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Strategy-Proof Aggregation Rules in Median Join-Semilattices and Arrowian Social Welfare Functions

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STRATEGY-PROOF AGGREGATION RULES IN MEDIAN JOIN-SEMILATTICES AND ARROWIAN SOCIAL WELFARE FUNCTIONS

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Abstract. Three characterizations of the whole class of strategy-proof aggregation rules on rich domains of locally unimodal total preorders in finite median join-semilattices are provided. In particular, it is shown that such class consists precisely of generalized weak consensus-sponsorship rules induced by certain families of order filters of the coalition poset. It follows that the co-majority rule and many other inclusive aggregation rules belong to that class. The co-majority rule for an odd number of agents is also characterized. The existence of strategy-proof anonymous neutral and unanimity-respecting social welfare functions which satisfy a suitably relaxed independence condition is shown to follow from our characterizations.

1. Introduction

The present work is devoted to characterizing those aggregation rules in finite median join-semilattices which are strategy-proof on rich domains of locally unimodal (or single peaked) total preorders. Moreover, the co-majority rule is characterized within the class of such strategy-proof rules as the only one that is anonymous and bi-idempotent. An application of our characterization results to social welfare functions is also provided: it is shown which specifically weakened form of independence is consistent with anonymity and neutrality of a unanimity-respecting Arrowian social welfare function.

2. Strategy-proof aggregation rules in finite median join-semilattices: model and results

Let $N = \{1, .., n\}$ denote the finite population of voters, X an arbitrary nonempty *finite* set of alternatives and $\leqslant$ a reflexive, transitive

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and antisymmetric binary relation on X. We assume that $n \geq 3$ in order to avoid tedious qualifications, denote by $\vee$ and $\wedge$ the *least-upper-bound* and *greatest-lower-bound* binary *partial* operations on X as induced by $\leqslant$, respectively, while for any $Y \subseteq X$, $\vee Y$ and $\wedge Y$ denote the least-upper-bound and greatest-lower-bound of Y (whenever they exist). We also posit -for any $x \in X$- $\uparrow x = \{y \in X : x \leqslant y\}$ i.e. the *(principal) order filter* generated by x. An element $x \in X$ is *meet-irreducible* if for any $Y \subseteq X$, $x = \wedge Y$ entails $x \in Y$, and *join-irreducible* if for any $Y \subseteq X$, $x = \vee Y$ entails $x \in Y$. The set of all meet-irreducible elements and join-irreducible elements of $\mathcal{X} = (X, \leqslant)$ will be denoted by $M_{\mathcal{X}}$ and $J_{\mathcal{X}}$, respectively.

The ordered pair $\mathcal{X} = (X, \leqslant)$ is a (finite) *median join-semilattice* if and only if

(i) $x \vee y$ is well-defined in X for all $x, y \in X$ so that $\vee : X \times X \rightarrow X$ is a function i.e. $\mathcal{X}$ is a *join-semilattice;*

(ii) for all $u \in X$, and for all $x, y, z \in X$ such that u is a lower bound of $\{x, y, z\}$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ (or, equivalently, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$) holds i.e. $(\uparrow u, \leqslant_{|\uparrow u})$ -where $\leqslant_{|\uparrow x}$denotes the restriction of $\leqslant$ to $\uparrow u$- is a *distributive lattice*[1] i.e. $\mathcal{X}$ itself is an *upper distributive join-semilattice*;

(iii) for all $x, y, z \in X$ if $x \wedge y$, $y \wedge z$ and $x \wedge z$ exist, then $(x \wedge y) \wedge z$ also exists i.e. $\mathcal{X}$ satisfies the *co-coronation (or meet-Helly) property.*

A well-known property of *finite* upper distributive join-semilattices that will be repeatedly used below is the following

Claim 1. *Let $m \in M_{\mathcal{X}}$ be a meet-irreducible element of an upper distributive finite join-semilattice $\mathcal{X}$ and $Y \subseteq X$ such that $\wedge Y$ exists. If $m > \wedge Y$ then there also exists some $y \in Y$ such that $m \geqslant y$ (see e.g. Monjardet (1990))*

Furthermore, it is easily checked that if $\mathcal{X} = (X, \leqslant)$ is a median join-semilattice then the partial function $\mu : X^3 \rightarrow X$ defined as follows: for all $x, y, z \in X$, $\mu(x, y, z) = (x \vee y) \wedge (y \vee z) \wedge (x \vee z)$

is indeed a *well-defined ternary operation* on X, the *median* of $\mathcal{X}$.

Relying on μ, a ternary **betweenness** relation

[1] A poset $(Y, \leqslant)$ is a *distributive lattice* iff, for any $x, y, z \in X$, $x \wedge y$ and $x \vee y$ exist, and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (or, equivalently, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$). Moreover, a (distributive) lattice $\mathcal{X}$ is said to be *lower (upper) bounded* if there exists $\bot \in X$ $(\top \in X)$ such that $\bot \leqslant x$ $(x \leqslant \top)$ for all $x \in X$, and *bounded* if it is both lower bounded and upper bounded. A bounded distributive lattice $(X, \leqslant)$ is *Boolean* if for each $x \in X$ there exists a *complement* namely an $x' \in X$ such that $x \vee x' = \top$ and $x \wedge x' = \bot$.

$B_\mu = \{(x,z,y) \in X^3 : x \wedge y \leqslant z \leqslant x \vee y\}$ is defined on $\mathcal{X}$,

and for any $x, y \in X$,

$I^\mu(x,y) := B_\mu(x,.,y) = \{z \in X : x \wedge y \leqslant z \leqslant x \vee y\}$ is the *interval* induced by x and y: therefore, *for any* $x, y, z \in X$, $z \in I^\mu(x,y)$ *if and only if* $(x,z,y) \in B_\mu$ (*also written* $B_\mu(x,z,y)$).

It follows that $\mathcal{I}^\mu = (X, I^\mu)$ is an **interval space**, i.e. I^μ is an *interval function on* X, namely $I^\mu : X^2 \to \mathcal{P}(X)$ is a function that satisfies the following conditions (see e.g. van de Vel (1993)):

I-(i) (**Extension**): $\{x, y\} \subseteq I^\mu(x,y)$ for all $x, y \in X$,

I-(ii) (**Symmetry**): $I^\mu(x,y) = I^\mu(y,x)$ for all $x, y \in X$.

Moreover, $\mathcal{I}^\mu = (X, I^\mu)$ also satisfies

(**Idempotence**): $I^\mu(x,x) = \{x\}$ for all $x \in X$,

(**Convexity**): $I^\mu(x,y)$ is $\mathcal{I}^\mu$-convex for all $x, y \in X$ i.e. $I^\mu(u,v) \subseteq I^\mu(x,y)$ for all $u, v \in I^\mu(x,y)$,

and

(**Median property**): for all $x, y, z \in X$, $\#(I^\mu(x,y) \cap I^\mu(y,z) \cap I^\mu(x,z)) = 1$.

Hence, in particular $\mathcal{I}^\mu = (X, I^\mu)$ is a **median interval space.** [2]

Let $\succcurlyeq$ denote a total preorder i.e. a reflexive, transitive and connected binary relation on X (we shall denote by $\succ$ and $\sim$ its asymmetric and symmetric components, respectively). Then, $\succcurlyeq$ is said to be **locally unimodal** with respect to interval space $\mathcal{I}^\mu = (X, I^\mu)$ - or $\mathcal{I}^\mu$-**lu** - if and only if

U-(i) there exists a *unique maximum* of $\succcurlyeq$ in X, its *top* outcome -denoted $top(\succcurlyeq)$- and

U-(ii) for all $x, y, z \in X$, if $z \in I^\mu(top(\succcurlyeq), y) \smallsetminus \{top(\succcurlyeq)\}$ then $z \succ y$.

We denote by $U_{\mathcal{I}^\mu}$ the set of all $\mathcal{I}^\mu$-*lu* preorders on X. An N-profile of $\mathcal{I}^\mu$-lu preorders is a mapping from N into $U_{\mathcal{I}^\mu}$. We denote by $U_{\mathcal{I}^\mu}^N$ the set of all N-profiles of $\mathcal{I}^\mu$-lu total preorders.

Moreover, A set $D \subseteq U_{\mathcal{I}^\mu}^N$ of locally unimodal preorders w.r.t. $\mathcal{I}^\mu$ is **rich** if for all $x, y \in X$ there exists $\succcurlyeq \in D_{\mathcal{X}}$ such that $top(\succcurlyeq) = x$ and the $UC(\succeq, y) = I^\mu(x,y)$ (where $UC(\succeq, y) := \{y \in X : x \succcurlyeq y\}$).

An **aggregation rule** for (N, X) is a function $f : X^N \to X$. An aggregation rule f is (simply) **strategy-proof** *on* $U_{\mathcal{I}}^N$ iff for all $\mathcal{I}$-unimodal N-profiles $(\succcurlyeq_i)_{i \in N} \in U_{\mathcal{I}}^N$, and for all $i \in N$, $y_i \in X$, and $(x_j)_{j \in N} \in X^N$ such that $x_j = top(\succcurlyeq_j)$ for each $j \in N$, $f((x_j)_{j \in N}) \succcurlyeq_i f((y_i, (x_j)_{j \in N \smallsetminus \{i\}}))$. Moreover, an aggregation rule f is **coalitionally strategy-proof** *on* $U_{\mathcal{I}}^N$ iff for all $\mathcal{I}$-unimodal N-profiles $(\succcurlyeq_i)_{i \in N} \in U_{\mathcal{I}}^N$, and for all $C \subseteq N$, $(y_i)_{i \in C} \in X^C$, and $(x_j)_{j \in N} \in X^N$ such that

[2] It is well-known that e.g. the interval spaces induced by trees or median semilattices (including distributive lattices) are median (see Sholander (1952), (1954)).

$x_j = top(\succcurlyeq_j)$ for each $j \in N$, there exists $i \in C$ with $f((x_j)_{j\in N}) \succcurlyeq_i f((y_i)_{i\in C},(x_j)_{j\in N\smallsetminus C}))$. Finally, an aggregation rule $f : X^N \to X$ is **$\mathcal{I}$-monotonic** iff for all $i \in N$, $y_i \in X$, and $(x_j)_{j\in N} \in X^N$, $f((x_j)_{j\in N}) \in I(x_i, f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))$.[3]

Non-trivial strategy-proof aggregation rules should be -at least to some extent- *input-responsive* and *output-unbiased.* A few requirements can be deployed to present several versions and degrees of input-responsiveness and output-unbiasedness of aggregation rules, namely

Inclusiveness: an aggregation rule for (N, X) is **inclusive** if and only if for each voter $i \in N$ there exist $x^N \in X^N$ and $y_i \in X$ such that $f(x^{N\smallsetminus\{i\}}, y_i) \neq f(x^N)$.

Anonymity: an aggregation rule f for (N, X) is **anonymous** if for each $x^N \in X^N$ and each permutation σ of N, $f(x^N) = f(x^{\sigma(N)})$ (where $x^{\sigma(N)} = (x_{\sigma(1)}, ..., x_{\sigma(n)})$).

Idempotence: an aggregation rule f for (N, X) is **idempotent** (or *unanimity-respecting)* if $f(x, ..., x) = x$ for each $x \in X$.

Sovereignty: an aggregation rule f for (N, X) is **sovereign** if for each $y \in X$ there exists $x^N \in X^N$ such that $f(x^N) = y$ i.e. f is an *onto* function.

Neutrality: an aggregation rule f for (N, X) is **neutral** if for each $x^N \in X^N$ and each permutation π on X, $f(\pi(x^N)) = \pi(f(x^N))$ (where $\pi(x^N) = (\pi(x_1), ..., \pi(x_k)))$.

Notice that both **Idempotence** and **Neutrality** imply **Sovereignty** (but not conversely), while **Anonymity** and **Sovereignty** jointly imply **Inclusiveness** (but not conversely). However, it is easily checked that *if Strategy-proofness holds,* **Sovereignty** and **Idempotence** are in fact equivalent.

In particular, let $\mathcal{X} = (X, \leqslant)$ be a finite *join-semilattice* and $M_{\mathcal{X}}$ the set of its meet-irreducible elements, and for any $x^N \in X^N$, and any $m \in M_{\mathcal{X}}$, posit $N_m(x^N) := \{i \in N : x_i \leqslant m\}$. Then, the following properties of an aggregation rule can also be introduced:

$M_{\mathcal{X}}$-Independence: an aggregation rule $f : X^N \to X$ is **$M_{\mathcal{X}}$-independent** if and only if for all $x_N, y_N \in X^N$ and all $m \in M_{\mathcal{X}}$: if $N_m(x_N) = N_m(y_N)$ then $f(x_N) \leqslant m$ if and only if $f(y_N) \leqslant m$.

Monotonic $M_{\mathcal{X}}$-Independence: An aggregation rule $f : X^N \to X$ is **monotonically $M_{\mathcal{X}}$-independent** if and only if for all $x_N, y_N \in$

[3] $\mathcal{I}$-monotonicity of f amounts to requiring all of its projections f_i to be *gate maps* to the image of f (see van de Vel (1993), p.98 for a definition of gate maps). The introduction of $\mathcal{I}$-monotonic functions in a strategic social choice setting is due to Danilov (1994).

X^N and all $m \in M_{\mathcal{X}}$: if $N_m(x_N) \subseteq N_m(y_N)$ then $f(x_N) \leqslant m$ implies $f(y_N) \leqslant m$.[4]

We are now ready to state the main result of this paper concerning strategy-proofness of aggregation rules on rich domains of locally unimodal profiles.

Theorem 1. *Let $\mathcal{X} = (X, \leqslant)$ be a finite median join-semilattice, $\mathcal{I}^\mu = (X, I^\mu)$ its median interval space as defined above, $D \subseteq U_{\mathcal{X}}$ a rich domain of locally unimodal total preorders on $\mathcal{I}^\mu$, and $f : X^N \to X$ an aggregation rule. Then, the following statements are equivalent:*
(i) f is strategy-proof on D^N ;
(ii) f is $\mathcal{I}^\mu$-monotonic;
(iii) f is monotonically $M_{\mathcal{X}}$-independent.

Proof. (i)$\Longrightarrow$(ii) Let us assume that $f : X^N \to X$ is *not* $\mathcal{I}^\mu$-monotonic: thus, there exist $i \in N$, $x'_i \in X$ and $x_N = (x_i)_{i \in N} \in X^N$ such that $f(x_N) \notin [x_i, f(x'_i, x_{N \smallsetminus \{i\}})]$. Then, consider the (total) preorder $\succcurlyeq^*$on X defined as follows: $x_i = top(\succcurlyeq^*)$ and for all $y, z \in X \smallsetminus \{x_i\}$, $y \succcurlyeq^* z$ iff (i) $\{y, z\} \subseteq [x_i, f(x'_i, x_{N \smallsetminus \{i\}})] \smallsetminus \{x_i\}$ or (ii) $y \in [x_i, f(x'_i, x_{N \smallsetminus \{i\}})] \smallsetminus \{x_i\}$ and $z \notin [x_i, f(x'_i, x_{N \smallsetminus \{i\}})]$ or (iii) $y \notin [x_i, f(x'_i, x_{N \smallsetminus \{i\}})]$ and $z \notin [x_i, f(x'_i, x_{N \smallsetminus \{i\}})]$. Clearly, by construction $\succcurlyeq^*$consists of three indifference classes with $\{x_i\}$, $[x_i, f(x'_i, x_{N \smallsetminus \{i\}})] \smallsetminus \{x_i\}$ and $X \smallsetminus [x_i, f(x'_i, x_{N \smallsetminus \{i\}})]$ as top, medium and bottom indifference classes, respectively. Now, observe that $\succcurlyeq^* \in U_{\mathcal{X}}$. To check this statement, take any $y, z, v \in X$ such that $y \neq z$ and $v \in [y, z]$ i.e. $\mu(y, v, z) = v$ (if $y = z$ then $v = y = z$ and there is in fact nothing to prove). If $\{y, z\} \subseteq [x_i, f(x'_i, x_{N \smallsetminus \{i\}})]$ then by definition $\mu(x_i, f(x'_i, x_{N \smallsetminus \{i\}}), y) = y$ and $\mu(x_i, f(x'_i, x_{N \smallsetminus \{i\}}), z) = z$. Therefore, by property (ii) -and permutation-invariance- of μ it follows that

$$\mu(\mu(x_i, f(x'_i, x_{N \smallsetminus \{i\}}), y), \mu(x_i, f(x'_i, x_{N \smallsetminus \{i\}}), z), v) =$$
$$\mu(\mu(y, z, v), y, z) = \mu(v, y, z) = v \text{ i.e. } v \in [x_i, f(x'_i, x_{N \smallsetminus \{i\}})].$$

Clearly, $\{y, z\} \neq \{x_i\}$ since $y \neq z$. Now, assume without loss of generality that $y \neq x_i$: thus $v \succcurlyeq^* y$ by definition of $\succcurlyeq^*$. If on the contrary $\{y, z\} \cap (X \smallsetminus [x_i, f(x'_i, x_{N \smallsetminus \{i\}})]) \neq \varnothing$ then clearly by definition of $\succcurlyeq^*$there exists $w \in \{y, z\}$ such that $v \succcurlyeq^* w$. Thus, $\succcurlyeq^* \in U_{\mathcal{X}}$ as claimed. Also, by assumption $f(x_N) \in X \smallsetminus [x_i, f(x'_i, x_{N \smallsetminus \{i\}})]$ whence by construction $f(x'_i, x_{N \smallsetminus \{i\}}) \succ^* f(x_N)$. But then, f is *not* strategy-proof on $U_{\mathcal{X}}^N$.

[4]The notions of $J_{\mathcal{X}}$-Independence and Monotonic $J_{\mathcal{X}}$-Independence are defined similarly by dualization for a finite median inf-semilattice $\mathcal{X} = (X, \leqslant)$.

(ii)$\Longrightarrow$(i) Conversely, let f be I^{μ}-monotonic. Now, consider any $\succcurlyeq = (\succcurlyeq_j)_{j\in N} \in U_{\mathcal{X}}^N$ and any $i \in N$. By definition of monotonicity $f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}}) \in [top(\succcurlyeq_i), f(x_i, x_{N\smallsetminus\{i\}})]$ for all $x_{N\smallsetminus\{i\}} \in X^{N\smallsetminus\{i\}}$ and $x_i \in X$. But then, since clearly $top(\succcurlyeq_i) \succcurlyeq_i f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}})$, either $f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}}) = top(\succcurlyeq_i)$ or $f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}}) \succcurlyeq_i f(x_i, x_{N\smallsetminus\{i\}})$ by local unimodality of $\succcurlyeq_i$. Hence, $f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}}) \succcurlyeq_i f(x_i, x_{N\smallsetminus\{i\}})$ in any case. It follows that f is indeed strategy-proof on $U_{\mathcal{X}}^N$.

(ii)$\Rightarrow$(iii) Suppose that f is $\mathcal{I}^{\mu}$-monotonic. Hence, for all $i \in N$, $y_i \in X$, and $(x_j)_{j\in N} \in X^N$, $f((x_j)_{j\in N}) \in I^{\mu}(x_i, f(y_i, (x_j)_{j\in N\smallsetminus\{i\}}))$ i.e. $f((x_j)_{j\in N}) = \mu(x_i, f((x_j)_{j\in N}), f(y_i, (x_j)_{j\in N\smallsetminus\{i\}}))$. Therefore, for any meet-irreducible element $m \in M_{\mathcal{X}}$, $m \leqslant f((x_j)_{j\in N})$ if and only if $m \leqslant \mu(x_i, f((x_j)_{j\in N}), f(y_i, (x_j)_{j\in N\smallsetminus\{i\}})) = (x_i \vee f((x_j)_{j\in N})) \wedge (f((x_j)_{j\in N}) \vee f(y_i, (x_j)_{j\in N\smallsetminus\{i\}})) \wedge (x_i \vee f(y_i, (x_j)_{j\in N\smallsetminus\{i\}}))$. It follows that if $m \geqslant f((x_j)_{j\in N})$ then [$m \geqslant x_i$ or $m \geqslant f(y_i, (x_j)_{j\in N\smallsetminus\{i\}})$]. Indeed, suppose that $m \geqslant f((x_j)_{j\in N})$ but [$m \ngeqslant x_i$ and $m \ngeqslant f(y_i, (x_j)_{j\in N\smallsetminus\{i\}})$]. Then, $m \ngeqslant (x_i \vee f((x_j)_{j\in N}))$, $m \ngeqslant f((x_j)_{j\in N}) \vee f(y_i, (x_j)_{j\in N\smallsetminus\{i\}})$, and $m \ngeqslant x_i \vee f(y_i, (x_j)_{j\in N\smallsetminus\{i\}})$. Hence, since $\mathcal{X}$ is upper distributive, $m \ngeqslant (x_i \vee f((x_j)_{j\in N})) \wedge (f((x_j)_{j\in N}) \vee f(y_i, (x_j)_{j\in N\smallsetminus\{i\}}))$ and therefore by upper distributivity again, $m \ngeqslant (x_i \vee f((x_j)_{j\in N})) \wedge (f((x_j)_{j\in N}) \vee f(y_i, (x_j)_{j\in N\smallsetminus\{i\}})) \wedge (x_i \vee f(y_i, (x_j)_{j\in N\smallsetminus\{i\}}))$

i.e. $m \ngeqslant \mu(x_i, f((x_j)_{j\in N}), f(y_i, (x_j)_{j\in N\smallsetminus\{i\}})) = f((x_j)_{j\in N})$, a contradiction.

Conversely, if $m \geqslant \mu(x_i, f((x_j)_{j\in N}), f(y_i, (x_j)_{j\in N\smallsetminus\{i\}}))$ -i.e. if $m \geqslant$
$\geqslant (x_i \vee f((x_j)_{j\in N})) \wedge (f((x_j)_{j\in N}) \vee f(y_i, (x_j)_{j\in N\smallsetminus\{i\}}) \wedge (x_i \vee f(y_i, (x_j)_{j\in N\smallsetminus\{i\}}))$- then $m \geqslant f((x_j)_{j\in N})$.

Thus, in particular, if $m \geqslant x_i$ and $m \geqslant f(y_i, (x_j)_{j\in N\smallsetminus\{i\}}))$ then $m \geqslant f((x_j)_{j\in N})$. Now, suppose that $m \in M_{\mathcal{X}}$, $m \geqslant f((x_i)_{i\in N})$ and $N_m((x_i)_{i\in N}) \subseteq N_m((y_i)_{i\in N})$. By $\mathcal{I}^{\mu}$-monotonicity of f, $m \geqslant x_i$ or $m \geqslant f(y_i, (x_j)_{j\in N\smallsetminus\{i\}})$. Thus, if $m \geqslant x_i$ then also $m \geqslant y_i$: hence, $m \geqslant f((x_i)_{i\in N})$ and $\mathcal{I}^{\mu}$-monotonicity of f entail $m \geqslant f(y_i, (x_j)_{j\in N\smallsetminus\{i\}})$. It follows that $m \geqslant f(y_i, (x_j)_{j\in N\smallsetminus\{i\}})$ in any case. From $\mathcal{I}^{\mu}$-monotonicity of f and $m \geqslant f(y_i, (x_j)_{j\in N\smallsetminus\{i\}})$, it follows that $m \geqslant x_{i+1}$ or $m \geqslant f((y_i, y_{i+1}, (x_j)_{j\in N\smallsetminus\{i,i+1\}}))$. But $m \geqslant x_{i+1}$ entails $m \geqslant y_{i+1}$ as well hence $m \geqslant f(y_i, (x_h)_{h\in N\smallsetminus\{i\}})$ and $\mathcal{I}^{\mu}$-monotonicity jointly imply $m \geqslant f((y_i, y_{i+1}, (x_j)_{j\in N\smallsetminus\{i,i+1\}}))$ in any case. Repeating the argument, we eventually obtain $m \geqslant f((y_i)_{i\in N})$, and f is indeed monotonically $M_{\mathcal{X}}$-independent as required.

(iii) $\Longrightarrow$(ii) Suppose that f is monotonically $M_{\mathcal{X}}$-independent but *not* $\mathcal{I}^{\mu}$-monotonic. Thus, there exist $i \in N$, $(x_j)_{j\in N} \in X^N$, $y_i \in X$ such that

$$f((x_j)_{j\in N}) \neq \mu(x_i, f((x_j)_{j\in N}), f(y_i, (x_j)_{j\in N\smallsetminus\{i\}}))$$

i.e. there must exist $m \in M_{\mathcal{X}}$ such that $m \geqslant f((x_j)_{j\in N})$ but $m \ngeqslant (x_i \vee f((x_j)_{j\in N})) \wedge (f((x_j)_{j\in N}) \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \wedge (x_i \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))$ or $m \geqslant (x_i \vee f((x_j)_{j\in N})) \wedge (f((x_j)_{j\in N}) \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \wedge (x_i \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))$ but $m \ngeqslant f((x_j)_{j\in N})$. Thus, suppose that $m \geqslant f((x_h)_{h\in N})$ and $m \ngeqslant (x_i \vee f((x_j)_{j\in N})) \wedge (f((x_j)_{j\in N}) \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \wedge (x_i \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))$. Then, it must be the case that $m \ngeqslant x_i$ and $m \ngeqslant f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})$ whence by construction $N_j((x_j)_{j\in N}) \subseteq N_j((y_i,(x_j)_{j\in N\smallsetminus\{i\}}))$ and therefore $m \geqslant f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})$ by monotonic $M_{\mathcal{X}}$-independence, a contradiction.

Next, suppose that $m \geqslant$
$\geqslant (x_i \vee f((x_j)_{j\in N})) \wedge (f((x_j)_{j\in N}) \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \wedge (x_i \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))$
and $m \ngeqslant f((x_h)_{h\in N})$.

Since, by upper distributivity of $\mathcal{X}$, it must be the case that $m \geqslant (x_i \vee f((x_j)_{j\in N}))$ or $m \geqslant (f((x_j)_{j\in N}) \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})$ or else $m \geqslant (x_i \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))$ it follows that $m \geqslant (x_i \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))$ hence in particular both $m \geqslant x_i$ and $m \geqslant f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})$. Thus, $N_j((y_i,(x_j)_{j\in N\smallsetminus\{i\}})) \subseteq N_j((x_j)_{j\in N})$ and $m \geqslant f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})$ and therefore, by monotonic $M_{\mathcal{X}}$-independence, $m \geqslant f((x_j)_{j\in N})$, a contradiction again, and the thesis is established. □

A similar argument is used for the case of (not necessarily finite) bounded distributive lattices in Savaglio and Vannucci (2019), and in Vannucci (2019).

Corollary 1. *Let $\mathcal{X} = (X, \leqslant)$ be a finite median join-semilattice, $\mathcal{I}^{\mu} = (X, I^{\mu})$ its median interval space as defined above, and $f : X^N \to X$ an aggregation rule. Then, the following statements are equivalent:*

(i) f is strategy-proof on $U^N_{\mathcal{X}}$;

(ii) for each $m \in M_{\mathcal{X}}$ there exists an order filter F_m of $(\mathcal{P}(N), \subseteq)$ such that $f(x_N) = \bigwedge \{m \in M_{\mathcal{X}} : N_m(x_N) \in F_m\}$ for all $x_N \in X^N$.

Proof. Immediate from Theorem 1 and dualization of Proposition 1.4 of Monjardet (1990). □

Thus, there are many idempotent and inclusive or anonymous strategy-proof aggregation rules, including quota rules and inclusive collegial rules, and the **co-majority rule** $f^{\partial maj}$ defined as follows:

for all $x_N \in X^N$,
$f^{\partial maj}(x_N) = \wedge_{S \in \mathcal{W}^{maj}} (\vee_{i\in S} x_i)$
where $\mathcal{W}^{maj} = \left\{ S \subseteq N : |S| \geq \lfloor \frac{|N|+2}{2} \rfloor \right\}$.

As a further Corollary we obtain a new characterization of the co-majority rule via strategy-proofness, anonymity as defined above and the following well-known properties for aggregation rules, namely

Bi-Idempotence: for any $x^N \in X^N$ and $y, z \in X$, if $x_i \in \{y, z\}$ for all $i \in N$, then $f(x^N) \in \{y, z\}$.

Corollary 2. *Let $\mathcal{X} = (X, \leqslant)$ be a finite median join-semilattice, B_μ its median betweenness relation, and $D \subseteq U_{\mathcal{X}}$ a rich domain of locally unimodal total preorders with respect to $B_{\mathcal{X}}$. Then, an aggregation rule $f : X^N \to X$ is strategy-proof on D^N and satisfies Anonymity and Bi-Idempotence if and only if f is the co-majority rule $f^{\partial maj}$.*

Proof. Immediate from Theorem 1 above and a straightforward dualization of Corollary 7.4 of Monjardet (1990). □

A most interesting application of the theorem involve aggregation rules for total preorders i.e. social welfare functions in the classic Arrowian sense, as made precise by the following

Proposition 1. *Let A be a nonempty finite set of alternative social states, $\mathcal{R}_A^T$ the set of all total preorders (i.e. reflexive, transitive and connected binary relations) on A, $\mathcal{X} = (\mathcal{R}_A^T, \subseteq)$ the set-inclusion poset on $\mathcal{R}_A^T$ and $N = \{1, ..., n\}$ such that $n = 2k + 1$ for some non-negative integer k. Then the co-majority rule $\widehat{f}^{\partial maj} : (\mathcal{R}_A^T)^N \to \mathcal{R}_A^T$ is the unique aggregation rule (i.e. social welfare function) which satisfies Anonymity and Bi-Idempotence, and is strategy-proof on $U_{\mathcal{X}}^N$.*

Proof. First, it can be shown that -if the join $\vee$ of two posets is defined as the transitive closure of their union- the poset $(\mathcal{R}_A^T, \subseteq)$ becomes a join-semilattice with respect to $\vee$, and satisfies both upper distributivity (by Claim (P.1) of Janowitz (1984), and co-coronation (by Claims (P.3) and (P.5) of Janowitz (1984)). Hence $(\mathcal{R}_A^T, \subseteq)$ is indeed a median join-semilattice whose meet-irreducibles are the total preorders having just two indifference classes, Theorem 1 and Corollary 2 apply to $\mathcal{X} = (\mathcal{R}_A^T, \subseteq)$, and the thesis follows. □

It is easily checked that $\widehat{f}^{\partial maj}$ also satisfies Neutrality. It follows that there exists an Arrowian social welfare function on the full domain of total preorders which is anonymous, neutral, idempotent (because Bi-Idempotence clearly implies Idempotence), satisfies a monotonic independence property w.r.t. the meet-irreducible total preorders (which are the co-atoms of the join-semilattice ($\mathcal{R}_A^T, \subseteq$) i.e. the total preorders having just two indifference classes) and is strategy-proof on any rich locally unimodal domain. Therefore, $\widehat{f}^{\partial maj}$ is in particular a *social welfare function that satisfies all the properties required by Arrow's (Im)Possibility Theorem except for* the *Independence of Irrelevant Alternatives (IIA) condition* [5]*,* and also enjoys a significant strategy-proofness property on a fairly large preference domain. What is then the relationship between $M_{\mathcal{X}}$-Independence ($M_{\mathcal{X}}$-I) and IIA? Clearly enough, under idempotence $M_{\mathcal{X}}$-I is definitely *weaker* than IIA because the former is consistent with anonymity and neutrality (as a consequence of Proposition 1), while the latter is not. Indeed, as established by Hansson (1969), IIA in combination with anonymity and neutrality provides a characterization of the *constant* social welfare function having the *universal indifference* relation $A \times A$ as its *unique* value[6].

It should also be noticed that -if the join of two preorders is defined as the *transitive closure of their union*- the poset $\mathcal{X}' = (\mathcal{R}_A, \subseteq)$ of *all preorders* (i.e. reflexive and transitive binary relations) on A (both total and not total), turns out to be a *lattice* whose set of join-irreducible elements is $J_{\mathcal{X}'} \simeq \{(x,y) : x, y \in A, x \neq y\}$. Hence IIA (in its binary version) is equivalent to $J_{\mathcal{X}'}$-I. However, $\mathcal{X}'$ is *not* distributive (see e.g. Barbut and Monjardet (1970)): indeed, as a join-semilattice it is not

[5]Recall that Arrow's IIA (in binary form) is a condition on social welfare functions $f : (\mathcal{R}_A^T)^N \rightarrow \mathcal{R}_A^T$ defined as follows: for every x,$y \in A$ and any $(\succcurlyeq_i)_{i \in N}$, $(\succcurlyeq'_i)_{i \in N} \in (\mathcal{R}_A^T)^N$ such that $x \succcurlyeq_i y$ if and only if $x \succcurlyeq'_i y$ for each $i \in N$, $(x, y) \in f((\succcurlyeq_i)_{i \in N})$ entails $(x, y) \in f((\succcurlyeq'_i)_{i \in N}$).

[6]Afriat (1987) perceptively stresses that defining a social welfare function ultimately amounts to 'voting for an order'. Thus -he argues- there can be no real obstacle to choosing a social preference by a democratic protocol whose only specific feature has to be that preferences about preferences must be considered. In that connection, Afriat forcefully suggests that the aforementioned Hansson's theorem exposes the unreasonable strenght of IIA. However, Afriat's critical discussion -and indeed rejection- of IIA as a requirement for 'democratic' aggregation/voting protocols omits to address squarely Arrow's own arguments *in favor* of IIA. To an extent, our discussion to follow confirms Afriat's perceptive observations, supplementing them with a clear and drastic circumscription of the actual scope of IIA as a compelling requirement. Our argument, however, is game-theoretic: it relies on the distinction between structural and strategic manipulation of a game form. Thus, it should not come as a surprise that none of these aspects is considered in Afriat (1987) which is the first widely accessible published version of a 1973 paper (1973 being precisely the year that arguably marks the very beginnings of *strategic* social choice theory thanks to the seminal contributions of Gibbard, Satterthwaite and Pattanaik published or written in that year).

upper distributive (and as a meet-semilattice it is not lower distributive), hence the median is not a well-defined ternary operation on $\mathcal{R}_A$, and Theorem 1 does not apply to it.

All of the above raises the following issue: if our focus is indeed on aggregation of *total* preference preorders why should we insist on IIA i.e. equivalently on $J_{\mathcal{X}'}$-I, and thus (at least implicitly) involve the lattice $\mathcal{X}' = (\mathcal{R}_A, \subseteq)$ of *all* preorders on A -whether total or not? The answer is suggested by Arrow himself as he motivates IIA as a condition that disallows dependence of social choices concerning pairs of alternatives from individual preferences involving *other* alternatives in A, which after all might be *not* actually feasible (see Arrow (1963), pp.26-28 and 109-111). Now, such an argument makes full sense only if it is precisely manipulation of the final outcome through manipulation of the *set of available alternatives* that is to be prevented. However, the latter amounts to manipulation of outcomes through changes of parameter A of the aggregation problem $(N, A, (\succcurlyeq_i)_{i\in N})$: manipulation of a *structural* type. But, arguably, such a structural manipulation should be firmly distinguished from *strategic* manipulation of outcomes of a *given* aggregation rule f for $(N, A, (\succcurlyeq_i)_{i\in N})$ through submission of *false* private information about individual preferential attitudes[7]. Thus, once the set of available alternatives is *fixed*, for any given population of agents it is precisely *strategic* manipulation that should be monitored and possibly prevented. The results of the present work show that in order to achieve the latter goal the full force of Arrowian IIA is not required, and the much weaker $M_{\mathcal{X}'}$-I suffices to secure it.

3. Related literature

The issue of strategy-proofness of preference aggregation rules has been already addressed in the previous literature, but never -to the best of the authors' knowledge- with respect to the 'full' domain of all total preorders. Under the heading 'social welfare functions', Bossert and Storcken (1992) study aggregation rules for *linear* orders on a finite set and their *coalitional strategy-proofness* properties with respect

[7]That proposition is not meant to imply that disentangling structural and strategic manipulation is always easy or indeed possible in actual practice. For instance, if alternative outcomes are candidates for an appointment or a political election then strategic candidacy is virtually always possible. But strategic candidacy may be regarded precisely as a structural manipulation of the aggregation rule that typically translates into a forced change of *available* strategies.

to topped metric total preference preorders (on the set of linear orders) as induced by the Kemeny distance. They prove an impossibility theorem for those coalitionally strategy-proof and sovereign social welfare functions (on linear orders) that also satisfy a certain condition of independence from extrema. Bossert and Sprumont (2014) offer several possibility results concerning *restricted* aggregation rules (mapping profiles of linear orders on a finite set A into total *preorders* on A) which are strategy-proof on the domain of topped preferences (on the set of total preorders) that are unimodal with respect to the median betweenness of the distributive lattice of binary relations on A.

Most recently, a significant weakening of IIA was proposed in Maskin (2020). It amounts to requiring invariance of aggregate preference between any two alternatives x, y for any pair of preference profiles whose restrictions to $\{x, y\}$ are identical *only if for every agent/voter the respective closed preference intervals having x and y as their extrema are also identical*. This particular weakened version of IIA is motivated in terms of strategy-proofness properties, namely resistance to certain sorts of 'vote splitting' effects (plus retention of some responsiveness to preference intensities). However, it should be emphasized that such a proposal concerns strategy-proofness of the *'maximizing' social choice function* attached to a social welfare function (as opposed to the social welfare function itself).

By contrast, the existence issue for strategy-proof social welfare functions as aggregation rules on the full domain of total preorders on a finite set has never been addressed explicitly in previously published work, to the best of the authors' knowledge. The results of the present work imply that anonymous and neutral social welfare functions on the full domain of total preorders on a finite set *do exist*, and are indeed *strategy-proof* on suitably defined single-peaked domains of 'preferences on preferences' (i.e. arbitrary rich locally unimodal domains). Quite remarkably, such social welfare functions may also be regarded as a *positive* solution to the classic Arrowian preference aggregation problem, once the focus is restricted to *strategic* as opposed to *structural* manipulation, and the Arrowian Independence condition IIA is accordingly replaced with a most 'natural' and milder independence requirement.

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