5]) the above critique can be interpreted in the sense that attention is only payed to expectations regarding the immediate future, as determined by current results; output expectations regarding the more distant future are completely ignored, presumably in view of their strong uncertainty[6].

[2] R. Harrod, "Notes on Trade Cycle Theory", *The Economic Journal*, June 1951, p.267.

[3] Nor is the validity of Harrod's remark necessarily related to differences in durability. Consider for example autonomous investment typically related technical progress; for any given flow of new ideas, the flow of such investment expenditures must bear some relation to the size of the economy, as measured particularly by the size of its capital stock; and capital stock is affected by output in the long-run. The point has important implications and will be taken up again.

[4] J. M. Keynes, *The Collected Writings of J. M. Keynes*, edited by D. E. Moggridge and E. Johnson, London, Macmillan, 1971 ff., vol.XIV, pp.152-3.

[5] In so doing we abstract from price expectations (including the prices of inputs) and from technological expectations (to be considered among the determinants of autonomous investment).

[6] The following quotation lends support to this interpretation, at least as far as Harrod is concerned.

"*You think I am wrong in making investment a function of current growth only. Granted. Suppose only half were governed by current growth, the rest by long period planning. My theory is substantially intact. It remains true that the growth of consumption cannot slow down without producing a great recession; but in this case the recession would only have to be such as to reduce savings to half their usual level. Personally I believe by far the greater part of investment rests on an immediate prospect of an increase of demand. People do not build new factories for use some years hence nor houses that will remain unwanted. Why should they? They increase equipment at the last feasible moment to save interest. Moreover if you try looking more than a year or so ahead everything becomes so violently uncertain.*" R. Harrod, letter to J. M. Keynes, in J. M. Keynes, *The Collected*, op. cit., p.175-6. This passage was brought to my attention by an unpublished paper of Prof. Heinz Kurz. One may observe, in passing, that Harrod's argument rests on the implicit assumption that autonomous investment has a sufficient degree of persistence over time. By

It should be noticed that if we introduce these expectations into the analysis also the usual notion of autonomous investment must be modified. The existence of components of investment expenditure (typically, innovation expenditure) bearing little or no relation to current levels of activity and rates of capacity utilization, by no means implies that such expenditures bear no relation with output expectations in general.
For the above reasons the definitions of autonomous and induced investment used in the present paper are more general than the usual ones. The label autonomous investment is used here to indicate a flow of expenditure which is independent of current rates of capacity utilization; by contrast, induced investment is responsive to current (output) results.

1.2.

In the real trade cycle models we are concerned with induced investment depends on current results and nothing else. This gives rise to a misleading alternative choice between dynamically stable or unstable types of adjustment between output and capacity. The choice for parameter values leading to dynamic instability, suitably supplemented with the introduction of floors and ceilings and/or of non-linearities, brings about what is here called the "automatic" character of the trade cycle. As long as fixed capital assets are of finite duration, and there are lower barriers to the fall of consumption and investment during depression, a recovery phase will necessarily follow a recession phase.
From this point of view, the difficulty lies not so much in explaining the lower turning point as such, but, as R. Goodwin insists, in explaining why the phase of depression is usually not as long as implied by the above mechanism. The problem is all the more evident with reference to long periods of sustained growth, when the troughs

sufficient degree of persistence we mean that the observed volatility in the flow of autonomous investment expenditure must be consistent with the idea that current realized results can be taken as a reliable guide to the near future. If this were not the case, expectations formulated on the basis of current results would be systematically disappointed by the effects of volatile autonomous expenditure on demand levels over the near future.

of the cycle may become short intervals in which the process of expansion loses momentum. The solution is to be found, in Goodwin's opinion, in the expendiditure for innovation in the guise of autonomous investment. According to this view, long periods of sustained growth tend to be associated with the upswings of technology-driven Schumpeterian long waves.
Although this Keynes and Schumpeter "marriage" may have its difficulties, some recently discussed by the present author[7], expenditure for innovation is probably an important part of the answer to the problem. Still, the fact that a positive level of innovation expenditure is consistent with widespread excess capacity in the existing lines of production should not be confused with the idea that the level of innovation expenditure is independent of expected future demand generally. Our definition of "autonomous" as "independent of current output" rather than "independent of output", allows for an influence of long-term output expectations on innovation expenditure. This is consistent, in a framework in which long-term expectations are assumed -as we shall do below- to be quite unresponsive to current output results.
A second difficulty in the application of trade-cycle theory to the analysis of long-period growth can be seen as symmetrical with respect to the one just outlined. Consider a long-period of relative stagnation in output and employment. Since stagnation is long-lasting, capacity has time to adjust to the level of output in most branches of production; the "automatic" character of the business cycle would then lead us to conjecture that the economy will soon experience a cyclical boom. Indeed, the automatic business cycle suggests that dynamic behaviour does not undergo important qualitative modifications during long-periods of fast growth as compared to long-periods of stagnation. The former would be those in which, thanks to the stimulus of autonomous investment, depressions are shorter and milder, while booms are longer and stronger; the trade cycle however would be at work in the latter as well as in the former.
The above feature seems to be at variance with economic "facts". One may ask, for example, why there appears to be no convincing evidence

[7]Cf. M. Caminati, "Technical Innovations, Life of Equipment and Effective Demand", *Political Economy*, vol.2, n.2, 1986.

for prosperity phases that could be intepreted as trade-cycle booms during the last fifteen years. Though we had instances of incipient recoveries, they did not lead to a proper take off; in other words, the ups and downs of economic activity in the last two decades can hardly be explained by the mechanism of Harrodian instability. If this occurrence is not merely accidental, it would seem that the "automatic" business cycle does not show up during long-period stagnations. In the present paper it is suggested that the above difficulty can be overcome by specifying the state of long-term expectations which determines the sensitivity of economic decisions to current results.

2. Endogenous cyclical growth with given long period expectations.

2.1.

As has been argued, the real models of the business cycle consider the investment decisions influenced by current realized results as being completely independent of factors not liable to be reduced to short term realized conditions. As Keynes observed, they come to depend on current realized results and nothing else.

In what follows we shall reverse this perspective and assume that long-term expectations are exogenously given. In so doing, we are probably going too far since long-term expectations are more often updated in the light of current experience. For this reason the above assumption will be qualified in due course. Our aim here is to emphasize the following related points:

(a) The sensitivity of investment decisions to current results cannot be convincingly specified without referring to a state of long-term expectations.

(b) It follows from (a) that long-term expectations have a definite influence on cyclical behaviour and long-run economic performance; more generally, the existence of persistent cycles is conditional on particular states of long-term expectations.

Statement (b) is proven here only in relation to a particular dynamic model, which borrows most of its basic formal structure from Kaldor's

1940 trade cycle model[8]. The following section deals with proposition (a); section 2.3 presents the formal model exhibiting properties (b); these properties and the methodological underpinnings of the model are informally discussed in the final section of part 2.. Part 3. contains an outline of the main mathematical proofs.

2.2.

The shape of the investment function proposed by Kaldor in his 1940 trade-cycle model was later interpreted by him in terms of volatility of expectations triggered by current results[9]. A cyclical endogenous dynamics for overall capital stock and output is obtained by Kaldor in this model. That a cyclical dynamics can also be obtained for relatively constant long-term expectations may seem, at first sight, rather dubious. In the attempt to clarify this point, it is probably best to start with a few definitions and assumptions.

With the term "investment plan" we refer below to a time oriented schedule of investment decisions made by a single firm. Each investment plan i available to firm j generates a time path of total returns Rij(t), which is in general unknown to the firm concerned. If Sj is the set of all investment plans available to firm j, let Rj be the set of all functions Rij(t), when i belongs to Sj.

With these definitions in mind, one can say that the influence of current results on investment decisions must depend on the extent to which they are allowed to influence the functions Rij(t), and thus the ordering on each Sj.

It is generally agreed that the current states of capacity and output

[8] N. Kaldor, "A Model of the Trade Cycle", *The Economic Journal*, vol. L, 1940, pp.78-92. The basic mathematical structure of Kaldor's model has been quite thoroughly investigated in the literature; cf. in particular S. Ichimura, "Toward a General Nonlinear Macrodynamic Theory of Economic Fluctuations", in K. K. Kurihara (ed.), *Post-Keynesian Economics*, London, Allen & Unwin, 1954; W. W. Chang and D. J. Smyth, "The Existence of Cycles in a Non-linear Model: Kaldor's 1940 Model Reexamined", *Review of Economic Studies*, vol. 38, 1971, pp. 37-44; H. R. Varian, "Catastrophe Theory and the Business Cycle", *Economic Inquiry*, 1979, pp. 14-28; F. Cugno and L. Montrucchio, "Stability and Instability in a Two Dimensional Dynamical System: A Mathematical Approach to Kaldor's Theory of the Trade Cycle", in G.P. Szego (ed.), *New Quantitative Techniques for Economic Analysis*, New York, Academic Press, 1982.

[9] N. Kaldor, "The Relation of Economic Growth and Cyclical Fluctuations", *The Economic Journal*, vol. LXIV, n. 253, pp. 53-71.

are likely to affect the ordering on Sj, in as far as they are bound to affect at least the initial part of some relevant Rij(t). Since most economic activities are time-consuming and are characterized by important elements of continuity, the observed rate of change of economic variables tends to be constrained within relatively narrow bounds. This implies that the current state of output must affect investment decisions even if entrepreneurs do not take the present as a guide to the distant future; the approach followed here separates the influence of current results on expectatons regarding the immediate future from the further influence they may exert on long-term expectations in general.

To make this simple idea more precise, let us assume a world of rigid prices, where monetary policy intervention holds the rate of interest constant. The firm planning horizon [0,T] is divided into two parts, the immediate future and the distant future. The two intervals are so defined that the following conditions[10] hold true:

(i) The sensitivity of output expectations (expectations at time t=0 about output to be produced at any date in the interval) to changes in current output and their total sensitivity to simultaneous changes in other determinants of expectations are of comparable size only in so far as the immediate future is concerned.

(ii) In addition, the ratio between the sensitivity of output expectations concerning the distant future to changes in current output, and the same sensitivity concerning expectations about the immediate future is, by assumption, sufficiently small. This assumption reflects the idea that when reliable exogenous information about the distant future is available, current output affects only very slowly, if at all, long-term output expectations, so that these can be taken as given exogenously, at least in a short-run[11] analysis[12].

---

[10]In mathematical terms, let $q^e_j(t)$ $t\in(0,T]$ be the output firm j expects -at time 0- to be willing to produce over its planning horizon of length T. Current output $q_j(0)$ can always be thought of as an argument of each of the -assumedly differentiable- expectation functions $q^e_j(t)=g_t()$, with $t\in(0,T]$. (Notice that the $g_t()$'s needn't have the same arguments.) The two above conditions can be expressed as follows. There exists $t_1>0$ such that the absolute value of the partial derivative $dg_t()/dq_j(0)$ is of comparable size relative to the Euclidian norm of the total derivative Dg_t if $0<t<t_1$. There exists $t_0>t_1$ s.t., if $t_0<t<T$, then $|dg_t()/dq_j(0)|$ is sufficiently small relative to $|Dg_t|$ and also relative to $|dg_\tau()/dq_j(0)|$, where $\tau<t_1$. The function $q^e_j(t)$ $t\in[t_0,T]$ is defined as the long-term expectation by firm j.

[11]Here the term short run does not refer to the Marshallian short run, although, like the latter,

Output expectations concerning the distant future (output one expects to be willing to produce over the distant future in the absence of capacity constraints) are defined here as "long-term (output) expectations". Our definition contrasts with Keynes's notion of a state of long-term expectation, which refers to the psychological expectation about the *determinants* of the prospective yields over the *whole life* of an investment project. To stress the difference, we remark how Keynes's notion refers to the *arguments* of an expectation function whose image is in yield space; the label long-term expectation refers here to the ***image*** of an expectation function in output space.

Still, the crucial claim has not yet been justified, namely that investment decisions may be highly responsive to current results even if these results are not allowed to alter long-term expectations. We cannot simply resort to Harrod's extreme argument that the distant future is *so violently uncertain*[13], for, in the absence of any further relevant information, one cannot see how the influence of current results on long-term expectations could be negligible, as assumed before. We shall therefore adopt a less extreem, hopefully more realistic point of view. Current results are highly relevant to form reliable expectations about the immediate future,but they are only slightly relevant where long-term expectations are concerned. When other subjectively-more-relevant sources of information about the distant future are available, the influence of current results on long-term expectations becomes negligible. Nonetheless, expectations about the immediate future are definitely more reliable than expectations about the distant future. This implies that, quite independently of and in addition to discounting, returns $R_{ij}(t)$ concerning the immediate future have a relatively larger weight in the choice of the profitable

---

it can not be identified with a definite interval of calendar time. It is defined only in terms of the relative reaction speed of the variables involved.

[12]Conditions under which the separation of the short-run from the long-run dynamics does not lead to serious mistakes can be rigorously expressed in the case of linear dynamic models. Cf. H. A. Simon and A. Ando, "Aggregation of Variables in Dynamic Systems", *Econometrica*, vol. 29, April 1961, where the behaviour through time of nearly-decomposable linear systems is first interpreted in terms of short run and long run. The above separation is also feasible under conditions regarding the parameter matrix that are less stringent than those indicated by Simon and Ando; see L. Boggio, "The Stability of Production Prices in a Model of General Interdependence", where such conditions are used for local stability analysis within a non-linear model.

[13]See above, footnote 6.

investment plan $S^*_{ij}(t)$ than returns concerning the more distant future. Thus, with given long-term expectations, the chosen $S^*_{ij}(t)$ shows a concentration of investment projects in the very early phase of the planning period if current activity levels are high (in accordance with the well known rule of harvesting till the sun shines), and a postponement of investment projects if current activity levels are relatively low. In this way, the current rate of accumulation can be well above or below the expected rate of long-term growth. Still, and here is a crucial point, the expected rate of long-term growth (if used as a proxy for long-term expectations) contributes to determine the current rate of accumulation, and the divergence between the two cannot grow indefinitely. The implication is that within the present economic scenario the sensitivity of current investment decisions to current results is conditional upon the given state of long-term expectations.

The approach followed is not so inaccurate whenever long-term expectations are rooted in some confidently relied upon exogenous knowledge or belief. This means here that individuals think they have reliable information about the likely values of the variables that are more relevant, in their view, to form long-term (output) expectations. Current results (concerning output) are only slightly relevant in this respect so that, although observable, they have in such conditions a minor influence on long-term expectations.

The same approach runs into problems when economic agents become less confident in their exogenous knowledge or belief (in the sense explained above), since in the presence of poor information about the more relevant variables, the less relevant ones, if known with certainty, may come to play an important role. Long-term expectations become more exposed to the influence of current results. It may be worth it to add that such expectations would be barely reliable, even to such an extent that they play little or even no role in investment decisions. The less entrepreneurs trust their predictions about the distant future, the more they must rely on myopic rules for decision making, based on the expected returns over the early life of investment projects[14]. Eventually, as Harrod insisted with Keynes, they

[14]Cf. R. C. O. Matthews, *The Trade Cycle*, Disgwell Place, J.Nisbet, Cambridge, Cambridge University Press, 1959, pp.44-5.

come to make their investment decisions in the light of reliable expectations about the immediate future (based on current results) and nothing else.

To sum up, current results, together with the exogenously given long-term expectations induce an ordering on S_j for all j, and determine the choice of an optimal $S^*_{ij}(t)$ for all j. We shall only be concerned with the aggregate expression of the above relationships; moreover we are not so much interested in specifying the desired aggregate investment plan $S^*(t)$ over the whole planning period, but rather in current aggregate investment (namely, $S^*(0)$). As a further simplification, the expected average growth rate (π) over the distant future will be assumed to synthesize the state of long-term expectations.

2.3.

The demand for investment goods comes from investment decisions A, which are independent of current output, and from decisions induced by the state of output and capacity. As we shall see, both decisions also depend upon the state of long-term expectations π. Thus we have,

(1) $I=I(Y,\ K,\ \pi)+A$, with $I'_Y>0$, $I'_K<0$, $I'_\pi>0$.

As is well known, the influence of activity levels and of capital stocks on the demand for investment is interpreted by Kaldor in terms of the incentive to invest determined by current profits. The non linearity of his investment function $I=I(X,K)$, where activity levels are measured in terms of employment (X) rather than income, is explained as follows. When activity levels are low with respect to the capital stock a rise in activity and hence in profits has a small impact on investment because of undesired excess capacity. The same is true when activity levels are very high, since increasing construction and borrowing costs imply that profits do not rise in line with activity[15]. The rising construction costs are to be explained with the declining marginal productivity of

[15]The increasing difficulty met by entrepreneurs in obtaining credit is also mentioned by Kaldor in this context; this, however, seems to have more to do with realized investment than with desired (ex-ante) investment.

labour in the face of short-run capacity constraints. The interpretation of the upper non linearity of Kaldor's investment function in terms of construction costs seems to require one of the following assumptions (or a mix of them)[16]. Smooth substitutability between capital and labour with constant commodity prices and wages; capacity constraints are more stringent in the capital-goods industries, and a smooth relative increase in the price of capital goods occurs as activity levels keep rising beyond certain levels (capacity constraints must here curb desired rather than realized investment).

Turning now to our measure of activity levels in terms of output, it is easy to see how an alternative interpretation of Kaldor's investment function can be given in terms of demand rather than profits, which amounts to adopting a non-linear stock-adjustment principle. Although we assumed constant prices, wages and a constant interest rate, we can avoid placing great emphasis on smooth substitutability (between capital and labour) in order to find solid grounds for the declining influence of output on investment at high activity levels (for a given capital stock). This is easily achieved if we consider the influence on investment decisions of non-linear borrowing costs, associated with Kalecki's principle of increasing risk[17].

Aside from these amendments and additions, investment function (1) differs from Kaldor's in one essential respect: the existence of one more independent variable, namely, π. As long as we hold π fixed, we can formalize the above properties assuming that I() is twice continuously differentiable and that I'_Y first increases and then decreases in Y, for a given K. Things get more complicated if we allow for parametric changes in π, since our main point is that I'_Y depends on π; but, again, this relationship is non-linear, as is argued below.

Let us consider how $I''_{Y\pi}$ changes with parametric increases of π at given levels of Y and K (see fig. 1). If the given Y is sufficiently low $I''_{Y\pi}>0$ no matter how high the level of π is. The idea here is that with

[16] The stress on increasing construction costs raises a further complication within Kaldor's model; labour productivity and profits must increase (at given prices), as the capital stock gradually increases when the system is in short-run stable effective-demand equilibrium at high levels of activity. Kaldor's implicit assumption is that the above positive effect of accumulation on investment is not sufficient to counteract the negative effect due to productive capacity rising faster than output and profits.

[17] M. Kalecki, "The Principle of Increasing Risk", *Economica*, 1937.

very unfavourable current results, the current rate of accumulation is always below the level at which the upper constraints imposed by capacity and increasing risk[18] start to bite. This is related to the postponement of investment projects alluded to in the previous section. At higher levels of Y we still have, initially (i. e. for low π), $I''_{Y\pi}>0$, but for a sufficiently large π, investment demand is eventually pushed up into the region where the above-mentioned upper constaints increasingly bite, so that $I''_{Y\pi}<0$. We now observe that the grounds for the upper non-linearity of the function $I(Y,K,\pi)$, for given K and π, are quite different depending on the level of π. For a sufficiently high π the sensitivity of investment demand to increases in output (i. e. I'_Y) is eventually curbed -as Y increases- only by borrowing and construction costs. For lower levels of π (i. e. for less optimistic long-term expectations), instead, I'_Y is eventually curbed by the increasing divergence between the current rate of accumulation I/K and the expected rate of long-term growth π. This constraint acts -at such lower levels of π- before the other upper constraints (borrowing and construction costs) start to bite; the lower π, the lower the level of Y at which the divergence between current growth and expected long-term growth is felt in its effects on I.

As suggested above, the distinction between fast and slow variables allows us to carry out the analysis for an exogenously given level of π; thus π is delt with, in what follows, as a parameter. In this case expression (1) boils down to

(1.b) $I={}_{\pi}I(Y, K)+A$, with ${}_{\pi}I'_Y>0$, ${}_{\pi}I'_K<0$;

to simplify our notation we shall often omit parameter π, unless it is strictly necessary.

From the view-point of economic interpretation, the main difference between (1.b) and Kaldor's investment function is as follows. While the demand for investment is postively affected by the level of output at a given capacity, the non-linearity of this relationship is not necessarily explained by rising excess capacity (on the one hand) or by increasing borrowing and construction costs (on the other). A further cause for

[18] We avoid reference to changes in the rate of interest, on the assumption of an accomodating monetary policy.

the non-linearity of the investment function, which possibly acts before the other one does, is that entrepreneurs must be reluctant to undertake rates of accumulation I/K that happen to be more and more at variance with the expected long-term growth π.

At the given and constant price structure, aggregate output Y results from the production of investment goods I_0 and from the production of consumption goods C_0; this yields the definition $Y\equiv C_0+I_0$.
By following a clear-cut Keynesian view of the relationship between savings and investment, we assume here that investment decisions are always realized, or, as Kaldor puts it, that *ex post saving is adjusted to ex ante investment*; this is formalized as

(2) $I_0=I$.

Assuming a constant rate of exponential depreciation δ we obtain

(3) $K=I-\delta K$.

Kaldor's savings function is now introduced, where savings depend positively on income levels, and negatively upon the level of capital stock[19].

(4) $S=S(Y, K) \quad S'_Y>0 \quad S'_K<0$.

Desired savings are always strictly lower than income and increase with the latter without (upper) bounds; taking also into account the upper non-linearity of the investment function $I(Y, K, \pi)$, for given K and π, we are led to formulate the following plausible Kaldorian assumption.

(5) Whatever the level of autonomous investment, savings are lower (higher) than investment if gross-output is sufficiently small[20] (large).

[19]The idea is that real wealth has a positive influence on consumption.

[20]Recall that, by definition, gross output cannot be lower than autonomous investment.

If, following Cugno and Montrucchio[21], we assume that I(Y,K) and S(Y,K) are homogeneous in the first degree in Y and K, using (1) and (4) we obtain $I/K=\pi I(Y/K,1)+A/K$, $S/K=S(Y/K,1)$. For simplicity of notation we define $C_0/K\equiv x$, $A/K\equiv z$, $Y/K\equiv y$, $I(Y/K, 1)\equiv I(y)$.
In the framework of the present paper the second Kaldorian assumption on the savings and investment functions holds true if and only if long-term expectations are sufficiently optimistic. Mathematically the assumption is expressed as follows.

(6) For each $\pi>\pi^*$ there exist unique real numbers y_1,y_2, with $0<y_1<y_2$ such that $\pi I'_y=S'_y$ at $y=y_1$ and $y=y_2$. Moreover $I'_y<S'_y$ at y, if $0<y<y_1$ or $y>y_2$, while the opposite holds true if $y_1<y<y_2$.

For given K and π the above assumption defines an intermediate region of activity levels (Y) where the (marginal) propensity to invest is higher than the (marginal) propensity to save. If savings equal investment at a point in this region, this equality can be preserved, after a small increase in activity, only through a simultaneous fall in autonomous investment. To make sure that this fall will suffice to do the job, we assume the following.

(7) Let y_1 and y_2 be the output/capital ratios defined in (6) for a given $\pi>\pi^*$. Then there exists $z_1>0$ sufficiently small s. t. $S(y)>I(y)+z$ for $y_1\le y\le y_2$, if $0\le z\le z_1$.

Although the responsiveness of induced investment to current demand depends on long-term expectations, an adequate level of current demand is still a necessary condition to obtain positive induced investment. Thus, whenever activity levels in the consumer-goods industries are infinitely small with respect to the overall capital stock, the existence of a positive flow of investment is conditional upon the existence of a positive flow of autonomous investment. This is formalized as

(8) If $x=0$, then $I(y)+z>0$ if and only if $z>0$.

[21]Cf. F. Cugno and L. Montrucchio, "Stability..." *op. cit.*.

Using the definition of Y together with (2), and recalling that $I'_y<1$, we can define the investment/capital ratio as a function of x and z[22]:

$$(9)\quad I/K={}_\pi i(x,\ z)\quad {}_\pi i'_x>0\quad {}_\pi i'_z>0.$$

Likewise, also taking into account (9),

$$(10)\quad S/K=S({}_\pi i(x,\ z)+x,\ 1),$$

which allows us to define the function ${}_\pi s()$ as:

$$(11)\quad S/K={}_\pi s(x,\ z)\quad {}_\pi s'_x>0\quad {}_\pi s'_z>0.$$

When the demand for consumer goods C=Y-S differs from the production C_0, consumers claims are disappointed; this drives producers to change the degree of capacity utilization, since they do not adjust their capital stock as fast as they adjust their output. Indeed, in accordance with the Marshallian distinction between short run and long run, changes in the output of consumer goods induced by demand take place faster than the capital stock is allowed to change as a result of current investment. Thus C_0/K increases, remains constant or decreases over time, depending on whether C is larger than, equal to, or lower than C_0. Following Cugno and Montrucchio[23], this is formalized as:

$$(12)\quad (dC_0/dt)/C_0\ -\ (dK/dt)/K\ =\ \mu(C/K\ -\ C_0/K),$$

where μ is a positive parameter.

So far nothing has been said about autonomous investment A. By definition the level of A is not decided with reference to current

[22]Using the definition of Y together with (2), we can define the functions g: $R_+ \to R_+$ and h: $R^2_+ \to R_+$ such that $g(y)=y-I(y)=x+z=h(x,z)$. Since $I'_y<1$ the function g^{-1} exists, and the composite function $g^{-1}h$ is such that $g^{-1}h(x,z)=y$. The induced-investment function c: $R^2_+ \to R_+$ is such that $c(x,z)=g^{-1}h(x,z)-h(x,z)=I(x+z+c(x,z))$. It follows that $c'_x=c'_z$ on R^2_{++}. We can eventually define: $I/K={}_\pi i(x,z)={}_\pi c(x,z)+z$, where ${}_\pi i'_x>0$, ${}_\pi i'_z>0$. It is easy to see that $i'_x=c'_x$ and $i'_z=c'_z+1=i'_x+1$.

[23]Cf. F. Cugno and L. Montrucchio, "Stability.." *op. cit.*, p.267.

output levels, or to current changes in output, in so far as it is not directly affected by the current state of demand. Nevertheless it would be highly unrealistic to assume that autonomous expenditures do not bear any relation to the size of the economy as emphasized by M. Kalecki and by J. Steindl after him[24]. Consider for example how the decision regarding plant capacity for the production of some new information technology would be influenced by the number and size of its potential users. The problem here seems to be to identify a macroeconomic variable related to the size of the economy in some long-run sense. On the basis of the empirical finding that capital stock is much less exposed to cyclical fluctuations than output, it could be claimed that capital stock is after all an appropriate candidate. Findings like the above should however be explained rather than simply taken for granted by a model of cyclical growth. Indeed, some trade-cycle models, and Kaldor's 1940 model is among them, do not produce that result and have been criticized for this inconsistency with stylized facts. In a model like ours, where production decisions are determined by demand only, but where investment decisions also depend on prior expectations about long-term prospects, there seems to be good reason for entrepreneurs to believe that fluctuations in capital stock will be less pronounced than fluctuations in output; thus, the influence of K on autonomous expenditure A has some rational foundation within our model. Still, autonomous investment is partly exogenous, in so far as it depends on technological and/or institutional factors. For a constant rate of technical progress there is a target ratio *A/K* which entrepreneurs want, on the average, to attain and which depends, *inter alia*, on the proportional destruction of capital caused by the prevailing forms and speed of technical change.

Autonomous expenditure is also influenced by long-term output expectations. To see why, suppose that the actual ratio A/K cincides, in the given initial conditions, with the target ratio *A/K*, while the expected long-term growth rate is π. Consistence between entrepreneurs' aims and expectations requires that they increase autonomous investment at the (proportional) rate π.

[24]Cf. M. Kalecki, "Trend and Business Cycles Reconsidered", *The Economic Journal*, vol. LXXVIII, n. 310, June 1968, pp. 263-76; J. Steindl, "Ideas and Concepts of Long Run Growth", *Banca Nazionale del Lavoro Quarterly Review*, vol. 34, March 1981, pp. 35-47.

Thus, autonomous expenditure is ruled by three factors: (i) the target ratio *A/K*=z^*, as determined in particular by technological expectations[25]; (ii) long-term (output) expectations; (iii) the gap between actual A/K and target *A/K* and the adjustment rule followed by entrepreneurs. $(dA/dt)/A$ is pushed more and more above (below) π, when the actual ratio A/K gets increasingly lower (higher) than the target ratio *A/K*. When the two ratios are equal, the proportional rate of change in autonomous expenditure will then reflect the state of long-term expectations[26] (fig. 2). The above assumption is formalized as

(13) $(dA/dt)/A=f(z)+\pi$, with $f'<0$; there exist $z>0$ s.t. $f(z)=0$.

In the above expression the target ratio *A/K* is implicitly defined $f^{-1}(0)$ and is constrained to be strictly positive. It is important to observe that, if the system state is not too far from equilibrium, the absolute value of f' should be expected to be relatively small, for the simple reason that autonomous investment is related to the size of the economy only in a long-run sense. Indeed, for A/K greater (lower) than *A/K* and thus $(dA/dt)/A<\pi$ ($>\pi$), any further increase (decrease) of A/K would be interpreted in the sense that K has temporarily departed from its expected long-term behaviour. Adjustments in the rate of growth of autonomous expenditure would be, as a consequence, relatively slow. This restriction imposed on the adjustment speed of $(dA/dt)/A$ is less significant when the flow of A becomes infinitely small. With a zero initial level of autonomous expenditure, any positive finite time rate of change A, however small, would lead to an infinite growth rate $(dA/dt)/A$. These remarks suggest that the relationship between $(dA/dt)/A$ and z is strongly non-linear for z close to zero. In particular, we are induced to formulate the following assumption.

(14) $\lim_{z\to 0} f(z)=\infty$.

[25] It should be noticed, in passing, that technological expectations may also contribute to shape long-term expectations as defined above.

[26] Although starting from Kalecki's idea that autonomous expenditure is influenced by the size of the economy, the above picture departs in some respects from Kalecki's view of the matter. Cf. M. Kalecki, "The Trend...", op. cit..

Reflecting the fact that the rate of growth of autonomous expenditure also depends on long-term expectations, $(dA/dt)/A=\pi$ if A/K=*A/K*, and the level of A increases through time if $A/K<f^{-1}(-\pi)$. $f^{-1}(-\pi)$ indicates the value of the ratio A/K such that autonomous expenditure is constant through time if the expected long-term growth is π. It is worth observing that if $\delta<f^{-1}(-\pi)$ a ratio $A/K \leq \delta$ cannot persist over time; the existence of an upward drift in the long-run growth path would already be explained by circumstances exclusively related to autonomous expenditure. This is not generally the case if $\delta>f^{-1}(-\pi)$.

After simple algebraic manipulations we obtain the following differential system:

$$(15)\quad \begin{aligned} (dx/dt)/x &= \mu[{}_{\pi}i(x, z) - {}_{\pi}s(x, z)] \\ (dz/dt)/z &= f(z) - {}_{\pi}i(x, z) + (\delta+\pi) \end{aligned}$$

2.4.

The formal analysis of the above model is left to the final part of the paper; here we present, in a very intuitive way, its main properties, with the aim of discussing some further issues of interpretation and relevance.

a) On the assumption that system (15) has a unique and unstable steady state position (x_e, z_e), and provided suitable restrictions are imposed on the speed of the adjustment of output to demand in the consumer goods industries, the endogenous variables x and z follow a path moving around (x_e, z_e), and converge to a so-called limit cycle. Akin to Kaldor's model, the existence of persistent cyclical behaviour (limit cycles) requires that -in equilibrium- the desired investment ratio I/K responds to a change in the output of consumer goods, and hence in C_0/K, faster than the desired savings ratio S/K. In mathematical terms we must have $i'_x-s'_x>0$ at equilibrium (x_e, z_e).
For a given δ, the condition is satisfied only if the value of π falls in

an "appropriate" interval[27]; indeed, the influence of π in this respect is twofold, since π affects both the sensitivity of induced investment to current results and the rate of growth of autonomous expenditure. In the third part of the paper we show how the joint action of the two influences is such that the steady state consumption ratio $C_0/K=x_e$ always increases with π (for a given δ).

If, starting from an "appropriate" value (meeting the condition), π keeps *increasing*, the difference $({}_\pi i'_x - {}_\pi s'_x)$ at each point in state space must also increase; still, the influence of the shift in steady state equilibrium must eventually prevail and make the equilibrium stable. The reason is that at sufficiently high consumption ratios x (and therefore income ratios y, since $y \geq x$) i'_x is constrained by borrowing and constuction costs.

Likewise, if π keeps *decreasing* -starting fom the same "appropriate" value- the steady state equilibrium eventually becomes stable. This can be easily seen in the following steps. (*i*) The propensity to invest i'_x is lower than the propensity to save s'_x if the activity ratio y declines below a lower bound depening on π; (*ii*) the equality between savings and investment (which must obviously hold in steady state) can only be obtained at low activity levels if the consumption ratio x is sufficiently low; (*iii*) the steady state consumption ratio x_e declines with a parametric fall in π; (*iiii*) the lower bound stated in (*i*) increases with a parametric fall in π.

Indeed, if long-term expectations are sufficiently pessimistic and hence π is sufficiently low, $({}_\pi i'_x - {}_\pi s'_x) < 0$ everywhere in the state space; persistent trade cycles cannot occur, no matter what the size of parameters other than π is.

The above features of the model suggest that the endogenous dynamics during long-period depressions may differ from the dynamics characterizing long-period expansions on qualitative and not only on quantitative grounds.

b) The steady state rate of growth g is here not necessarily equal to zero, as in Kaldor's 1940 article, nor is it exogenously determined

[27]In the context of a Kaldor model with exogenous growth rate, persistent cycles occur only if this rate falls in an appropriate interval (more generally, in an appropriate set of intervals). Cf. F. Cugno and L. Montrucchio, *op. cit.*, pp. 276-7.

through the growth rate of autonomous expenditure as suggested by Hicks, or as is the case for recent mathematical formulations of Kaldor's model. To emphasize this point we show that, with a (long-run) target ratio A/K and parameter values for π and δ that would make entrepreneurs happy with a flow of autonomous investment which falls short of replacing the capacity destroyed by depreciation (this is the case if $f^{-1}(-\pi)<\delta$), a positive g may still be obtained as a result of the endogenous dynamics.

c) A long-period theory of output has recently been defined by J.Eatwell to be one where long-term expectations are quite independent of short-term realized conditions, but are more closely related to economic and/or social fundamentals characterized by a relative degree of persistence[28]. Keynes, on the contrary, used the term long-period in a notional sense; a long-period level of employment, a magnitude not necessarily constant through time [29], is one which would be brought about if a state of expectation were

"to continue for a sufficient length of time for the effect on employment to have worked itself out so completely that there is, broadly speaking, no piece of employment going on which would not have taken place if the new state of expectation had always existed..."[30]

It would seem that our model is compatible with both definitions, but a close scrutiny reveals that this is not the case. For expectations to be relatively independent of current results they must be rooted in some exogenous items of knowledge which are believed by economic agents to be sufficiently persistent. If I believe I know long-term prospects, I must also believe that tomorrow I will hold the same belief. But if my grounds for rational judgement are ever changing, I would then hold a systematically falsified belief; and when I come to realize that my

[28]Cf. J. Eatwell, "The Long-Period Theory of Employment", *Cambridge Journal of Economics*, vol.7, 1983, pp.269-85.

[29]J.M. Keynes, *The General Theory of Employment,Interest and Money*, London, Macmillan, 1936, p.48, n.1.

[30]*Ibidem*, p.48.

expectations are not sustained by solid arguments, I will not disregard the information about the future conveyed by current results, however poor it may be. Thus, the relative weight of immediate prospects (as determined in the present framework by current results) and of long-term prospects in investment decisions is not independent of the degree of volatility of the latter. The investment behaviour considered in our model rests on the assumption of relatively firm long-term expectations and would not be consistent with highly volatile (long-term) expectations.

Quite in contrast to Eatwell's opinion, however, the economic magnitudes determined by a theory of output where long-term expectations stand still in the face of current results need not be independent of the endogenous process of adjustment[31]. This may be the case if one has in mind a situation where long-term expectations are held with certainty, so that current results have only a minor influence on investment decisions. The present paper refers to the less extreme and more realistic case in which long-term expectations, though not highly volatile, are held with uncertainty, and are for this reason less reliable than expectations about immediate prospects, as determined by current results.

d) After publication of *A Contribution to the Theory of the Trade Cycle* by J. Hicks consent grew regarding the idea that long-term growth and fluctuations could both be explained by the instability of the adjustment between output and capacity, if coupled with the idea that economic booms had to be curbed sooner or later by the ceiling of full employment. In this way, a theory of output incorporating the Keynesian principle of effective demand was reconciled with the traditional view that economic growth depends, in the long run, upon the supply of the scarce factors of production and upon their productivity. The above type of outcome can be generated by the present model only as a very particular case, following upon specific states of expectations. Thus, the rate of long-term growth g is, in general, lower than the rate of potential growth.

[31] Cf. J. Eatwell, *op. cit.*, p.275.

e) What has been argued in the previous pages has an important corollary which is worth stressing. The explanation of induced investment in terms of current results, and of current results only, is not fully consistent with the simultaneous imposition of an exogenous path on autonomous investment. One can hardly see how present and future levels of autonomous investment, be it directed towards technical progress or the provision of infrastructures, can be independent of any expectation regarding long-term growth. If this is true, a given long-term path for autonomous investment implies a state of expectation about long-term prospects which is bound to influence the sensitivity of induced investment to current realized results. Thus, pure-cycle models cannot be regarded as what is left after de-trending, inasmuch as long-term growth prospects may have a crucial influence on the prevailing form of endogenous dynamics. As has been argued, pessimistic long-term prospects may imply that there is no persistent regular cycle at all.

3. Equilibrium and stability analysis of the model.

3.1.

In order to solve (15), we specify the domain of definition of the functions i(), s() and f() and impose the standard smoothness conditions. To this purpose we introduce the open intervals E_π and E_δ representing the set of admissible long-term expectations and rates of depreciation, respectively. E_δ is a subset of R_+.

(16) For each $\pi \in E_\pi$ ${}_\pi i(x,z)$ and ${}_\pi s(x,z)$ are defined and continuous on R^2_+ and twice continuously differentiable on R^2_{++}; they depend smoothly on π. $f(z)$ is defined and continuous on R_+ and twice continuously differentiable on R_{++}.

Recalling that $i(x,z)=I(i(x,z)+x)+z$ and $s(x,z)=S(i(x,z)+x)$ we obtain

$i'_x=I'_y\cdot(i'_x+1)$ and $s'_x=S'_y\cdot(i'_x+1)$. Thus $i'_x=s'_x$ at (x,z) if $I'_y=S'_y$ at $y=i(x,z)+x$. Assumptions (5), (6), (7) and (8) can therefore be expressed as follows:

(17) For each $\pi\in E_\pi$ and each $z\in R_+$ we can find a number $N>0$ s.t. for any couple (x_1,z) for which $x_1>N$ the following condition $_\pi s(x_1,z)-{}_\pi i(x_1,z)>0$ holds true. For each $\pi\in E_\pi$ and each $z\in R_+$ we can find a number $x_2>0$ s.t. $_\pi s(x,z)<{}_\pi i(x,z)$ for any $0\leq x\leq x_2$[32] .

(18) For each $\pi>\pi^*$ there exist unique real numbers y_1, y_2, with $0<y_1<y_2$ such that $_\pi i'_x={}_\pi s'_x$ at (x,z) if (x,z) belongs to the union of the two separated sets $\{(x,z)\in R^2_{++}$ s.t. $_\pi i(x,z)+x=y_1\}$, $\{(x,z)\in R^2_{++}$ s.t. $_\pi i(x,z)+x=y_2\}$.

Assumption (18) tells us that the set of points (x,z) in R^2_{++} s.t. $i'_x=s'_x$, is the sum of two separated sets; each of them can be seen as the graph of a decreasing linear function, namely, $q_1(x)$ and $q_2(x)$ (see fig. 3). Linearity is shown as follows. graph $q_j(x)=\{(x,z)\in R^2_{++}$ s.t. $i(x,z)+x=y_j\}$ $j=1,2$. Thus graph $q_j(x)$ is a one dimensional submanifold of R_{++} and $(i'_x dx + i'_z dz + dx) = 0$ on graph $q_j(x)$ $j=1,2$. Recalling that $i'_z=i'_x+1$ we obtain $(dq_j(x)/dx)=-(i'_x+1)/i'_z=-1$ $j=1,2$. These graphs identify on R_{++} the three separated open regions Q_1^+, Q_2^+ and Q_3^+: $Q_1^+=\{(x,z)\in R^2_{++}$ s.t. $x<q_1^{-1}(z)\}$, $Q_2^+=\{(x,z)\in R^2_{++}$ s.t. $q_1^{-1}(z)<x<q_2^{-1}(z)\}$, $Q_3^+=\{(x,z)\in R^2_{++}$ s.t. $q_2^{-1}(z)<x\}$. We also define the sets Q_1, Q_2 and Q_3 by replacing R^2_+ for R^2_{++} in the definitions of Q_1^+, Q_2^+ and Q_3^+ respectively. Assumptions (7) and (8) are now expressed as follows.

(19) Let y_1 and y_2 be the output/capital ratios defined in (18) for a given $\pi>\pi^*$. There exists $z_1>0$ sufficiently small s. t. for each $z\in[0,z_1]$ $s(x,z)>i(x,z)$ if $q_1^{-1}(z)\leq x\leq q_2^{-1}(z)$.

(20) $i(0,z)>0$ if and only if $z>0$.

Since $s'_z= S'_Y\cdot i'_z$, as long as the marginal propensity to save is positive and lower than one we can impose:

[32]Cf. F. Cugno and L. Montrucchio, *op. cit.*, p.268.

(21) $_\pi i'_z > {}_\pi s'_z$ everywhere on R^2_{++}, for all $\pi\in E_\pi$.

In R^2_+ we define the following sets:

$$D = \{(x,z)\in R^2_+ \text{ s.t. } i(x,z)=\delta\}$$
$$B = \{(x,z)\in R^2_+ \text{ s.t. } i(x,z)-\delta=f(z)+\pi\}$$
$$P = \{(x,z)\in R^2_+ \text{ s.t. } i(x,z)=s(x,z)\}$$

D is the set of points in R^2_+ where the capital stock is constant, since gross investment equals depreciation. It is the graph of the strictly decreasing function $d(x)$, twice continuously differentiable on the interior of its connected domain, as a consequence of (16). Assumption (20) assures us that $d(x)$ is defined and strictly positive at $x=0$. As shown in fig.4, set D divides R^2_+ into two half-regions: $D_1=\{(x,z)\in R^2_+$ s.t. $z>d(x)\}$, where the capital stock increases, $D_2=\{(x,z)\in R^2_+$ s.t. $z<d(x)\}$, where the capital stock decreases over time.

B is the set of points where A/K is constant. It is the graph of the strictly decreasing function $b(x)$, twice continuously differentiable on the interior of its connected domain as a consequence of (16). A/K increases on set $B_1=\{(x,z)\in R^2_+$ s.t. $z>b(x)\}$ and decreases on set $B_2=\{(x,z)\in R^2_+$ s.t. $z<b(x)\}$. Since $f'<0$, as assumed in (13), it follows that $b'(x)>d'(x)$, wherever they are both defined (see fig. 4).

With a view to make the present analysis more significant, it is useful to consider the case where the level of z sustaining a constant flow of autonomous expenditure at given π, i. e. $z=f^{-1}(-\pi)$, is lower than the level of z sustaining, at $x=0$, a gross investment to capital ratio i() equal to the rate of depreciation δ; this amounts to assuming that the existence of a positive rate of long-term growth is not merely explained by conditions affecting autonomous expenditure. When the above holds true $d(0)>b(0)$ and $\{B\cap D\}$ is the point $(x=d^{-1}(f^{-1}(-\pi))$, $z=f^{-1}(-\pi))$ (see fig. 4). As π keeps increasing, we eventually obtain $d(0)<b(0)$, while $\{B\cap D\}$ becomes empty.

P is the set of points in R^2_+ where C_0/K is constant; it is the graph of the function $p(x)$. Before we can say something more about set P, further restrictions must be imposed on functions i(), s() and on the differential system (15).

(22) For each $\pi \in E_\pi$ s.t. $\pi > \pi^*$ there exists $(x_s,z_s) \in Q_2^+$ s.t. $i(x_s,z_s)=s(x_s,z_s)$.

(23) For each $\pi > \pi^*$ let $z(\pi)$ be Sup. $\{z \in R_{++}$ s.t. $s(x,z) > i(x,z)$ when $q_1^{-1}(z) \leq x \leq q_2^{-1}(z)\}$; let also $Z=\bigcap_{\pi>\pi^*}[0,z(\pi)]$. There exists $z_2 \in Z$ s.t. $[(s'_x \cdot i'_z)-(i'_x \cdot s'_z)+f' \cdot (i'_x - s'_x)]>0$ for each $\pi \in E_\pi$ and each (x,z) in the Cartesian product of the sets $\{x \in R^2_{++}\}$, $\{z \in R_{++}$ s.t. $z>z_2\}$.

Taken together, assumptions (17), (18), (19) and (22) imply that the equality $i(x,z)=s(x,z)$ holds at one point in $Q_1 \cap \{(x,z) \in R^2_+$ s.t. $z=0\}$ and at no other point (x,z) s.t. $z=0$. This implies that $p^{-1}(0)$ is well defined and $p(x)>0$ if $x>p^{-1}(0)$; moreover, $p^{-1}(0)>0$ as a consequence of (17).
With the addition of assumption (23) we can rule out the possibility that an equilibrium for the differential system (15) is a saddle point. More generally, the slope of $b(x)$ is lower than the slope of $p(x)$ wherever they are both defined, or, in other words, set B can only intersect set P from above (see fig. 4).
This observation completes the preliminaries necessary to state the relevant properties of the function $p(x)$ and of its graph P (see fig. 3 and 4). $p(x)$ is defined and continuous on $\{x \geq p^{-1}(0)>0\}$ and (twice) continuously differentiable on $\{x>p^{-1}(0)\}$. $p(x)$ increases at x if $(x,p(x)) \in Q_1^+ \cup Q_3^+$ and decreases at x if $(x,p(x)) \in Q_2^+$; for any arbitrarily-large real number M there exists x_M s.t. $p(x)>M$ if $x>x_M$ (as a consequence of assumption (17)). The following proposition can now be easily proved:

(24) If assumptions (13), (14) and (16) to (23) hold true, there exists $(x_e,z_e) \in R^2_{++}$ s.t. $z_e=p(x_e)=b(x_e)$. P and B have no other intersection on R^2_+.

To make sure that the equilibrium point (x_e,z_e) can be unstable for appropiate parameter values we assume the following:

(25) Let $\pi_0 \in E_\pi$ and $\delta_0 \in E_\delta$; at the unique equilibrium point $(x_0,z_0)=(x_e(\pi_0,\delta_0),z_e(\pi_0,\delta_0))$ $i'_x(x_0,z_0)>s'_x(x_0,z_0)$.

By considering the linear part of (15) at (x_e,z_e) the following

proposition is obtained:

(26) The equilibrium point (x_e,z_e) of (15) is unstable if and only if $\mu \cdot (i'_x - s'_x)x_e > (i'_z - f')z_e$ at (x_e,z_e).

It is easy to see that the inequality in (26) is satisfied, for a sufficiently high μ, if (25) holds true. In such conditions, application of the Poincaré-Bendixon theorem yields the existence of a limit cycle in the phase space of (15).

3.2.

System (15) can be shown to be structurally stable in the sense that if the functions i(), s() and f() meet assumptions (13), (14), (16) to (23) and (25) for parameter values π_0 and δ_0, then functions *i(), s(), f()* sufficiently close to i(), s(), f() in the appropriate perturbation space, will also meet the same assumptions for parameter values π and δ sufficiently close to π_0 and δ_0 in the parameter space $E_\pi x E_\delta$.[33]
The study of the set of parameter values where the system becomes structurally unstable has direct economic implications. For the sake of simplicity such a study is carried out here for a given value of the parameter μ and in particular for $\mu_0=+\infty$. Using the instability condition in (26), it is easy to see that, as long as $\pi>\pi^*$, the equilibrium point (x_e,z_e) in the phase space, corresponding to the critical point (π,δ,μ_0) in the parameter space $E \equiv E_\pi x E_\delta x \{\mu_0\}$, must belong to the set {graph ${}_\pi q_1(x)$ U graph ${}_\pi q_2(x)$} defined in (18). Indeed, it can be easily shown that the set of critical points $S(q_1)=\{(\pi,\delta,\mu) \in E$ s.t. $\pi>\pi^*$, $\mu=\mu_0$ and $(x(\pi,\delta,\mu_0),z(\pi,\delta,\mu_0)) \in$ graph ${}_\pi q_1(x)\}$ is a smooth one-dimensional manifold. The smooth one-dimensional manifold $S(q_2)$ can be defined analogously. We want to give a qualitative description of the non-contiguous sets $S(q_1)$ and $S(q_2)$ (see fig. 5).
At the equilibrium point (x_e,z_e) corresponding to the point (π,δ,μ_0) in parameter space the following conditions hold true:

[33]The main difficulty concerns the proof of stability of assumptions (23) and (25). It can be obtained following the hints given by F. Cugno and L. Montrucchio, *op. cit.*, pp. 273-275.

(27) $\quad i(x_e,z_e) = s(x_e,z_e) = f(z_e) + \pi + \delta;$

By differentiating (27), we can express dx and dz as functions of $d\pi$ and $d\delta$ as follows:

$$dx = \{[(i'_x-s'_x)+s'_z i'_\pi - i'_z s'_\pi - f'(i'_\pi - s'_\pi)]d\pi \quad + (i'_x-s'_x)d\delta\}\cdot$$
$$\cdot[1/(s'_z i'_x - i'_x s'_z)+f'(i'_x-s'_x)]$$

$$dz = -\{[(i'_x-s'_x)(1-i'_\pi)+i'_x(i'_\pi-s'_\pi)]d\pi \quad + (i'_x-s'_x)d\delta\}\cdot$$
$$\cdot[1/(s'_x i'_z - i'_x s'_z)+f'(i'_x-s'_x)].$$

If we define the coefficients multiplying $d\pi$ and $d\delta$ in the expression for dx (dz) as α (σ) and β (θ) respectively, we can write:

(28) $\quad dx = \alpha d\pi + \beta d\delta,\ \alpha>0,\ \beta>0;\ dz = \sigma d\pi + \theta d\delta,\ \sigma<0$ if $i'_z \geq s'_x$, $\mathrm{sign}(\theta)=\mathrm{sign}(i'_x-s'_x)$.

The restrictions for β and θ are direct consequences of assumptions (21) and(23); to derive the restrictions for α and σ we need some supplementary algebraic remarks. First recall that ${}_\pi s(x,z)=S({}_\pi i(x,z)+\pi, 1)$; hence, after defining ${}_\pi i(x,z)+\pi=Y/K\equiv y$, we obtain:

(29) $\quad s'_x=S'_y(i'_x+1) \qquad s'_z=S'_y i'_z \qquad s'_\pi=S'_y i'_\pi.$

Substitution of the right-hand sides for the left-hand sides of (29) in the expressions defining α and σ completes the proof.
It may be worth it to stress that $dx/d\delta$, for a given π, is positive; that is, the x coordinate of equilibrium point (x_e,z_e) increases in δ for given π and μ.
If $(\pi,\delta,\mu_0) \in S(q_1)US(q_2)$ the following condition must also hold at the corresponding equilibrium point.

(30) $\quad i'_x = s'_x.$

At this point $dz/d\delta=0$. Differentiating (30) and substituting for dx and dz from expressions (28) -suitably simplified by taking into account

(30)- and finally substituting from (29) we can express the following differential condition holding on $\{S(q_1)US(q_2)\}$.

(31) $\quad i'_x(1-S'_y)^2[i'_z i''_{x\pi} \quad - i'_\pi i''_{xz}]d\pi \quad + (i''_{xx}-s''_{xx})\alpha d\pi \quad + (i''_{xx}-s''_{xx})\beta\, d\delta$
$= 0$

The economic interpretation of function i() suggested in sections 2.2. and 2.3. of the present paper allows us to impose the following restriction on its first and second partial derivatives:

(32) $\quad i'_z i''_{x\pi}-i'_\pi i''_{xz}>0$ at the equilibrium point (x_e,z_e) corresponding to each $(\pi,\delta,\mu_0)\in S(q_1)$; the reverse holds true for $(\pi,\delta,\mu_0)\in S(q_2)$, provided π is sufficiently large.

Recalling that $(i''_{xx}-s''_{xx})>0$ (<0) at each point (x,z) in graph ${}_\pi q_1(x)$ (graph ${}_\pi q_2(x)$), we are eventually in a position to give a qualitative description of the bifurcation diagram (see fig. 5).

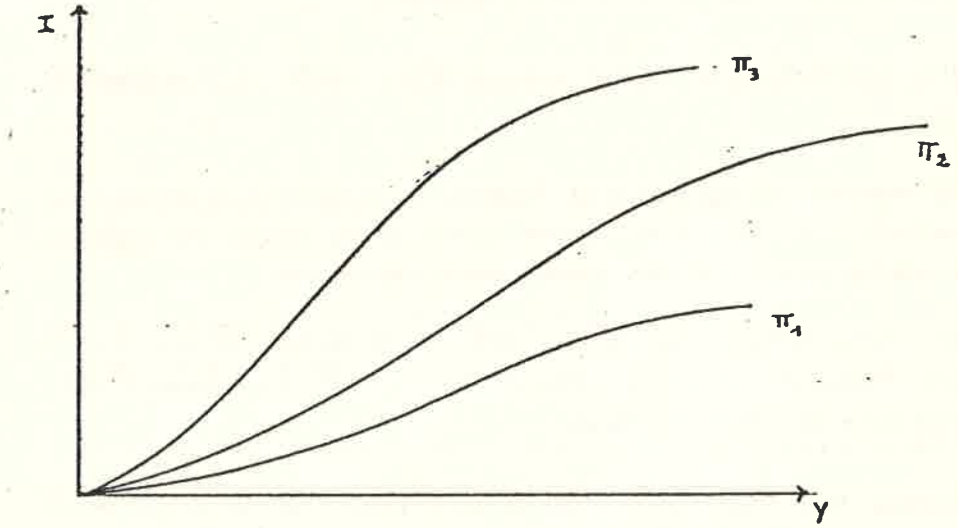

Fig. 1

Graphs of the investment functions $_{K\pi}I(Y)=I(Y,K,\pi)$ traced out for a given and constant K and for alternative, fixed levels of π, $\pi_1<\pi_2<\pi_3$.

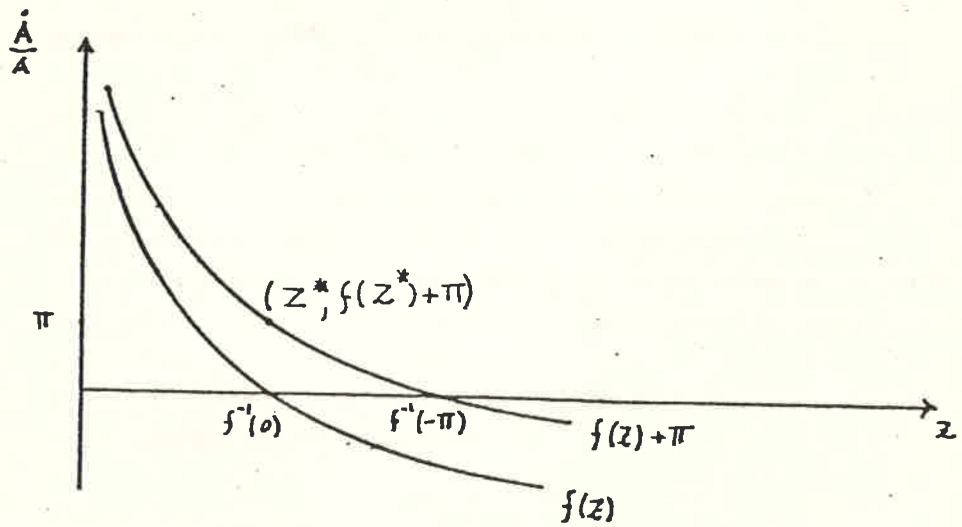

Fig. 2

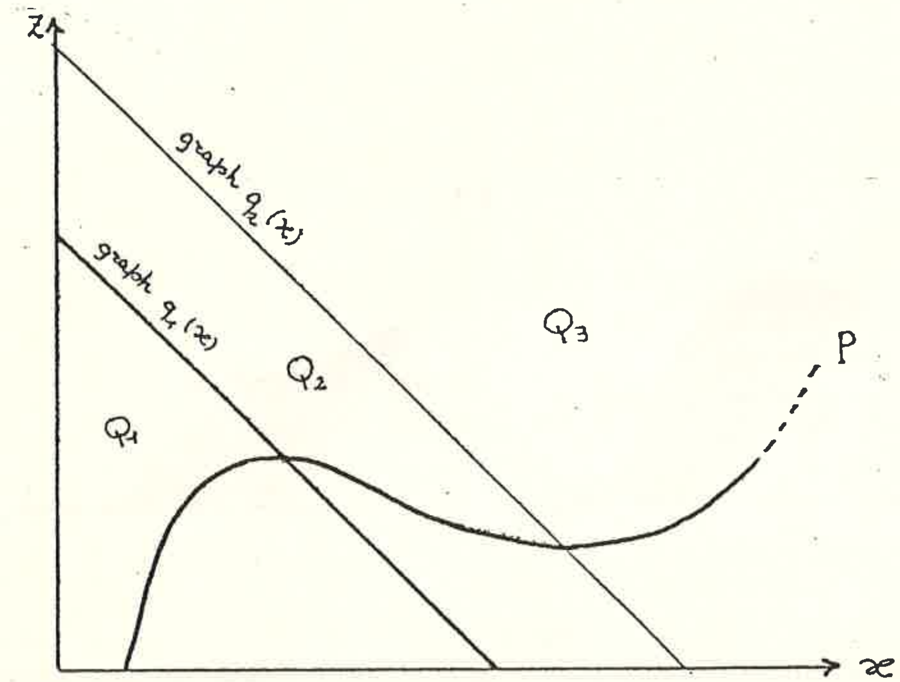

Fig. 3

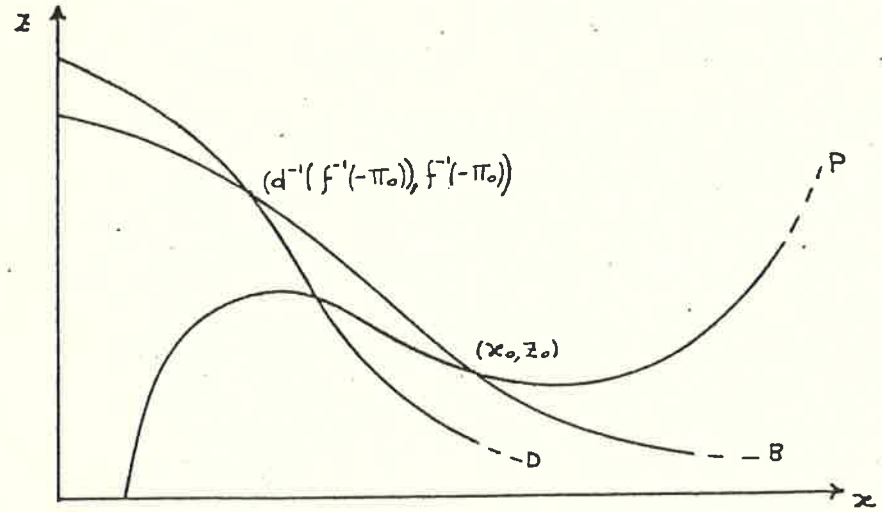

Fig. 4

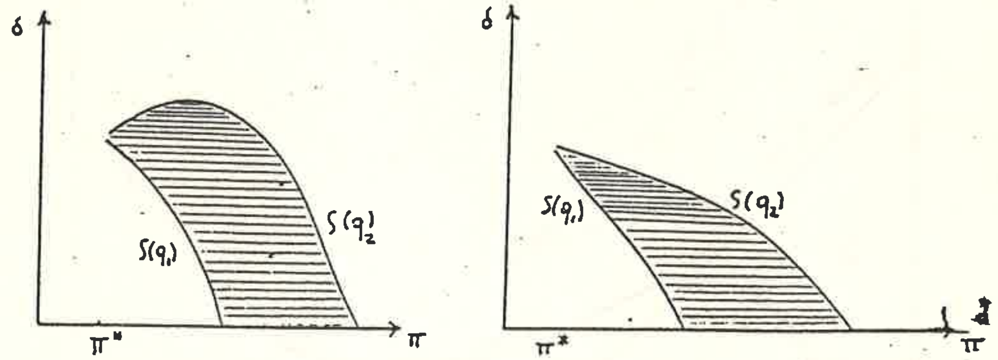

Fig. 5

The shaded regions describe configurations for the set of points in parameter space $R_\pi x R_\delta$ such that the equilibrium point (x,z) of (10) is unstable if $\mu=+\infty$; $S(q_1) \cup S(q_2)$ is the set of critical points.

Elenco dei Quaderni pubblicati

n. 1 (febbraio 1979) MASSIMO DI MATTEO, Alcune considerazioni sui concetti di lavoro produttivo e improduttivo in Marx

n. 2 (marzo 1979) MARIA L.RUIZ, Mercati oligopolistici e scambi internazionali di manufatti. Alcune ipotesi e un'applicazione all'Italia

n. 3 (maggio 1979) DOMENICO MARIO NUTI, Le contraddizioni delle economie socialiste: una interpretazione marxista

n. 4 (giugno 1979) ALESSANDRO VERCELLI, Equilibrio e dinamica del sistema economico-semantica dei linguaggi formalizzati e modello keynesiano

n. 5 (settembre 1979) A. RONCAGLIA - M. TONVERONACHI, Monetaristi e neokeynesiani: due scuole o una?

n. 6 (dicembre 1979) NERI SALVADORI, Mutamento dei metodi di produzione e produzione congiunta

n. 7 (gennaio 1980) GIUSEPPE DELLA TORRE, La struttura del sistema finanziario italiano: considerazioni in margine ad un'indagine sull'evoluzione quantitativa nel dopoguerra (1948-1978)

n. 8 (gennaio 1980) AGOSTINO D'ERCOLE, Ruolo della moneta ed impostazione antiquantitativa in Marx: una nota

n. 9 (novembre 1980) GIULIO CIFARELLI, The natural rate of unemployment with rational expectations hypothesis. Some problems of estimation

n. 10 (dicembre 1980) SILVANO VICARELLI, Note su ammortamenti, rimpiazzi e tasso di crescita

n. 10 bis (aprile 1981) LIONELLO F. PUNZO, Does the standard system exist?

n. 11 (marzo 1982) SANDRO GRONCHI, A meaningful sufficient condition for the uniqueness of the internal rate of return

n. 12 (giugno 1982) FABIO PETRI, Some implications of money creation in a growing economy

n. 13 (settembre 1982) RUGGERO PALADINI, Da Cournot all'oligopolio: aspetti dei processi concorrenziali

n. 14 (ottobre 1982) SANDRO GRONCHI, A Generalized internal rate of return depending on the cost of capital

n. 15 (novembre 1982) FABIO PETRI, The Patinkin controversy revisited

n. 16 (dicembre 1982) MARINELLA TERRASI BALESTRIERI, La dinamica della localizzazione industriale: aspetti teorici e analisi empirica

n. 17 (gennaio 1983) FABIO PETRI, The connection between Say's law and the theory of the rate of interest in Ricardo

n. 18 (gennaio 1983) GIULIO CIFARELLI, Inflation and output in Italy: a rational expectations interpretation

n. 19 (gennaio 1983) MASSIMO DI MATTEO, Monetary conditions in a classical growth cycle

n. 20 (marzo 1983) M. DI MATTEO - M.L. RUIZ, Effetti dell'interdipendenza tra paesi produttori di petrolio e paesi industrializzati: un'analisi macrodinamica

n. 21 (marzo 1983), ANTONIO CRISTOFARO, La base imponibile dell'IRPEF: un'analisi empirica

n. 22 (gennaio 1984) FLAVIO CASPRINI, L'efficienza del mercato dei cambi. Analisi teorica e verifica empirica

n. 23 (febbraio 1984) PIETRO PUCCINELLI, Imprese e mercato nelle economie socialiste: due approcci alternativi

n. 24 (febbraio 1984) BRUNO MICONI, Potere prezzi e distribuzione in economie mercantili caratterizzate da diverse relazioni sociali

n. 25 (aprile 1984) SANDRO GRONCHI, On investment criteria based on the internal rate of return

n. 26 (maggio 1984) SANDRO GRONCHI, On Karmel's criterion for optimal truncation

n. 27 (giugno 1984) SANDRO GRONCHI, On truncation "theorems"

n. 28 (ottobre 1984) LIONELLO F. PUNZO, La matematica di Sraffa

n. 29 (dicembre 1984) ANTONELLA STIRATI, Women's work in economic development process

n. 30 (gennaio 1985) GIULIO CIFARELLI, The natural rate of unemployment and rational expectation hypotheses: some empirical tests.

n. 31 (gennaio 1985) SIMONETTA BOTARELLI, Alcuni aspetti della concentrazione dei redditi nel Comune di Siena

n.32 (febbraio 1985) FOSCO GIOVANNONI, Alcune considerazioni metodologiche sulla riforma di un sistema tributario

n. 33 (febbraio 1985) SIMONETTA BOTARELLI, Ineguaglianza dei redditi personali a livello comunale

n. 34 (marzo 1985) IAN STEEDMAN, Produced inputs and tax incidence theory

n. 35 (aprile 1985) RICHARD GOODWIN, Prelude to a reconstruction of economic theory. A critique of Sraffa

n. 36 (aprile 1985) MICHIO MORISHIMA, Classical, neoclassical and Keynesian in the Leontief world

n. 37 (aprile 1985) SECONDO TARDITI, Analisi delle politiche settoriali: prezzi e redditi nel settore agroalimentare

n. 38 (maggio 1985) PIETRO BOD, Sui punti fissi di applicazioni isotoniche.

n. 39 (giugno 1985) STEFANO VANNUCCI, Schemi di gioco simmetrici e stabili e teoremi di possibilità per scelte collettive.

n. 40 (luglio 1985) RICHARD GOODWIN, The use of gradient dynamics in linear general disequilibrium theory.

n. 41 (agosto 1985) M. MORISHIMA and T. SAWA, Expectations and the life span of the regime.

n. 42 (settembre 1985) ALESSANDRO VERCELLI, Keynes, Schumpeter, Marx and the structural instability of capitalism.

n. 43 (ottobre 1985) ALESSANDRO VERCELLI, Money and production in Schumpeter and Keynes: two dichotomies

n. 44 (novembre 1985) MARCO LONZI, Aspetti matematici nella ricerca di condizioni di unicità per il Tasso Interno di Rendimento.

n. 45 (dicembre 1985) **NIKOLAUS K.A. LAUFER**, Theoretical foundations of a new money supply hypothesis for the FRG..

n. 46 (gennaio 1986) **ENRICO ZAGHINI**, Una dimostrazione alternativa dell'esistenza di equilibri in un modello di accumulazione pura

n. 47 (febbraio 1986) **ALESSANDRO VERCELLI**, Structural stability and the epistemology of change: a critical appraisal

n. 48 (marzo 1986) **JOHN HICKS**, Rational behaviour: observation or assumption?

n. 49 (aprile 1986) **DOMENICO MARIO NUTI**, Merger conditions and the measurement of disequilibrium in labour-managed economies

n. 50 (maggio 1986) **SUSAN SENIOR NELLO**, Un'applicazione della Public Choice Theory alla questione della riforma della Politica Agricola Comune della CEE.

n. 51 (maggio 1986) **SERENA SORDI**, Some notes on the second version of Kalecki's Business Cycle Theory

n. 52 (giugno 1986) **PIERANGELO GAREGNANI**, The classical theory of wage and the role of demand schedules in the determination of relative prices

n. 53 (luglio 1986) **FRANCO BIRARDI**, Crescita, sviluppo ed evoluzione: l'affermazione dei nessi strutturali sistematici e la combinazione di scienza, tecnologia ed economia nelle prime industrializzazioni

n. 54 (agosto 1986) **M. AMENDOLA - J.L. GAFFARD**, Innovation as creation of technology: a sequential model

n. 55 (settembre 1986) **GIULIO CIFARELLI**, Exchange rates, market efficiency and "news". Model specification and econometric identification

n. 56 (settembre 1986) **ALESSANDRO VERCELLI**, Technological flexibility, financial fragility and the recent revival of Schumpeterian entrepreneurship

n. 57 (ottobre 1986) **FABIO PETRI**, The meaning of the Cambridge controversies on capital theory: an attempt at communication

n. 58 (ottobre 1986) **DANIELE CHECCHI**, The international coordination of counter-cyclical policies

n. 59 (novembre 1986) **ALESSANDRO VERCELLI**, Probabilistic causality and economic models: Suppes, Keynes, Granger

n. 60 (dicembre 1986) **LIONELLO F. PUNZO**, J. Von Neumann and K. Menger's mathematical colloquium

n. 61 (dicembre 1986) **MICHAEL BACHARACH**, Three scenarios for fulfilled expectations

n. 62 (gennaio 1987) **J. ESTEBAN** and **T. MILLAN**, Competitive equilibria and the core of overlapping generations economies

n. 63 (gennaio 1987) **ROBERT W. CLOWER**, New directions for keynesian economics

n. 64 (febbraio 1987) **MARCO P. TUCCI**, A simple introduction to flexible functional forms and consumer behaviour theory

n. 65 (marzo 1987) **NICOLA DIMITRI**, Generalizations of some continuos time epidemic models

n. 66 (aprile 1987) **MASSIMO DI MATTEO**, Goodwin and the Evolution of a Capitalistic Economy: An Afterthought

n. 67 (maggio 1987) **MASSIMO DI MATTEO**, Early Discussions on Long Waves

n. 68 (giugno 1987) **MASSIMO DI MATTEO**, Warranted, Natural, and Actual Rates of Growth: Reflections of a Perplex

n. 69 (settembre 1987) **LUISA MONTUSCHI**, Inward-Looking development and import substitution in the Argentine Economy 1950-1980

n. 70 (dicembre 1987) **GIANCARLO GANDOLFO** and **PIETRO CARLO PADOAN**, The Mark V version of the Italian continuos time model

n. 71 (dicembre 1987) **SILVANO VICARELLI**, Impatience to consume, the utility of wealth and the rate of interest

n. 72 (gennaio 1988) **OMAR CHISARI**, Worker's assets, foreign debt and the dynamics of an economy

n. 73 (febbraio 1988) **STEFANO VANNUCCI**, On exactly strong-Nash-implementable power allocations

n. 74 (marzo 1988) **PIER LUIGI LANDI**, Aspetti economici della società anonima della strada ferrata Empoli-Siena o Centrale Toscana fino al 1859

n. 75 (aprile 1988) **FRANCO BIRARDI**, Generating the space of physical characteristics by means of elements which are discrete and limited over time (A note on innovation in chemical technologies)

n. 76 (maggio 1988) **LILIA COSTABILE**, Politica monetaria, debito pubblico e tasso di interesse

n. 77 (giugno 1988) **GIUSEPPE DELLA TORRE**, L'evoluzione dei sistemi finanziari nell'approccio contabile di Raymond Goldsmith.
Alcune considerazioni analitiche

n. 78 (luglio 1988) **FRANCESCO FARINA**, Fattori tecnologici e transazionali nella divisione interindustriale del lavoro

n. 79 (ottobre 1988) **FRANCESCO FARINA**, Gli obiettivi della Banca d'Italia e i tassi d'interesse sul debito pubblico

n. 80 (novembre 1988) **GIULIO CIFARELLI**, La razionalità del tasso di cambio a termine e le "notizie": una analisi empirica

Alla presente Collana "gialla" di quaderni il Dipartimento di Economia Politica affianca una Collana "celeste" di monografie. I numeri pubblicati a tutt'oggi sono i seguenti:

1) **GIUSEPPE DELLA TORRE**, Istituzioni creditizie ed Accumulazione del Capitale in Italia (1948-81), 1984.

2) **SANDRO GRONCHI**, Tasso Interno di Rendimento e Valutazione dei Progetti: una Analisi Teorica, 1984.

3) **ALDINO MONTI**, La Proprietà Immobiliare a Bologna in Età Napoleonica (1797-1810), 1984.

4) **GIULIO CIFARELLI**, Equilibrium and Disequilibrium Interpretations of Inflation, 1985.