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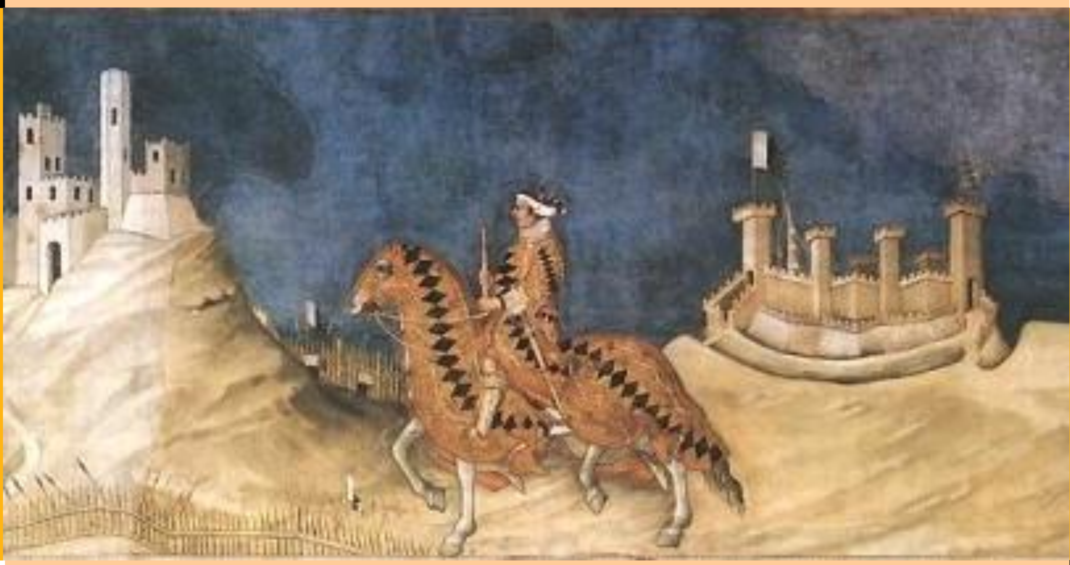

Strategy-Proof Aggregation of Approximate and Imprecise Judgments

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Abstract

The present work is devoted to the study of *aggregation rules* for several types of *approximate judgments* and their strategy-proofness properties when the relevant judgment space is lattice-ordered and endowed with a natural metric, and the agents/experts have *single-peaked preferences* consistent with it. In particular, *approximate probability estimates* as modeled by intervals of probability values, *numerical measurements with explicit error bounds*, *approximate classifications,* and *conditional judgments* that are amenable to composition by means of a set of logical connectives are considered. Relying on (bounded) *distributivity* of the relevant lattices, we prove the existence of a large class *of inclusive and unanimity-respecting strategy-proof aggregation rules for approximate assessments or conditional judgments,* consisting of *sup-projections* and *sup-inf polynomials* as parameterized by certain families of locally winning coalitions called *committees.* Amongst them, the *majority aggregation rule* is characterized as the only one that ensures both *anonymity* (i.e. an equal treatment of agents) and *bi-idempotence* (i.e. a definite choice between the only two judgments nominated by a maximally polarized body).

JEL classification: D71; D81

1 Introduction

In spring 2011, the President of United States, Barak Obama, decided to send the SEAL TEAM SIX to Abbottabad, Pakistan, to hunt Osama bin Laden. His decision arrived

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after 40 reports from Intelligence Agencies and many meetings held in the White House Situation Room. However, intelligence analysts' and officials' estimates fixed the chances that Osama bin Laden was living in Abbottabad from a low of 10 percent to a high of 95 percent.[1] The President stated that he found this discussion confusing and even misleading, nevertheless he had to interpret and combine these estimated probabilities and articulate a single point estimate. Moreover, he also was very doubtful on which kind of action to take: large bombing, surgical bombing, an attack from the ground or simply doing nothing because of the uncertainty of finding Osama in Abbottabad and the hazard of a confrontation if Pakistan sounded an alarm. So, he said: "Look guys, this is fifty-fifty, this is a flip of a coin. I can't base this decision on the notion that we have any greater certainty than that". This somewhat provocative attempt at summarizing the practical import of the information provided by so many advisors conveys very effectively the presidential discomfort and frustration. And that frustration, arguably, has at least *two* sources: not only the evidence that the available information is very imprecise and unreliable, but also the feeling of impotence engendered by lack of any clear and consistent amalgamation protocol enabling to make the best of it anyway. Indeed, in such a highly dramatic situation, president Obama had to take a decision without the support of any consistent rule to aggregate such an impressively large variety of *imprecise*, *approximate assessments* or *conditional judgements* of so many discordant advisors.

The foregoing momentous and exceedingly difficult decision problem is in fact just a very remarkable and comparatively simple instance of a wide class of similar problems that arise whenever aggregation of *possibly imprecise assessments* and *judgements* provided by a manifold of qualified sources is required.

Motivated by such a broad class of prominent examples (see below for more

[1] In fact, according to those who investigated such a national security dilemma, the lead analyst at CIA put the probability that bin Laden was there in the interval [90%-95%], the deputy director of the CIA was around 60%, the CIA Red Team, a bunch of independent analysts charged to offer a 'devil's advocate' position, put their position in [30%-40%] range, other analyst assessed probability around 10%, others around 40% and some assessed it as being close to 80%. Further, one week before the final meeting, John Brennan, assistant to the President Obama for Homeland Security, had asked to the Counterterrorism Center director, Mike Leiter, to assemble another team to assess the probability that Osama bin laden was right in Abbottabad. So the final meeting began with the Leiter's findings, which were deflating. Leiter told the President that this new group could arrive at only 40 percent certainty that bin Laden was there (see e.g. Bowden (2012); Fiedman and Zeckhauser (2015)).

details), the present article studies and singles out one possible median-based solution to the problem of aggregating judgments in a very general framework that covers both *exact* and *approximate assessments* and *conditional judgments*. Indeed, such an issue arises quite regularly whenever a Decision Making Agency (DMA) has to collect the information coming from a group of agents who independently vote or simply express evaluations on a set of complex propositions with some interdependency constraints (as e.g. transitivity when describing preferences) between them.

A DMA, for instance, needs to amalgamate the judgments of the group members on an issue, as it is the case for a judge that needs to evaluate if the defendant is guilty after that a jury of different jurors has expressed its own judgements or for the European Central Bank council that might need to form a decision about the monetary policy to be implemented after a committee of experts has been audited. These processes of aggregating information matter in practice and therefore finding suitable protocols, namely compelling aggregation rules is certainly an exercise that is worth pursuing. A juror or an expert could express *exact* judgments (guilty or not-guilty, increase the money stock of 2% etc.) that can be ordered in a natural way or that can even be representable by real-valued ordinal scales (but not necessarily by interval or ratio numerical scales). However, reliable judgements are often available in an *approximate* form or as *imprecise* and multiple measurements and classifications. Moreover, agents may also be required to provide *approximate conditional judgements*, namely evaluations that involve two, not necessarily distinct, propositions, one specifying the condition and whose fulfilment then leads to the second being taken as asserted. In both cases, the possibility of coherently aggregating different conditional judgments into a single one that may be applied to the decision problem at hand is surely a issue worth investigating.

In what follows, we consider the entire class of *inclusive aggregation rules*, namely protocols to amalgamate individual assessments' profiles in which each expert's proposal might be pivotal. In particular, we put a special emphasis on *median-based rules*, in view of their very interesting combination of *unbiased input-responsivity, output-unbiasedness,* and *decisiveness* properties . In particular, we show that:

(i) if the relevant judgments are endowed with a natural order, and such an order is a *(bounded) lattice*, namely it admits both a top and a bottom element and each pair of judgments has both a least upper bound (or *joint*) and a greatest lower bound (or *meet*), then *both exact and approximate judgments* can be *aggregated* by the *median* *rule*, which in turn can be computed by the simple *majority rule*, if and only if the judgment lattice is also *distributive*[2], and

(*ii*) if the evaluating agents' preferences on judgments are *single-peaked, i.e.* have a unique maximum and consistently favor 'closeness' to that maximum, then simple majority is in fact a *strategy-proof* aggregation rule, namely it makes it impossible for any agent to achieve a better aggregate outcome by making a proposal which is actually *not the best outcome* according to her *true* preferences.

It should be emphasized that point (*i*) entails a considerable extension of the scope for robust and inclusive aggregation of approximate judgments far beyond the standard averaging-based rules, while (*ii*) ensures (on a very comprehensive class of *single-peaked* preference domains) a safeguard against *strategic manipulation* which is much *more direct* and simpler (and arguably more effective) than reliance on the *proper* or *strictly proper scoring rules* which are usually adjoined to averaging aggregation rules precisely to discourage strategic transmission of false information.

Some observations are needed here to justify and clarify such statements. Concerning point (*i*), notice that (bounded) distributive lattices are quite pervasive, or at least very common: arbitrary (bounded) linearly ordered sets[3], sets of admissible answers to a finite list of yes/no questions, admissible classifications of certain items by means of an ordered list of binary characters or (bounded) grids as endowed with their respective 'natural' component-wise orders are all positive examples of (bounded) distributive lattices.

Moreover, concerning point (*ii*), observe that single-peakedness is a very natural and comparatively mild restriction on preferences when the underlying outcome space is endowed with a *metric.* Now, that is precisely the case with bounded distributive lattices which typically admit an intrinsic and most 'natural' metric based on the length of the shortest path connecting the meet and the join of any pair of elements.

To be sure, probability distributions ordered by the natural component-wise order are not even lattices, while discrete probability distributions ordered by dominance (namely by concentration or, dually, by dispersion), or information partitions ordered

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[2] Namely, its least upper bound and greatest lower bound behave (and mutually interact) very much like propositional conjunctions 'or' and 'and', respectively. See section 2 for a formal definition of (bounded) distributive lattices.

[3] Indeed, the structure underlying the 'Obama's problem' mentioned above belongs precisely to that subclass of distributive lattices.

by refinement (or, dually, by coarsening) are definitely *not* distributive lattices. But that is in fact part of the point the present paper purports to establish: the judgment format can make a significant difference when it comes to judgment elicitation and aggregation, and a judgment space that is a bounded distributive lattice may be very helpful in that connection.

Thus, in order to fully *appreciate the potentially wide scope and relevance* of the proposed setting and thus furtherly motivate our paper, we also present some prominent classes of examples in which A DMA needs to amalgamate the evaluations of experts who have exact, approximate, imprecise, multiple measurements and classifications or have to assess conditional judgements. In particular, we consider the following issues:

Aggregation of exact proposals when a DMA faces the problem of aggregating the alternative proposals advanced by members of a panel committee who select an appropriate profile of binary criteria to be satisfied by candidates in order to qualify for a certain position.

Aggregation of graded evaluations achieved by a population of students in different subjects, of assessments of wines according to several alternative graded criteria or of the graded performances of participants in a multi-trial competition or of *computing reputation systems* both offline and online.[4]

Aggregation of approximate classifications when an ensemble of *rough experts,* trained on different training sets, predict, for instance, cases of *viral infections* from a collection of easily detected symptoms. This kind of selection corresponds to define an *approximate classification* of *infected units* from the sample, namely the *rough set* analytically expressed by an order interval. The final ensemble decision is supposed to produce a unique approximate classification by aggregating the rough sets of profile of experts' order intervals.

Aggregation of approximate measurements with error bounds when a finite number of experts have to estimate, for example, the global mean surface atmospheric temperature (SAT) increase for selected emission scenarios whose realization depends on the possible future states of the world. More in general, experts provides estimates, i.e. *approximate (numerical) measurements* that are *approximate numbers,* namely an ordered pair of the proposed numerical *estimate* with its *error bound*. In such a case, a DMA

[4] We observe that the latter issue is in fact a problem studied in an analytical setting recently proposed by Balinski and Laraki (2010) in order to advance their case for *majority judgment.*

needs to amalgamate all those approximate measurements in order to implement her environmental policy.

Aggregation of interval probability estimates if a finite number of experts have to make judgments about, for instance, the probability of different pandemic event occurrence, such as the global increase of the fatality rate as a function of the mean of infected, expressed as subjective probability intervals. A DMA then has the problem to aggregate these probability intervals in order to fix her future intervention policy aimed to reduce as much as possible the probability that fatality rate increases too much in the future.

Aggregation of conditional judgments if a DMA is interested, for example, to evaluate climate response to alternative future trajectories of radiative forcing, but the behavior of climate system is very uncertain and ambiguous probabilistic estimations of equilibrium climate sensitivity result from models of different complexity and statistical methods. Experts' judgments about global mean temperature are then considerably different and the relevant conditional judgments may be represented by *conditional assertions*.

The foregoing set of examples is of course not meant to be an exhaustive list, and some of them may well refer to comparatively more uncommon or hypothetical decision problems than others. However, that list provides in our view a quite representative sample of the wide class of interesting aggregation problems to which our results on strategy-proof aggregation rules of judgements in bounded distributive lattices do in fact apply. We notice here that the examples on approximate and conditional judgements will further be analyzed in detail below as applications of our main result (Theorem 2).

The paper is organized as follows. Section 2 introduces the model. Section 3 and 4 provide the main results of the paper. In Section 5, some applications emphasize the significance and possible practical import of our results. The last section includes a discussion of the related literature and concludes, while Appendix 1 collects some more specific technical notions and all the proofs.

2 The model: notation and definitions

Let $\mathcal{X} = (X, \leqslant)$ be a partially ordered set (poset) of *judgements* which is also a *bounded distributive lattice*, namely a set X endowed with a partial order $\leqslant$ (i.e. a reflexive, transitive and antisymmetric binary relation) such that (i) X includes both a maximum $\top$ and a minimum $\bot$; (ii) for any $x, y \in X$ both the *least-upper-bound* (*l.u.b.*) or *join* $\vee$ and the *greatest-lower-bound (g.l.b.)* or *meet* $\wedge$ of x and y with respect to $\leqslant$ are well-defined binary operations on X, and (iii) the join and meet operation satisfy the *distributive* identities namely $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, for every $x, y, z \in X$.[5] As mentioned in the Introduction, examples abound: judgment set X may consist of assignments of probability values to a given event (with $\leqslant$ given by the natural total order on real numbers), gradings of certain items by means of a common bounded linearly ordered set of grades (with $\leqslant$ given by the component-wise partial order induced by grades), or conditional bets (with $\leqslant$ given by the partial order induced by the betting preorder on betting-equivalence classes of conditional statements)[6].

To be sure, in many situations reliable judgments of the relevant sort are only available in an *approximate* version. But then, *approximate judgments* in $\mathcal{X} = (X, \leqslant)$ admit a convenient and natural representation via its *order intervals*, where, for any $x, y \in X$ such that $x \leqslant y$, an order interval $[x, y]$ of $\mathcal{X}$ is defined as $[x, y] := \{z \in X : x \leqslant z \leqslant y\}$. Thus, *the set of order intervals* of $\mathcal{X}$ is $I_{\mathcal{X}} := \{[x, y] : x, y \in X,\ x \leqslant y\}$, which in turn can be partially ordered component-wise, namely for any $[x, y]$, $[v, z] \in I_{\mathcal{X}}$, $[x, y] \mathrel{\widehat{\leqslant}} [v, z]$ if and only if $x \leqslant v$ and $y \leqslant z$. As a result, we end up with a new poset $\mathcal{I}_{\mathcal{X}} = (I_{\mathcal{X}}, \widehat{\leqslant})$ of *approximate judgments* of type $\mathcal{X}$, to be adjoined to the original poset $\mathcal{X}$ of (exact) judgments.

The present work is mainly devoted to the study of *aggregation rules* for such judgments, both *exact* and *approximate*. Accordingly, we denote by $N = \{1, ..., n\}$ a finite population of *agents/experts* who are those who express *judgments* that are to be aggregated and by $J \in \{X, \mathcal{I}_{\mathcal{X}}\}$ the relevant set of judgments which may be indeed either exact or approximate, and are endowed with the respective order $\leq \in \left\{\leqslant, \widehat{\leqslant}\right\}$. In any case, agents are only required to propose a single judgment in J as the one

[5]Moreover, both $\vee$ and $\wedge$ satisfy Associativity, Commutativity, Idempotency and are mutuality related to Absorption as detailed in the Appendix. In particular, $x \vee y = y$ and $x \wedge y = x$ hold if and only if $x \leqslant y$ (hence the order $\leqslant$ is easily and immediately recovered from $\vee$ or $\wedge$). Conversely, every structure $(X, \vee, \wedge)$ where $\vee$ and $\wedge$ are binary operations on X which satisfy the foregoing gives rise to a lattice $(X, \leqslant)$ where $x \leqslant y$ holds if and only if $x \vee y = y$ (or equivalently $x \wedge y = x$). Those facts shall be repeatedly used in the present work.

[6]The *betting preorder* $\preccurlyeq$ on conditional statements (denoted as p/q, s/t,...) is such that $p/q \preccurlyeq s/t$ if and only if [(s and t) is a logical consequence of (p and q) *and* (q and *not* p) is a logical consequence of (t and *not* s).

they *endorse* or equivalently the one they most *prefer* in the relevant sense. We first consider the case of exact judgments, which turns out to be very helpful to address the aggregation problem for approximate judgments.

We also assume that each agent/expert $i \in N$ is endowed with a *total preference preorder* (namely a reflexive, transitive and connected binary relation) $\succcurlyeq$ on X having a *unique maximum* denoted *top*$(\succcurlyeq)$.[7] Hence, the experts' preferences belong to the set T_X of all *topped* total preorders on X. In particular, we focus on *single-peaked total preorders* a wide subclass of preference relations with a unique top element that naturally arises whenever each agent's representation of the outcome space is endowed with some *ternary betweenness relation,* establishing, for any two judgments $x, y \in X$, whether an arbitrary $z \in X$ lies between x and y as a 'genuine compromise' between them, in that *everyone* agrees that z is not strictly worse than *both* x and y. In other words, the betweenness relation is meant to represent a shared structure of compromises between outcomes: such a shared structure may (though, of course, need not) be induced by a common 'natural' metric. Hence, single peaked preferences are aptly described as those total preorders with a unique best outcome such that an outcome located between the maximum and another distinct outcome is invariably regarded as *not worse* than the latter. Or, simply put, single peaked preferences are *consistent* with the 'compromise-structure' represented by the (latticial) **betweenness** relation $B_{\mathcal{X}}$ of $\mathcal{X} = (X, \leqslant)$ defined as follows:

$$B_{\mathcal{X}} := \left\{(x, z, y) \in X^3 : x \wedge y \leqslant z \leqslant x \vee y\right\}.$$

All of this is made precise by the following:

Definition 2.1. *A topped total preorder* $\succcurlyeq \in T_X$*, with top outcome* $top(\succcurlyeq) = x^*$*, is* ***single-peaked*** *if and only if, for each* $y, z \in X$*,* $z \in B_{\mathcal{X}}(x^*, ., y)$ *implies that* $z \succcurlyeq y$*.*

We denote by $U_{\mathcal{X}} \subseteq T_X$ the set of all *single-peaked total preorders* (with respect to $B_{\mathcal{X}}$), while $U_{\mathcal{X}}^N$ is the corresponding set of all N-*profiles* of single peaked total preorders.

We shall be mostly concerned with single-peaked total preorders domains that satisfy the following *richness* condition:

Definition 2.2. *A set* $D_{\mathcal{X}} \subseteq U_{\mathcal{X}}$ *of single-peaked total preorders (with respect to* $B_{\mathcal{X}}$*) is* ***rich*** *if, for any* $x, y \in X$*, there exists* $\succcurlyeq \in D_{\mathcal{X}}$ *such that top*$(\succcurlyeq) = x$ *and*

[7] The asymmetric and symmetric components of $\succcurlyeq$ are denoted with $\succ$ and $\sim$, respectively.

$UC(\succcurlyeq, y) = \{z \in X : B_{\mathcal{X}}(x, z, y)\}$ where the set $UC(\succcurlyeq, y) := \{z \in X : z \succcurlyeq y\}$ is the upper contour of $\succcurlyeq$ at $y \in X$.

Thus, a single peaked domain $D_{\mathcal{X}}$ is rich whenever for each pair of outcomes $x, y \in X$ there exists a preference relation in $D_{\mathcal{X}}$ having x as its top outcome and such that the subset of outcomes (weakly) preferred to y is precisely the interval between x and y.

In what follows, we mainly focus on a class of well-behaved **aggregation rules** (**AR**) of individual judgements, namely functions $f : X^N \to X$ satisfying the following benchmark properties:

Inclusiveness. For every $i \in N$, there exist $x_N \in X^N$ and $y \in X$ such that $f(y, (x_N)_{-i}) \neq f(x_N)$;

Idempotence. $f(x_N) = x$ for any $x \in X$ and $x_N \in X^N$ such that $x_i = x$ for every $i \in N$;

Strategy-proofness. on $\Pi_{i \in N} D_i \subseteq U_{\mathcal{X}}^N$ if and only if, for all $x_N \in X^N$, $i \in N$ and $x'_i \in X$, and for all $\succcurlyeq_i \in D_i$, $f(top(\succcurlyeq_i), x_{N \setminus \{i\}}) \succcurlyeq_i f(x'_i, x_{N \setminus \{i\}})$.

An aggregation rule is *Inclusive* if for each agent there exists at least one profile of proposed judgments at which her own proposal turns out to be pivotal; *Idempotent* if whenever all the proposed judgments coincide, the aggregate outcome is indeed the unanimously proposed judgment; *Strategy-proof* if it is immune to advantageous individual manipulations through submission of false information. Arguably, those three properties are minimal requirements to ensure that an aggregation rule has indeed the potential to take advantage of any ‘diversity bonus’ arising from information gradients among the involved agents, by providing the latter with some basic incentives to share their private information.

More specific and demanding requirements on an aggregation rule f will also be considered, namely:

Anonymity. For all $x_N = (x_i)_{i \in N} \in X^N$ and all permutations $\sigma : N \to N$, $f(x_N) = f(x_{\sigma(N)})$ (where $(x_{\sigma(N)} = (x_{\sigma(i)})_{i \in N})$;

Neutrality. For all $x_N \in X^N$, and all $y, z \in X$: if $N_y(x_N) = N_z(x_N)$ then $y \leqslant f(x_N)$ if and only if $z \leqslant f(x_N)$ (where for any $x \in X$, $N_x(x_N) = \{i \in N : x \leqslant x_i\}$);

Bi-idempotence. For all $x_N = (x_i)_{i \in N} \in X^N$ and all $y, z \in X$, if $x_i \in \{y, z\}$ for all $i \in N$ then $f(x_N) \in \{y, z\}$.

Anonymity requires an equal weight for all agents, Neutrality is an equal treatment requirement for alternative judgments, while Bi-idempotence ensures that at maximally polarized profiles the aggregation rule is able to select one of the advanced proposals.

In what follows, we first consider the case of exact judgments, which turns out to be very helpful to address the aggregation problem for approximate judgments.

3 Aggregation of exact judgments

We aim at characterizing three *nested classes of aggregation rules* that are, respectively:

(i) *Inclusive, Idempotent* and *Strategy-proof*;

(ii) *Anonymous*, *Neutral* and *Strategy-proof*;

(iii) *Anonymous*, *Bi-idempotent* and *Strategy-proof*;

on rich single peaked domains.

In order to describe such **AR**, we need to introduce some further definitions.

Definition 3.1. ***(Decisive Coalition)*** *We call $S \subseteq N$ a* ***coalition*** *of agents/experts. We say that S is decisive with respect to $z \in X$ for aggregation rule $f : X^N \to X$ if there exists a profile $x_N \in X^N$ such that $N_z(x_N) = S$ and $z \leqslant f(x_N)$, and* ***decisive*** *for f if there exists some $z \in X$ such that S is decisive with respect to z for f.*

Thus, a coalition is decisive if it can 'force' the outcome to be consistent with a certain judgment. We denote by $\mathcal{W}^f$ the set of all decisive coalitions for an aggregation rule f, and by $\mathcal{W}^f_m \subseteq \mathcal{W}^f$ the set of all *minimal* decisive coalitions for f, (namely those decisive coalitions for f such that all of their *proper* subcoalitions turn out to be *not* decisive).

Definition 3.2. ***(Generalized Committee)*** *A* ***generalized committee*** *in N is a set of coalitions $\mathcal{W} \subseteq \mathcal{P}(N)$ such that $T \in \mathcal{W}$ if and only if $T \subseteq N$ and $S \subseteq T$ for some $S \in \mathcal{W}$. (In particular, a* ***committee*** *in N is a non-empty generalized committee in N which does not include the empty coalition). The set $\mathcal{W}_m \subseteq \mathcal{W}$ of minimal coalitions of a generalized committee $\mathcal{W} \subseteq \mathcal{P}(N)$ is called its basis.*

We say that a generalized committee $\mathcal{W} \subseteq \mathcal{P}(N)$ is:

Inclusive if $\bigcup \mathcal{W}_m = N$; moreover, a family $\left\{\mathcal{W}^i : \mathcal{W}^i \subseteq \mathcal{P}(N)\right\}_{i\in I}$ of (generalized) committees is said to be *inclusive* if $\bigcup_{i\in I} \mathcal{W}^i_m = N$.

Anonymous if, for any $S, T \subseteq N$ such that $|S| = |T|$, $S \in \mathcal{W}$ if and only if $T \in \mathcal{W}$.

Accordingly, an aggregation rule $f : X^N \to X$ is said to be **Inclusive** if and only if $\bigcup_{S\in\mathcal{W}^f_m} S = N$. Therefore, an aggregation rule is inclusive if every agent turns out to be *decisive* or *pivotal* under certain circumstances.

To each generalized committee $\mathcal{W}$ several aggregation rules on the bounded distributive lattice $\mathcal{X} = (X, \leqslant)$ of judgments can be attached in a natural way. For instance, the *maxmin* (or disjunctive-normal-form) aggregation rule of $\mathcal{W}$ is the function $f : X^N \to X$ such that $f(x_N) = \bigvee_{T\in\mathcal{W}} \bigwedge x_{i\in T}$ for every $x_N \in X^N$: thus, its outcome is the join of the consensus outcomes of the coalitions in $\mathcal{W}$.

Moreover, aggregation rules can also be attached to a family $\{\mathcal{W}^i\}_{i\in I}$ of generalized committees, including the *max* (or disjunctive) aggregation rule of $\{\mathcal{W}^i\}_{i\in I}$, which is the function $f : X^N \to X$ such that $f(x_N) = \bigvee \{z \in X : N_z(x_N) \in \mathcal{W}^i \text{ for some } i \in I\}$ for every $x_N \in X^N$ to the effect of selecting the join of outcomes that are accepted by all the members of some coalition of some generalized committee $\mathcal{W}_i$. This sort of rule can also be further specified by attaching distinct generalized committees to certain distinguished outcomes (see e.g. Monjardet (1990)).

We are now ready to state our first characterization result of aggregation rules for **exact** judgments in bounded distributive lattices.

THEOREM 3.1. *Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice with $|X| \geq 3$ and $B_{\mathcal{X}}$ be its latticial betweenness relation, $D_{\mathcal{X}} \subseteq U_{\mathcal{X}}$ a rich domain of single-peaked total preorders on X (with respect to $B_{\mathcal{X}}$), and N a finite set with $|N| \geq 3$. Then, an aggregation rule $f : X^N \to X$ is:*

(i) ***inclusive***, ***idempotent*** *and* ***strategy-proof*** *on $D_{\mathcal{X}}^N$ if and only if there exists an inclusive family of committees $\{\mathcal{W}^i\}_{i\in I}$ on N such that for any $x_N \in X^N$:*

$$f(x_N) = \bigvee \{x \in X : N_x(x_N) \in \mathcal{W}^i \text{ for some } i \in I\}; \tag{1}$$

(ii) ***anonymous***, ***neutral*** *and* ***strategy-proof*** *on $D_{\mathcal{X}}^N$ if and only if there exists a* ***fixed-quota-committee*** *$\mathcal{W}^q = \{S \subseteq N : |S| \geq q\}$, for some positive integer $q \leq n$, on N such that $f = f_{\mathcal{W}^q}$*

$$f(x_N) = \bigvee_{S\in\mathcal{W}^q} \bigwedge_{i\in S} x_i$$

for all $x_N \in X^N$;

(iii) ***anonymous***, ***bi-idempotent*** *and* ***strategy-proof*** *on $D_{\mathcal{X}}^N$ if and only if $|N|$ is odd and f is the simple majority rule f^{maj}, namely for all $x_N \in X^N$,*

$$f^{maj}(x_N) = \bigvee_{S\in\mathcal{W}^{maj}} \bigwedge_{i\in S} x_i$$

where $\mathcal{W}^{maj} = \left\{ S \subseteq N : |S| \geq \lfloor \frac{|N|+2}{2} \rfloor \right\}$.

Thus, we have three distinct algebraic closed-form representations of two significant classes of aggregation rules and of a version of the simple majority rule for those judgements which amount to points of a bounded distributive lattice. Specifically, the first and most comprehensive class collects all the aggregation rules which respect unanimity, ensure a locally pivotal role to every agent while being at the same time strategy-proof on a large class of single peaked domains. The second class is the subclass of the previous one consisting of aggregation rules which are also anonymous and neutral: they are given in terms of certain lattice polynomials in disjunctive-normal form. Since polynomials can be regarded as efficient algorithms, it follows that the outputs of such polynomial aggregation rules are by definition *'easily' computed*. Within the latter class the simple majority rule is characterized by bi-idempotence, the property requiring selection of an uncompromising proposal in maximally polarized situations[8].

4 Aggregation of approximate judgments

In what follows, we *apply Theorem 1 to the aggregation of approximate assessments of several sorts*. Indeed, as previously observed, on many occasions approximate versions of assessments and judgements most typically amount to specifications of *order intervals* of such values, namely of the values comprised between the smallest and the largest one of a pair of ordered values. This is clearly the case for approximate probability estimates, measurements with explicit error bounds, approximate classifications of objects according to a somewhat imprecise type. As a matter of fact, it also applies (as we shall see below) even to *conditional judgments*, since they too are representable by certain order interval.

Thus, in order to apply Theorem 1, the following Claim is key in that connection, because it ensures that *the partially ordered sets of order intervals* alluded to above *are* indeed *bounded distributive lattices*.

Claim 1. *Let* $(L, \vee, \wedge)$ *be a lattice, and* I_L *the set of its order intervals, i.e.* $I_L := \{[x, y] : x, y \in L,\ x \leqslant y\}$, *with* $x \leqslant y$ *if and only if* $x \wedge y = x$ *or equivalently* $x \leqslant y$

[8] Indeed, a strategy-proof and bi-idempotent aggregation rule on a bounded distributive lattice is also neutral (see Vannucci (2019), Theorem 1).

if and only if $x \vee y = y$ *and* $[x, y] := \{z \in L : x \leqslant z \leqslant y\}$, *and for any* $[x, y], [w, z] \in I_L$*:*

$$[x, y]\widehat{\vee}[w, z] \quad : \quad = \{u \vee v : u \in [x, y], v \in [w, z]\},$$
$$[x, y]\widehat{\wedge}[w, z] \quad : \quad = \{u \wedge v : u \in [x, y], v \in [w, z]\}.$$

Then, the following statements are equivalent:

(i) $(L, \vee, \wedge)$ *is a (bounded) distributive lattice;*

(ii) $(I_L, \widehat{\vee}, \widehat{\wedge})$ *is a (bounded) lattice;*

(iii) $(I_L, \widehat{\vee}, \widehat{\wedge})$ *is a (bounded) distributive lattice.*[9]

We are now ready to prove the main result of the present paper, namely:

THEOREM 4.1. *Let* $\mathcal{X} = (X, \leqslant)$ *be a (bounded) lattice with* $|J| \geq 3$, $\mathcal{I}_{\mathcal{X}} = (I_X, \widehat{\leqslant})$ *the (bounded) poset of its order intervals,* $B_{\mathcal{I}_{\mathcal{X}}}$ *its latticial betweenness relation,* $D_{\mathcal{I}_{\mathcal{X}}} \subseteq U_{\mathcal{I}_{\mathcal{X}}}$ *a rich domain of single-peaked total preorders on* X *(with respect to* $B_{\mathcal{I}_{\mathcal{X}}}$*), and* $f : I_X^N \to I_X$ *an* anonymous *and* bi-idempotent *aggregation rule which is* strategy-proof *on* $D_{\mathcal{I}_{\mathcal{X}}}^N$ *with* $n = |N| \geq 3$ *and odd. Then, the following statements are equivalent:*

(i) $\mathcal{I}_{\mathcal{X}} = (I_X, \widehat{\leqslant})$ *is a (bounded)* ***lattice;***

(ii) $\mathcal{I}_{\mathcal{X}} = (I_X, \widehat{\leqslant})$ *is a (bounded)* ***distributive lattice;***

(iii) f *is the* ***majority rule*** f^{maj}*, namely for all* $x_N \in I_X^N$,

$$f(x_N) = \bigvee_{S \in \mathcal{W}^{maj}} \bigwedge_{i \in S} x_i$$

where $\mathcal{W}^{maj} = \left\{ S \subseteq N : |S| \geq \lfloor \frac{|N|+2}{2} \rfloor \right\}$.

Proof. $(i) \iff (ii)$: Immediate from Claim 1 by taking $L = X$.

$(ii) \implies (iii)$: If $\mathcal{I}_{\mathcal{X}}$ is a (bounded) distributive lattice then f^{maj} is well-defined and is the unique anonymous bi-idempotent and strategy-proof on $D_{\mathcal{I}_J}^N$ with $n = |N|$ odd by *Theorem 1 (iii)*.

$(iii) \implies (ii)$: If f^{maj} is a well-defined aggregation rule with domain I_J^N then $\mathcal{I}_{\mathcal{X}} = (I_J, \widehat{\leqslant})$ must be a *distributive* lattice, because otherwise $\mathcal{W}^{f^{maj}}$ has a *unique* minimal element (or 'winning coalition'), a contradiction since by definition $\mathcal{W}^{f^{maj}} = \mathcal{W}^{maj}$ and (with $n = 2k+1$) $\mathcal{W}^{maj}$ has in fact $\binom{n}{k+1} > 1$ minimal elements (or 'winning coalitions'). □

[9] To the best of the authors' knowledge, Claim 1 has never been stated and proved in previous works, though some (but by no means all) of its constitutive implications have been established by Fitting (1991) or Milne (2004) (more details on that issue are available from the authors upon request).

Thus, if the set of agents is odd, the majority aggregation rule can be characterized as the only anonymous and bi-idempotent rule which is strategy-proof on any single-peaked domain and applies both to a latticial judgment space and to its respective approximate version. As mentioned above, the foregoing theorem can be immediately applied to the aggregation of approximate assessments of several sorts.

5 Applications

In what follows, we provide some applications of Theorem 2 to the very general *problem of aggregation* of profiles of approximate, imprecise multiple measurements and classifications or approximate conditional judgements, a problem that typically arises whenever a DMA must collect information that comes from the evaluation on a certain issue by a group of experts. All the expert elicitation protocols, as for instance Delphi, Q-methodology, Nominal Group technique etc., suffer from problems of polarization, strategic manipulation, overconfidence, self-censorship, pressure to conform, anchoring, adjustment etc. The proposed aggregation rules help to avoid at least some of the possible aforementioned drawbacks and easily apply to cases whose outcome space is possible to model as a bounded distributive lattice.

In what follows, we go back to the examples presented in the Introduction, we formally state them and we show that our results easily apply to those prominet aggregation issues.

5.1 Approximate classifications

An ensemble E of *rough classifiers,* trained on different training sets, predict cases of *viral infections* from a collection of easily detected symptoms. The available cases are characterized by a profile of values $\{0,1\}$ for each of the following health troubles:

$$U = \left\{ \begin{array}{c} \text{Runny Nose (RN), Nausea (N), Headache (H), Sore Throat (ST), Cough (C),} \\ \text{Fever (F), Feeling of Being Unwell (FBU), Shortness of Breath (SB), Infected (I)} \end{array} \right\},$$

where each attribute takes value 0 if the patient monitored is not affected, for instance by cough, and 1 otherwise. For a subset $D \subset U$, an expert may be required to produce a D-based classification of those who are infected. This kind of selection corresponds to define an *approximate classification* of *infected units* from the sample, namely the *rough set* analytically expressed by an order interval. The final ensemble decision is supposed to produce a unique approximate classification by aggregating the rough sets of profile of experts' order intervals.

We show here that the outcome space of the different approximate classifications of the elements of a set, based on some criteria and due to a committee of *rough* experts, is a bounded distributive lattice. In order to do that, we rely on a result due to Milne (2004), that allows us to directly apply Theorem 2 to this remarkable case.

To start with, we denote with X the universal set of objects to be classified according to some prescribed type(s), U the universal finite set of criteria/attributes, V_p the set of possible values of criterion p, for any $p \in U$, and $f : X \to \prod_{p\in U} V_p$ the information function (see Comer (1991)) that just classifies the objects according to their own characteristics. The structure $\mathcal{I} = (X, U, (V_p)_{p\in U}, f)$ is usually called an *information system*. For any $P \subseteq U$, we say that $\sim_P$ is an equivalence relation on X if, for each $x, y \in X$, $x \sim_P y$ if and only if $(f(x))_p = (f(y))_p$ for each $p \in P$. The pair $(X, \sim_P)$ is said to be an *approximation space* for knowledge P, and the equivalence classes of $\sim_P$, written as $[x]^{\sim_P}$with $x \in X$, are called the P-elementary concepts.

Both a closure operator $\overline{P} : \mathcal{P}(X) \to \mathcal{P}(X)$ (i.e. a monotonic, idempotent, extensive function), and an interior operator $P^\circ : \mathcal{P}(X) \to \mathcal{P}(X)$ (i.e. a monotonic, idempotent, function) can be attached to $(X, \sim_P)$ by the following definitions: for any $A \subseteq X$,

$$\overline{P}(A) := \bigcup \{[x]^{\sim_P} : x \in A\} \text{ and } P^\circ(A) := \bigcup \{[x]^{\sim_P} : [x]^{\sim_P} \subseteq A\}\,.$$

By construction, $P^\circ(A) \subseteq A \subseteq \overline{P}(A)$, with $P^\circ(A)$ that is also denoted as the P-*lower approximation* of A, and $\overline{P}(A)$ as the P-*upper approximation* of A. A is said to be P-*definable* if and only if it is $\overline{P}$-closed, i.e. if and only if it is a fixed point of $\overline{P}$.

A *rough set* is defined as the order interval $[P^\circ(A), \overline{P}(A)]$ of the poset $(\mathcal{P}(X), \subseteq)$ and can be regarded as a P-based *approximate classification* of the objects of X in terms of some target label having A as its extension in X.[10] Now, let $I_{\mathcal{P}(X)}$ be the set of all order intervals of $(\mathcal{P}(X), \subseteq)$ and $I^P_{\mathcal{P}(X)} \subseteq I_{\mathcal{P}(X)}$ the set of all such P-based *approximate classifications* of a certain type of tokens in X. Suppose now that a set of experts may be required to produce a P-based classification of a certain target type by selecting a suitable set $A \subseteq X$. This sort of classification amounts precisely to a P-based approximate classification, i.e. an order interval $[P^\circ(A), \overline{P}(A)] \in I^P_{\mathcal{P}(X)}$. Again, define, for any $[P^\circ(A), \overline{P}(A)]$,$[P^\circ(B), \overline{P}(B)] \in I^P_{\mathcal{P}(X)}$, the partial order $\leqslant$ on $I^P_{\mathcal{P}(X)}$ as

[10] Roughness or uncertainty of a rough set is represented by its boundary region: the larger the boundary region is, the greater the roughness is.

follows:

$$[P^\circ(A), \overline{P}(A)] \leqslant [P^\circ(B), \overline{P}(B)] \text{ if and only if } P^\circ(A) \subseteq P^\circ(B) \text{ and } \overline{P}(A) \subseteq \overline{P}(B).$$

Now, by Milne (2004) (Theorem 3.4.1, p. 515) the poset $(I^P_{\mathcal{P}(X)}, \leqslant)$ is a bounded distributive lattice. It follows that Theorem 2 applies to the poset $(I^P_{\mathcal{P}(X)}, \leqslant)$ of approximate classifications and then suitable aggregation rules for approximate classifications exist and satisfy some compelling properties including that making the aggregation of approximate classifications of a set of objects by a team of experts protected against strategic manipulations.

5.2 Approximate measurements with error bounds

Let $i = \{1, 2, 3\}$ be a finite number of experts that has to estimate the global mean surface atmopheric temperature (SAT) increase for the $\{s_1, s_2, s_3)$ selected emission scenarios of possible future states of the world. Assume that each expert i has a probability distribution function on each possible scenario. As a consequence of the variety, different quality and origin of the information, an expert defines the mean of SAT by considering the lower and upper probabilities for every scenario (see Kriegler et al. (2009)) as follows:

$$\begin{array}{lccc} & s_1 & s_2 & s_3 \\ 1 := \{ & (0.25; 0.45) & (0.22, 0.38) & (0.34, 0.36) \ \} \\ 2 := \{ & (0.34; 0.36) & (0.38, 0.40) & (0.28, 0.38) \ \} \\ 2 := \{ & (0.36; 0.50) & (0.40, 0.50) & (0.42, 0.52) \ \} \end{array}$$

Thus, the mean value for every scenario can be elicited with an error bound, i.e.:

$$\begin{array}{lccc} & s_1 & s_2 & s_3 \\ 1 := \{ & (0.35 \mp 0.10) & (0.30 \mp 0.08) & (0.35 \mp 0.01) \ \} \\ 2 := \{ & (0.35 \mp 0.01) & (0.39 \mp 0.01) & (0.33 \mp 0.05) \ \} \\ 2 := \{ & (0.43 \mp 0.07) & (0.45 \mp 0.05) & (0.47 \mp 0.05) \ \} \end{array}$$

Then, for each scenario $j = 1, 2, 3$ we have the following profiles to be amalgamated:

$$[(0.35 \mp 0.01) \leq (0.35 \mp 0.10) \leq (0.43 \mp 0.07)];$$

$$[(0.30 \mp 0.08) \leq (0.39 \mp 0.01) \leq (0.45 \mp 0.05)]\,;$$

$$[(0.33 \mp 0.05), (0.35 \mp 0.01), (0.47 \mp 0.05)]\}.$$

More in general, a set of experts provides estimates, i.e. *approximate (numerical) measurements* that are *approximate numbers,* namely an ordered pair $(m,\delta) \in \mathbb{R}_+ \times \mathbb{R}_+$, where m is the proposed numerical *estimate* and δ (with $m \geq \delta \geq 0$) its *error bound* (see e.g. Markov (2016)).

Clearly, there is an obvious one-to-one correspondence between approximate numbers and *order intervals* $[x,y] \subseteq \mathbb{R}_+$ of $(\mathbb{R}_+, \leq)$ namely $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{P}(\mathbb{R}_+)$ as defined by the rule $\varphi(m,\delta) = [m-\delta, m+\delta]$, and the inverse of its corestriction $\psi : \varphi(\mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}_+ \times \mathbb{R}_+$ as defined by the rule $\psi([x,y]) = (\frac{y-x}{2}, \frac{x+y}{2})$. In order to check validity of that statement, observe that $(\psi \circ \varphi)(m,\delta) = \psi(\varphi(m,\delta)) = \psi([m-\delta, m+\delta]) = (m,\delta))$.

It follows that an aggregation rule for N-profiles of approximate numbers, or order intervals, of $(\mathbb{R}_+, \leq)$ essentially amounts to an aggregation rule $f : I^N_{[0,1]} \to I_{[0,1]}$, where $I_{[0,1]}$ denotes the set of order intervals of the bounded distributive lattice $([0,1], \leq)$.

But then, by Claim 1, the poset $(I_{[0,1]}, \leqslant)$, with the partial order $\leqslant$ defined to as $[x,y] \leqslant [w,z]$ if and only if $[x,y]\widehat{\vee}[w,z] = [w,z]$, is a bounded distributive lattice, hence Theorem 2 applies and again we obtain a wide class of interesting aggregation rules for the problem of amalgamating numerical *estimates with error bounds* which are provided by a set of experts concerning a certain issue.

5.3 Aggregation of interval probability estimates

Let $i = \{1,2,3\}$ be a finite number of experts that have to make judgments about the probability of different pandemic event occurrence, such as the global increase of the fatality rate as a function of the mean of infected, as espressed as subjective probability intervals. They have different available global datasets and any information about past conditions is a compromise between other data. As a result of their analysis, they expect that the impact of the lockdown withdrawn will increase the global pandemic of 1.5% between the month of April and June. They estimate that this state can be reached with a certain probability that can vary in an interval $[a,b]_i$, where $i = 1,2,3$ denotes the expert concerned and $a, b \in [0,1]$, as follows:

$[a,b]_1 = [0.55, 0.6]$, low probability;

$[a,b]_2 = [0.62, 0.7]$, middle probability;

$[a,b]_3 = [0.68, 0.72]$, high probability;

A DMA has the problem to aggregate these probability intervals in order to fix her future intervention policy aimed to reduce as much as possible the probability that fatality rate increases too much in the future.

In general let $I_{[0,1]} := \{[a,b] : a,b \in [0,1], a \leq b\}$ be the set of all closed probability intervals defined on the ordered real unit interval $[0,1]$, that is in fact a bounded distributive lattice. A panel of $N = \{1,...,n\}$ experts have to make judgements about the probability of the occurrence of an event. Those n subjective probability judgments are collected in the set J and expressed as intervals, an expert judgment on an uncertain event is therefore a closed interval of possible probability values $[a_j, b_j] \in I_{[0,1]}$, where a_j and b_j respectively denote the lower and the higher probabilities that an event happens, hence, $J = I_{[0,1]}^N$. Such intervals can be compared by a partial order $\leqslant$ saying that, for any $[a,b],[c,d] \in I_{[0,1]}$, $[a,b] \leqslant [c,d]$ if and only if $a \leq c$ and $b \leq d$. As a consequence of Claim 1, $(J, \leqslant)$ is also a bounded distributive lattice, and then Theorem 2 applies.[11]

5.4 Conditional judgments and conditional bets

A DMA is interested to evaluate climate response to alternative future trajectories of radiative forcing. Unfortunately, the behavior of climate system is very uncertain and ambiguous probabilistic estimations of equilibrium climate sensitivity result from models of different complexity and statistical methods. Then, experts' judgments about global mean temperature are considerable different.

Consider the following list of plausible levels of global warming within a certain conveniently fixed time frame:

$q_1 =$ negligible increase ΔT of global mean temperature $(0 \leq \Delta T \leq 2\ °C)$
$q_2 =$ medium increase ΔT of global mean temperature $(2\ °C < \Delta T \leq 4\ °C)$
$q_3 =$ high increase ΔT of global mean temperature $(4\ °C < \Delta T)$.

Then, consider the following list of large-scale events in the Earth system ('tipping points': see e.g. Kriegler et al. (2009)):

$p_1 =$ reorganisation of the Atlantic Meridional overturning circulation (drastic reduction in deep water overflow across the Greenland-Scotland ridge),
$p_2 =$ dieback of the Amazon rainforest,
$p_3 =$ melt of the Greenland ice sheet,
$p_4 =$ disintegration of the West Antarctic ice sheet.

[11] It is worth remarking here that if the probability intervals elicited by Obama's experts had been compared by a partial order in order to get a bounded distributive lattice as judgment space, then Obama could have taken his decision by simply applying the median rule.

The relevant conditional judgments may be represented by *conditional assertions* (e.g. if the global mean temparature increases of 4 $^\circ C$, then the Greenland ice sheet will melt). Conditional assertions are informally identified with certain *conditional events.* They are represented as conditional propositions p/q (namely 'p given q' in a suitably defined propositional calculus $\mathbf{CA}$[12]) which may be regarded as the objects of conditional bets. Conditional assertions can be ordered in a natural way by the 'betting' *entailment* (or *betting preorder*) saying that $p/q \preccurlyeq s/t$ if and only if [(s and t) is a logical consequence of (p and q) *and* (q and *not* p) is a logical consequence of (t and *not* s)].[13]

In particular, two conditional assertions p/q, s/t are 'betting'-*equivalent* -written $p/q \approx s/t$- if and only if both $p/q \preccurlyeq s/t$ *and* $s/t \preccurlyeq p/q$ hold. Of course, the resulting equivalence classes $[p/q]_\approx$ of conditional assertions are a partially ordered set or poset with the partial order $\leqslant$ defined in the obvious way, namely for any pair of conditional assertions p/q, s/t, $[p/q]_\approx \leqslant [s/t]_\approx$ if and only if $p/q \preccurlyeq s/t$.

The relevant conditional judgments in the example may be represented by conditional assertions p_{ih}/q_j, $i = 1, 2, 3, 4$, $h = 1, ..., k$, $j = 1, 2, 3$ *and their disjunctions, conjunctions, implications and negations.*

Each expert is asked to select the conditional judgment she regards as most plausible (to repeat, multiple submissions are admissible but treated as a disjunction of the single items). Finally, a DMA needs to amalgamate all the information received. We identify the set J of *conditional judgments* precisely with such equivalence classes of conditional assertions with respect to the betting preorder. Hence, we end up with a *poset* $(J, \leqslant)$ *of conditional judgments* of the conditional assertion calculus $\mathbf{CA}$ that is precisely the outcome space of our aggregation problem. It turns out that, thanks to the structure imposed on conditional assertions by their propositional calculus and the corresponding 'betting' entailments, $(J, \leqslant)$ inherits a very rich algebraic structure denoted as *de Finetti algebra* (see Milne (2004) for details). Milne (2004) in particular shows that the poset $(J^{CA}, \leqslant)$ of conditional judgments of $\mathbf{CA}$ is a de Finetti algebra and that the latter is a bounded distributive lattice with some supplementary structure. Of

[12] That kind of calculus is outlined in de Finetti (1936), and made more precise by Milne (2004), but its details need not detain us here.

[13] Thus, the 'betting' entailment preserves *truth* from premises to conclusions of deductive inferences, and retransmits *falsity* from a conclusion to some of the corresponding premises. Or, to put in 'betting' terms all the way, the bet on s/t is won whenever the bet on p/q is won, and the bet on p/q is lost whenever the bet on s/t is lost (see e.g. de Finetti (1936), Milne (2004)).

course, the key point of the foregoing representation theorem is, from our own present perspective, that conditional judgments are in fact a special instance of a bounded distributive lattice. Hence, Theorem 2 does apply to $(J^{CA}, \leqslant)$ and we have shown that the aggregation of conditional judgments that is a process that requires an effective elicitation of individual (conditional) judgments, might produce a reliable and therefore strategy-proof (conditional) judgment as an output.

6 Related literature and concluding remarks

Indeed, the extant literature on aggregation of (expert) opinions on uncertain events is vast. But the format of the relevant opinions may be *not* that of intervals of bounded distributive lattices or conditional judgments as presented above. Moreover, the aggregation protocols considered may not be aggregation operations.

To be sure, experts are sometimes asked *probability estimates* on a certain family of events, possibly including conditional events (see e.g. Dalkey (1975)): in particular, a list of events to be classified as 'probable' or 'not probable' by experts is given in order to get an aggregate assessment with the same format. Hence, we are given a bounded distributive lattice and an aggregation problem that fits our present model. In some cases experts are asked to submit *positive probability values* -to be chosen from a *pre-fixed finite set* $\{p_1, ..., p_m\}$- to a finite set of independent uncertain events (see e.g. Cooke (1991), ch. 12). If the possible outcomes are arbitrary combinations of those probability values for each event then again our results do apply.

Interestingly for our paper, Cooke (1991) concludes that "point prediction as medians of experts' combined distributions outperform combined medians" by inducing a 65% improvement with performance weighted combinations and 46% improvement with equally weighted combinations.

By contrast, allowing for probability estimates of *distinct* subfamilies on the part of distinct experts (see e.g. Osherson,Vardi (2006)) gives rise to a *partial* aggregation rule (which is not covered by our results).

Moreover, it is also common to consider opinions consisting of *probability distributions* (see e.g. Lehrer and Wagner (1981), Rubinstein and Fishburn (1986), Barrett and Pattanaik (1987), Clemen and Winkler (1999, 2007), Basili and Pratelli (2015)). But of course, under the component-wise natural order probability distributions are not a lattice. To be sure, under the majorization or dominance order (discrete) probability distributions do constitute a lattice: but such a lattice is definitely nondistributive (actually it does not admit a rank function hence it is not even semimodular: see e.g. Brylawski (1973)).

More recently, several authors consider the aggregation of *imprecise probability assessments* modeled by *convex sets of admissible probability measures* or by *probability intervals* to produce a more precise aggregate probability estimate (see among others Nau (2002), Crès, Gilboa and Vieille (2011), Gajdos and Vergnaud (2013), Basili and Chateauneuf (2020)). Of course, that approach amounts to a *codomain-constrained aggregation* exercise, which would not fit our present framework even if the domain of imprecise probability assessments were a bounded distributive lattice (which is typically *not* the case anyway). Conversely, Stewart and Quintana (2018) consider aggregation of probability distributions into *a set of admissible probability distributions*, i.e. an imprecise aggregate opinion: but this kind of pooling exercise amounts of a *restricted aggregation* exercise on sets of probability distributions (by taking the class of singleton sets of probability distributions as its restricted domain).

Other works model expert opinions by *opinion functions,* i.e. *functions on a complementation-closed set $\mathcal{E}$ of events* that are extendable to probability measures on the σ-algebra generated by $\mathcal{E}$ (see Dietrich, List (2017)). Even *qualitative probability relations* have been considered (Weymark (1997)) as the target of the opinion aggregation exercise: however, it is easily checked that in general qualitative probability relations ordered by set inclusion are not even a lattice since the trivial top preorder (the one consisting of a single indifference class) is not a qualitative probability relation (recall that a qualitative probability relation is a total preorder $\succcurlyeq$ on a Boolean algebra of subsets with state space X and such that $X \succ \varnothing$, hence $\mathcal{P}(X) \times \mathcal{P}(X)$ does not qualify).

Any such model of expert opinions has its own merits. When it comes to aggregation problems, however, most of those formats do not support inclusive and strategy-proof rules on suitably rich preference domains. Our results then (i) highlight some distinctive advantages of those formats that in fact, thanks to their bounded distributive latticial structure, do support use of median rules hence in particular of majority rule along several other inclusive aggregation rules that turn out to be strategy-proof on arbitrary rich single-peaked domains; (ii) show that such 'majority-friendly' formats include conditional judgements and conditional bets, probability intervals, approximate measurements with explicit error bounds, and approximate classifications.

Summing up, the main message of the present work is that whenever the domain of relevant judgments is a bounded distributive lattice, the *aggregation of such judgments* - as provided e.g. from a committee of experts - *can always be performed by computing their median* by means of a suitably defined majority rule. Moreover, *the same holds for approximate versions of such judgments as representable by ordered intervals* of the relevant values. Furthermore, even if each agent/expert wishes her opinion to prevail such a median-based aggregation is *strategy-proof* provided that *her preferences are single-peaked.* A distinctive advantage of strategy-proofness of the aggregation rule - thus secured - is that it dispenses with the whole apparatus of *proper and strictly proper scoring rules* that is commonly deployed in order to induce truth-telling when expert judgment aggregation is performed through *weighted means* (see e.g Cooke (1991)). Furthermore, the single peakedness requirement is arguably *not* particularly demanding in the present context, since (as previously observed) a natural metric on opinions is available in the case under consideration.

However, the feasibility of such robust median-based judgment aggregation rule relies on two basic structural requirements: (i) the space of alternative outcomes must be a *bounded distributive lattice* and (ii) the aggregation rule must be a well-defined *operation on the space of alternative outcomes*. It remains to be seen if and to what extent aggregation problems that have been traditionally approached with other formats are amenable to reformulations that are consistent with the framework advanced here.

Appendix 1: Proofs

To begin with, it should be recalled that a partially ordered set (poset) $\mathcal{X} = (X, \leqslant)$ is a lattice precisely when the structure $(X, \vee, \wedge)$ with $x \vee y = y$ and $x \wedge y = x$ iff $x \leqslant y$ satisfies the following four properties which shall be freely and repeatedly used in the following proofs:

- *Associativity*: $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ for all $x, y, z \in X$;
- *Commutativity*: $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ for all $x, y \in X$;
- *Idempotency*: $x \vee x = x$ and $x \wedge x = x$ for all $x \in X$;
- *Absorption*: $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$ for all $x, y \in X$.

A few basic notions to be used in the proofs of the main results are now to be introduced.

- An **order filter** of a lattice $\mathcal{X}$ is a set $F \subseteq X$ such that $z \in F$ if and only if $z \in X$ and $y \leqslant z$ for some $y \in F$: it is said to be *non-trivial* if $F \neq \varnothing$ and *proper* if $F \neq X$.

- An order filter F of $\mathcal{X} = (X, \leqslant)$ is a (latticial) **filter** if $x \wedge y \in F$ for any $x, y \in F$, and a **prime filter** if $x \vee y \in F$ implies that $x \in F$ or $y \in F$.

- A (latticial) filter F of $\mathcal{X} = (X, \leqslant)$ is an **ultrafilter** if it is proper and *maximal* (i.e. there is no proper filter $F'$of $\mathcal{X}$ such that $F \subset F'$) and is **principal** if there exists an $x \in X$ such that $F = [x) := \{y \in X : x \leqslant y\}$.

Clearly, as it is easily checked, *any proper principal filter is an ultrafilter*. We denote with $\mathcal{F}_P$ the set of all *non-trivial and proper* prime filters of $\mathcal{X}$.

We report here some important and well-known results that we use in the main text:

PROPOSITION 6.1 (*see e.g. Davey, Priestley* (1990), *ch.*10). *There is an isomorphism between the elements of a bounded distributive lattice $\mathcal{X}$ and the sets of prime filters of $\mathcal{X}$ they belong to, namely the function $\phi : X \to 2^{\mathcal{F}_P}$ defined by the rule $\phi(x) = \{F \in \mathcal{F}_P : x \in F\}$ is both injective and surjective.*

Bijection ϕ is in fact the basis of Priestley's representation theorem, establishing that any bounded distributive lattice $\mathcal{X}$ is isomorphic to the lattice of all superset-closed clopen sets of the ordered topological space $(\mathcal{F}_P, \tau, \subseteq)$ where τ is the smallest topology on $\mathcal{F}_P$ which includes the set-theoretic union of $\{\{F \in \mathcal{F}_P : x \in F\} : x \in X\}$ and $\{\{F \in \mathcal{F}_P : x \notin F\} : x \in X\}$ (see e.g. Davey and Priestley (1990), Theorem 10.18).

PROPOSITION 6.2 (*see Davey, Priestley* (1990), *Theorem* 9.7). *Every ultrafilter (hence in particular every (proper) principal filter) of a bounded distributive lattice is a (proper) prime filter.*

Next, we introduce two key properties of aggregation rules which also have a key role in the proofs. We say that an aggregation rule $f : X^N \to X$ is:

- $B_{\mathcal{X}}$**-monotonic** if and only if for all $x_N = (x_j)_{j \in N} \in X^N$, $i \in N$ and $x'_i \in X$,

$$f(x_N) \in B_{\mathcal{X}}(x_i, ., f(x'_i, x_{N \setminus \{i\}});$$

- **Monotonically Independent** if and only if for all $x_N, y_N \in X^N$ and all $F \in \mathcal{F}_P$,

$$\text{if } N_F(x_N) \subseteq N_F(y_N), \text{ then } f(x_N) \in F \text{ implies } f(y_N) \in F.$$

with the obvious meaning that an aggregation rule is $B_{\mathcal{X}}$*-monotonic* if for any agent and any profile of proposals, proposing a certain outcome brings closer to that outcome than any other alternative proposal, the outcome is a compromise of all individual assessments and *Monotonically Independent* whenever social acceptance of an outcome is never disrupted by an increasing support for that outcome.

The following two Lemma extending some previous results concerning strategy-proofness of aggregation rules on trees and on bounded distributive lattices (see Danilov (1994), Vannucci (2016)), will be used in the proof of our main results.

LEMMA 6.1 (VANNUCCI (2019)). *Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice, $f : X^N \to X$ an aggregation rule for (N, X), and $D_{\mathcal{X}}$ a rich domain of single-peaked total preorders (with respect to $B_{\mathcal{X}}$). Then, the following statements are equivalent:*

(i) f is strategy-proof on $D_{\mathcal{X}}^N$;

(ii) f is $B_{\mathcal{X}}$-monotonic;

(iii) f is Monotonically Independent.

REMARK 6.1. *Notice that richness of the preference domain is only required to prove that strategy-proofness implies $B_{\mathcal{X}}$-monotonicity, while the reverse implication holds anyway. Without the richness restriction strategy-proofness is a weaker condition than $B_{\mathcal{X}}$-monotonicity.*

LEMMA 6.2 (VANNUCCI (2016), SAVAGLIO, VANNUCCI (2019)). *Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice and $B_{\mathcal{X}}$ its latticial betweenness relation. Then, a generalized committee aggregation rule $f : X^N \to X$ is $B_{\mathcal{X}}$-monotonic.*

We are now ready to prove our main results.

Proof of Theorem 1

(i) Let $f : X^N \to X$ be an inclusive and idempotent aggregation rule that is strategy-proof on $D_{\mathcal{X}}^N$ and $F \in \mathcal{F}_P$ an arbitrary nonempty and proper prime filter of $\mathcal{X}$. To begin with, observe that by Lemma 1 f is also *monotonically-independent,* hence, in particular, independent. By idempotence of f, there exists a coalition $S \subseteq N$ which is F-decisive for f. Moreover, $S \neq \varnothing$ by monotonic-independence of f (since by the latter property $S = \varnothing$ entails $f(x_N) \in F$ for all $x_N \in X^N$, contradicting idempotence of f).

Then, for any $F \in \mathcal{F}_P$, consider:

$$\begin{aligned} \mathcal{W}_F^f &= \{T \subseteq N : T \text{ is } F\text{-decisive for } f\} = \\ &= \{T \subseteq N : T = N_F(x_N) \text{ and } f(x_N) \in F\}\,. \end{aligned}$$

Accordingly, $(\mathcal{W}_F^f)_m$ denotes its subset of *minimal* coalitions, while $\mathcal{W}^f = \bigcup_{F \in \mathcal{F}_P} \mathcal{W}_F^f$, and $(\mathcal{W}^f)_m = \bigcup_{F \in \mathcal{F}_P} (\mathcal{W}_F^f)_m$. Clearly, $\mathcal{W}_F^f$ is an order filter of $(\mathcal{P}(N), \subseteq)$ for all $F \in \mathcal{F}_P$ since f is monotonically-independent, hence, by construction $\mathcal{W}^f$, is also an order filter of $(\mathcal{P}(N), \subseteq)$.

Also, observe that for every $x_N \in X^N$ and $x \in X$, $x \leqslant f(x_N)$ if on only if $x \in F$ entails $f(x_N) \in F$ for all $F \in \mathcal{F}_P$ (since any such F is by definition an order filter, and every (proper) principal filter $[x)$ is known to be in particular prime (see e.g. Davey, Priestley (1990), Theorem 9.7). Therefore,

$$\begin{aligned} f(x_N) &= \bigvee \{x \in X : \ x \leqslant f(x_N)\} = \\ &= \bigvee \{x \in X : \text{ for all } F \in \mathcal{F}_P \text{ s.t. } x \in F,\ f(x_N) \in F \text{ also holds}\} = \\ &= \bigvee \left\{x \in X : \text{ for all } F \in \mathcal{F}_P \text{ s.t. } x \in F,\ N_F(x_N) \in \mathcal{W} \equiv \mathcal{W}^f \right\}, \end{aligned}$$

where $\mathcal{W}^f$ is by hypothesis an inclusive committee.

Conversely, let $f : X^N \to X$ be such that, for all $x_N \in X^N$,

$$f(x_N) = \bigvee \{x \in X : \text{ for all } F \in \mathcal{F}_P \text{ s.t. } x \in F,\ N_F(x_N) \in \mathcal{W}\}\,,$$

for some *inclusive* order filter $\mathcal{W} \subseteq \mathcal{P}(N)$.

For every $z \in X$, and every $F \in \mathcal{F}_P$ such that $z \in F$, $N_F(x_N^z \equiv (z, ..., z)) = N \in \mathcal{W}$: indeed, $\mathcal{W} \neq \varnothing$ since $\mathcal{W}$ is inclusive, and by definition $N \in \mathcal{W}'$ for each nontrivial order filter of $(\mathcal{P}(N), \subseteq)$. Thus, $f(x_N^z) = z$, namely f is idempotent (hence, it is in particular idempotent). Moreover, let $F \in \mathcal{F}_P$, $x_N, y_N \in X^N$ such that $f(x_N) \in F$ and $N_F(x_N) \subseteq N_F(y_N)$. By definition of f, $N_F(x_N) \in \mathcal{W}$ whence $N_F(y_N)$ since $\mathcal{W}$ is an order filter. It follows that f is independently monotonic and therefore, by Lemma 1, strategy-proof on $D_{\mathcal{X}}^N$.

(*ii*) Let $f : X^N \to X$ be an anonymous and neutral aggregation rule that is strategy-proof on $D_{\mathcal{X}}^N$. Then f is monotonically-independent by Lemma 1.

Thus,

$$\begin{aligned} f(x_N) &= \bigvee \{x \in X : \ x \leqslant f(x_N)\} = \\ &= \bigvee \left\{x \in X : \text{ for all } F \in \mathcal{F}_P \text{ s.t. } x \in F,\ N_F(x_N) \in \mathcal{W}^f \right\} \end{aligned}$$

where for all $F, F' \in \mathcal{F}_P$ and $x_N \in X^N$ such that $N_F(x_N) = N_{F'}(x_N)$, $f(x_N) \in F$ if and only if $f(x_N) \in F'$, and $\mathcal{W}^f$ is anonymous. Therefore, there exists a positive integer $q \leq n = |N|$ such that for each $x \in X$ and $F \in \mathcal{F}_P$ s.t. $x \in F$, $f(x_N) \in F$ if and only if $N_F(x_N) \in \mathcal{W}_q \equiv \{T \subseteq N : |T| \geq q\}$, namely:

$$f(x_N) = \vee_{S \in \mathcal{W}_q} (\wedge_{i \in S} x_i).$$

Conversely, let $q \leq n = |N|$ be a positive integer. Then, $f_{\mathcal{W}_q}$ is by construction anonymous, and neutral (hence of course idempotent). Moreover, it is -again by construction- monotonically-independent hence strategy-proof on $D^N_{\mathcal{X}}$ by Lemma 1.

(iii) Clearly, f^{maj} is anonymous by definition, and strategy-proof on $D^N_{\mathcal{X}}$ by Lemmas 1 and 2. Moreover, if $|N|$ is odd then f^{maj} is bi-idempotent, by definition.

Conversely, suppose that $f : X^N \to X$ is anonymous, bi-idempotent and strategy-proof on $D^N_{\mathcal{X}}$. Since f is strategy-proof on $D^N_{\mathcal{X}}$, it follows by Lemma 1 that f is monotonically independent. But it can also be easily shown that (a) if f is (monotonically) independent and bi-idempotent then it is also neutral, and (b) if f is monotonically independent and neutral then f is a generalized committee aggregation rule,[14] i.e. there exists an order filter $\mathcal{W}$ of $(\mathcal{P}(N), \subseteq)$ such that for all $x_N \in X^N$:

$$f(x_N) = \bigvee_{A \in \mathcal{W}} \bigwedge_{i \in A} x_i.$$

Finally, notice that anonymity of f implies that there exists a positive integer $k \leq |N|$ such that $\mathcal{W} = \{A \subseteq N : |A| \geq k\}$, and bi-idempotence of f implies that $|N| - k = k - 1$, whence $n = |N| = 2k - 1 = 2(k-1) + 1$ and $k = \frac{n+1}{2}$. Therefore, $\mathcal{W} = \left\{A \subseteq N : |A| \geq \frac{n+1}{2}\right\} = \mathcal{W}^{maj}$, namely $f = f^{maj}$. □

Remark 6.2. *Notice that inclusivity and anonymity are mutually independent conditions. Indeed, by definition the constant aggregation rules are trivially anonymous but not inclusive while for any collegial* $\mathcal{W} \subseteq \mathcal{P}(\mathcal{N})$ *with* $\bigcup \mathcal{W}_m = N$ *(such as e.g.* $\mathcal{W} = \{\{1, 2, ..., i\}, \{i, i+1, ..., n\}\}$*)* $f_{\mathcal{W}}$ *is inclusive but not anonymous. However, it is easily checked that any idempotent aggregation rule is anonymous only if it is also inclusive (but not conversely).*

[14] Broadly speaking, the proof of points (a) and (b) sketched above is an adaptation and extension of a similar proof provided by Monjardet (1990) for *finite* distributive lattices (see Vannucci (2019) for more details).

□

Proof of Claim 1

To begin with, observe that, by construction, for each $x, y, w, z \in L$:

$$\{u \vee v : u \in [x, y], v \in [w, z]\} \subseteq [x \vee w, y \vee z],$$

$$x \vee y \in \{u \vee v : u \in [x, y], v \in [w, z]\},$$

$$y \vee z \in \{u \vee v : u \in [x, y], v \in [w, z]\}.$$

Thus, $\widehat{\vee}$ and $\widehat{\wedge}$ are well-defined binary operations on I_L if and only if:

$$[x \vee w, y \vee z] \subseteq \{u \vee v : u \in [x, y], v \in [w, z]\},$$

and

$$[x \wedge w, y \wedge z] \subseteq \{u \wedge v : u \in [x, y], v \in [w, z]\}$$

as well.

Let us now show that if L is distributive then $[x, y]\widehat{\vee}[w, z] \in I_L$ holds for any $[x, y], [w, z] \in I_L$.

To see that, suppose that L is distributive and $u' \in [x \vee w, y \vee z]$. Thus, $x = x \wedge (x \vee w) \leqslant x \wedge u' \leqslant y \wedge u' \leqslant y \wedge (y \vee z) = y$, hence $x \wedge u' \in [x, y]$. Similarly, $w = w \wedge (w \vee x) \leqslant w \wedge u' \leqslant z \wedge u' \leqslant z \wedge (y \vee z) = z$, hence $w \wedge u' \in [w, z]$. But then, $u' = u' \wedge (x \vee w) = (u' \wedge x) \vee (u' \wedge w) = (x \wedge u') \vee (w \wedge u')$, namely $u' \in \{u \vee v : u \in [x, y], v \in [w, z]\}$, whence $[x \vee w, y \vee z] \subseteq \{u \vee v : u \in [x, y], v \in [w, z]\}$. Therefore, $\{u \vee v : u \in [x, y], v \in [w, z]\} = [x \vee w, y \vee z]$. It follows that $[x, y]\widehat{\vee}[w, z] \in I_L$, as claimed.

By the dual argument, $[x, y]\widehat{\wedge}[w, z] \in I_L$ also holds for any $[x, y], [w, z] \in I_L$. Hence, $\widehat{\vee}$ and $\widehat{\wedge}$ are well-defined binary operations on I_L. It remains to be checked that both $\widehat{\vee}$ and $\widehat{\wedge}$ do indeed satisfy Associativity, Commutativity, Idempotency, and Absorption.

To see this concerning $\widehat{\vee}$, notice that for any $x_1, y_1, x_2, y_2, x_3, y_3 \in L$,

$$\begin{aligned}
([x_1, y_1]\widehat{\vee}[x_2, y_2])\widehat{\vee}[x_3, y_3] &= [(x_1 \vee x_2), (y_1 \vee y_2)]\widehat{\vee}[x_3, y_3] = \\
&= [(x_1 \vee x_2) \vee x_3, (y_1 \vee y_2) \vee y_3] = \\
&= [x_1 \vee (x_2 \vee x_3), y_1 \vee (y_2 \vee y_3)] = \\
&= [x_1, y_1]\widehat{\vee}[x_2 \vee x_3, y_2 \vee y_3] = \\
&= [x_1, y_1]\widehat{\vee}([x_2, y_2]\widehat{\vee}[x_3, y_3]).
\end{aligned}$$

Moreover,

$$[x_1, y_1]\widehat{\vee}[x_2, y_2] = [x_1 \vee x_2, y_1 \vee y_2] = [x_2 \vee x_1, y_2 \vee y_1] = [x_2, y_2]\widehat{\vee}[x_1, y_1],$$

$$[x_1, y_1]\widehat{\vee}[x_1, y_1] = [x_1 \vee x_1, y_1 \vee y_1] = [x_1, y_1],$$

and

$$[x_1, y_1]\widehat{\vee}([x_1, y_1]\widehat{\wedge}[x_2, y_2] = [x_1, y_1]\widehat{\vee}([x_1 \wedge x_2, y_1 \wedge y_2] =$$

$$= [x_1 \vee (x_1 \wedge x_2), y_1 \vee (y_1 \wedge y_2)] = [x_1, y_1].$$

A dual argument establishes the same properties for $\widehat{\wedge}$. It follows that $(I_L, \widehat{\vee}, \widehat{\wedge})$ is indeed a lattice.

Furthermore, suppose that $(L, \vee, \wedge)$ is bounded, with top element 1 and bottom element 0. Then, it is easily checked that $(I_L, \widehat{\vee}, \widehat{\wedge})$ is also bounded, with $[1, 1]$ and $[0, 0]$ as its top and bottom elements, respectively.

Conversely, suppose that $(I_L, \widehat{\vee}, \widehat{\wedge})$ is a lattice, i.e. both $\widehat{\vee}$ and $\widehat{\wedge}$ are well-defined binary operations on I_L (namely, for any $[x, y], [w, z] \in I_L$, both $[x, y]\widehat{\vee}[w, z] \in I_L$ and $[x, y]\widehat{\wedge}[w, z] \in I_L$), and satisfy Associativity, Commutativity, Idempotency and Absorption as defined above.

Since $\widehat{\vee}$ is a well-defined operation, it follows from the definition of $\widehat{\vee}$ and the observation mentioned at the very beginning of the present proof that, for any $x, y, w, z \in L$,

$$[x \vee w, y \vee z] \subseteq \{u \vee v : u \in [x, y], v \in [w, z]\}.$$

Thus, for every $t \in L$ such that $x \vee w \leqslant t \leqslant y \vee z$ it must be the case that $t = u \vee v$ for some $u \in [x, y]$, $v \in [w, z]$. Hence, $t \geqslant (y \wedge t) \vee (z \wedge t) \geqslant (u \wedge t) \vee (v \wedge t) = (u \wedge (u \vee v)) \vee (v \wedge (u \vee v)) = u \vee v = t$. It follows that $t = (u \wedge t) \vee (v \wedge t)$, for every $t \in [x \vee w, y \vee z]$: in particular, $t = (y \wedge t) \vee (z \wedge t)$ also holds.

But then, since $y \vee z \geqslant t \geqslant t \wedge (y \vee z)$, $t \wedge (y \vee z) = t$ whence $t \wedge (y \vee z) = (y \wedge (t \wedge (y \vee z)) \vee (z \wedge (t \wedge (y \vee z) = (y \wedge t) \vee (z \wedge t) = (t \wedge y) \vee (t \wedge z)$, and distributivity of L is thus established (since t, y, z are in fact arbitrarily chosen).

Finally, suppose that $(I_L, \widehat{\vee}, \widehat{\wedge})$ is a lattice. By the previous argument, L is a distributive lattice. Then, for any $x_1, y_1, x_2, y_2, x_3, y_3 \in L$,

$$
\begin{aligned}
([x_1, y_1]\widehat{\vee}[x_2, y_2])\widehat{\wedge}([x_1, y_1]\widehat{\vee}[x_3, y_3]) &= [(x_1 \vee x_2), (y_1 \vee y_2)]\widehat{\wedge}[(x_1 \vee x_3), (y_1 \vee y_3)] = \\
&= [(x_1 \vee x_2) \wedge (x_1 \vee x_3), (y_1 \vee y_2) \wedge (y_1 \vee y_3)] = \\
&= [x_1 \vee (x_2 \wedge x_3), y_1 \vee (y_2 \wedge y_3)] = \\
&= [x_1, y_1]\widehat{\vee}[x_2 \wedge x_3, y_2 \wedge y_3] = \\
&= [x_1, y_1]\widehat{\vee}([x_2, y_2]\widehat{\wedge}[x_3, y_3].
\end{aligned}
$$

Therefore, $(I_L, \widehat{\vee}, \widehat{\wedge})$ is indeed a distributive lattice, as claimed.

Furthermore, suppose that $(I_L, \widehat{\vee}, \widehat{\wedge})$ is bounded, and denote by $[x^\top, y^\top]$ and $[x^\perp, y^\perp]$ its top and bottom elements, respectively. Then, for any $[x, y] \in I_L$, $[x, y]\widehat{\vee}[x^\top, y^\top] = [x^\top, y^\top]$ and $[x, y]\widehat{\wedge}[x^\perp, y^\perp] = [x^\perp, y^\perp]$. In particular, $[y^\top, y^\top]\widehat{\vee}[x^\top, y^\top] = [x^\top, y^\top]$ and $[x^\perp, x^\perp]\widehat{\wedge}[x^\perp, y^\perp] = [x^\perp, y^\perp]$.

It follows that for any $z \in L$, $z \leqslant x^\top = y^\top$ and $y^\perp = x^\perp \leqslant z$, hence L is indeed bounded with $x^\top$and $x^\perp$ as its top and bottom elements, respectively.

□

References

[1] Balinski M, R. Laraki, (2010): *Majority judgment. Measuring, ranking, and electing.* MIT Press, Cambridge.

[2] Barrett, C., P. Pattanaik, (1987): Aggregation of Probability Judgments, *Econometrica* 55, 1237-41.

[3] Basili M., and A. Chateauneuf (2020). Aggregation of experts' opinions and conditional consensus opinion by the Steiner point. *International Journal of Approximate Reasoning.* 123, 17–25.

[4] Basili M., L. Pratelli (2015): Aggregation of not independent experts' opinions under ambiguity, *Structural Safety* 52, 144-149.

[5] Bowden M. (2012): The Finish. The killing of Osama bin Laden. Grow Press, London.

[6] Brylawski T. (1973): The lattice of integer partitions, *Discrete Mathematics* 6, 201-219.

[7] Clemen R.T., R.Winkler (1999): Combining probability distributions from experts in risk analysis, *Risk Analysis* 19, 187-203.

[8] Clemen R.T., R.Winkler (2007): Aggregating probability distributions, in W. Edwards, R.F. Miles, D. von Winterfeldt, eds. *Advances in Decision Analysis: From Foundations to Applications.* Cambridge University Press, Cambridge.

[9] Crès H., I.Gilboa, N.Vieille (2011): Aggregation of multiple prior opinions, *Journal of Economic Theory* 146, 2563-2582.

[10] Comer S. (1991): An algebraic approach to the approximation of information, *Fundamenta Informaticae* XIV, 492-502.

[11] Cooke R.M. (1991): *Experts in Uncertainty. Opinion and Subjective Probability in Science.* Oxford University Press, Oxford.

[12] Dalkey N.C. (1975): Toward a theory of group estimation, in H.A. Linstone, M. Turoff (eds.): *The Delphi Method. Techniques and Applications.* Addison-Wesley, Reading MA.

[13] Danilov V.I. (1994): The structure of non-manipulable social choice rules on a tree, *Mathematical Social Sciences* 27, 123-131.

[14] Davey B.A., H.A. Priestley (1990): *Introduction to Lattices and Order.* Cambridge University Press, Cambridge UK.

[15] de Finetti B. (1936): La logique de la probabilité, *Actes du Congrès International de Philosophie Scientifique, Sorbonne Paris 1935.* Hermann et C. Éditeurs, Paris.

[16] Dietrich F., C. List (2017): Probabilistic opinion pooling generalized. Part one: general agendas, *Social Choice and Welfare* 48, 747-786.

[17] Fitting M. (1991): Kleene's logic, generalized, *Journal of Logic and Computation* 1, 797-810.

[18] Friedman, J. A., and R. Zeckhauser (2015): Handling and Mishandling Estimative Probability: Likelihood, Confidence, and the Search for Bin Laden. *Intelligence and National Security.* 30.1, 77-99.

[19] Gajdos T., J.-C. Vergnaud (2013): Decisions with conflicting and imprecise information, *Social Choice and Welfare* 41, 427-452.

[20] Kriegel E., Hall J.W., Held H., Dawson R., Schellnhuber H.J. (2009): Imprecise probability assessment of tipping points in the climate system, *Proceedings of the National Academy of Sciences of the USA* 106, 5041-5046.

[21] Lehrer K., C. Wagner (1981): *Rational Consensus in Science and Society.* Reidel, Dordrecht.

[22] Markov S. (2016): On the algebra of intervals, *Reliable Computing* 21, 80-108.

[23] Milne P. (2004): Algebras of intervals and a logic of conditional assertions, *Journal of Philosophical Logic* 33, 497-548.

[24] Monjardet B. (1990): Arrowian characterizations of latticial federation consensus functions, *Mathematical Social Sciences* 20, 51-71.

[25] Nau R.F. (2002): The aggregation of imprecise probabilities, *Journal of Statistical Planning and Inference* 105, 265-282.

[26] Osherson D., M.Y.Vardi (2006): Aggregating disparate estimates of chance, *Games and Economic Behavior* 56, 148-173.

[27] Rubinstein A., P.C. Fishburn (1986): Algebraic aggregation theory, *Journal of Economic*

Theory 38, 63-77.

[28] Savaglio E., S. Vannucci (2019): Strategy-proof aggregation rules and single peakedness in bounded distributive lattices, *Social Choice and Welfare* 52, 295-327.

[29] Stewart R.T., I. O. Quintana (2018): Probabilistic opinion pooling with imprecise probabilities, *Journal of Philosophical Logic* 47, 17-45.

[30] Vannucci S. (2016): Weakly unimodal domains, anti-exchange properties, and coalitional strategy-proofness of aggregation rules, *Mathematical Social Sciences* 84, 56-67.

[31] Vannucci S. (2019): Majority judgment and strategy-proofness: a characterization, *International Journal of Game Theory* 48, 863-886.

[32] Weymark, J. A. (1997): Aggregating Ordinal Probabilities on Finite Sets, *Journal of Economic Theory* 75, 407-432.

[33] Wilson K.J., Farrow M. (2018): Combining Judgements from Correlated Experts. In: Dias L., Morton A., Quigley J. (eds) *Elicitation*. International Series in Operations Research & Management Science, vol 261, 211-240. Springer, Cham.