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Strategy-Proof Aggregation Rules in Median Semilattices with Applications to Preference Aggregation

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STRATEGY-PROOF AGGREGATION RULES IN MEDIAN SEMILATTICES WITH APPLICATIONS TO PREFERENCE AGGREGATION

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Abstract. Two characterizations of the whole class of strategy-proof aggregation rules on rich domains of locally unimodal preorders in finite median join-semilattices are provided. In particular, it is shown that such a class consists precisely of generalized weak sponsorship rules induced by certain families of order filters of the coalition poset. It follows that the co-majority rule and many other inclusive aggregation rules belong to that class. The co-majority rule for an odd number of agents is characterized and shown to be equivalent to a Condorcet-Kemeny rule. Applications to preference aggregation rules including Arrowian social welfare functions are also considered. The existence of strategy-proof anonymous neutral and

unanimity-respecting social welfare functions which are defined on *arbitrary* profiles of total preorders and satisfy a suitably relaxed independence condition is shown to follow from our characterizations.

JEL Classification D71

1. Introduction

The present work is devoted to characterizing those aggregation rules in finite median join-semilattices which are strategy-proof on rich domains of locally unimodal (or single peaked) total preorders. Moreover, the co-majority (or median) rule is characterized within the class of such strategy-proof rules as the only one that is anonymous and bi-idempotent when the number of agents is odd. Two applications of our characterization results to preference aggregation rules including social welfare functions are also provided, and their connections to two classical ways out of Arrow's 'impossibility theorem' suggested from the

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social choice-theoretic literature are emphasized and discussed. In particular, a new characterization of generalized Condorcet-Kemeny rules for an odd number of agents is established.

Thus, the foregoing results *jointly* address *several* related issues: the characterization of the *entire class* of strategy-proof aggregation rules in *median* finite semilattices, a specific characterization of 'median' rules amongst them, and -as a by–product- both a general characterization of strategy-proof preference aggregation rules including social welfare functions, and a specific characterization of the Condorcet-Kemeny aggregation rule in a fixed population setting.

Now, such issues have been previously considered in the literature, but most typically from mutually 'disconnected' perspectives. For instance, aggregation rules in semilattices and lattices have been studied in depth in seminal contributions mostly due to Monjardet and his co-workers (see e.g. Monjardet (1990)), but with no reference to strategy-proofness issues. Condorcet-Kemeny aggregation rules have been characterized at least for the case of *linear orders* and in a *variable population* setting (see e.g. Young, Levenglick (1978)), but again with no reference to strategy-proofness properties. By contrast, Bossert, Sprumont (2014) *does* consider strategy-proofness issues, and also provides characterizations of *some* strategy-proof preference aggregation rules in a variable population setting, but focusses in fact on *restricted aggregation rules* which admit total preorders as possible outputs while being *only* defined on profiles of *linear orders*. Most recently, Bonifacio, Massó (2020) essentially characterizes the *sub-class of anonymous and unanimity-respecting aggregation rules* in arbitrary join-semilattices which are *strategy-proof* on 'single-peaked' domains of *total preorders* (according to a notion of 'single-peakedness' that reflects the structure of the underlying join-semilattice). Since the relevant join-semilattice may not be median, however, 'median' rules such as Condorcet-Kemeny rules are in general not available, and a *specific* application to the case of total preorders and consequently to classical Arrowian social welfare functions is in fact out of reach in that general framework without some further structure.

On the contrary, focussing on the case of *median* join-semilattices makes it possible and natural to address *jointly* all of the previous issues: that is precisely what is done in the present work. The application of its main results to two prominent examples of median join-semilattices (namely, the median semilattice of total preorders and the distributive lattice of reflexive binary relations on a finite set) provides, as a by-product, a simple concrete instance for each one of two classical ways out of Arrow's 'impossibility theorem'. The first one consists in a considerable *relaxation of Arrow's Independence of Irrelevant Alternatives*, while the second one consists in *dropping entirely connectedness and transitivity-like requirements* on preferences *both as an input and as an output* for the relevant preference aggregation rule.

The rest of the paper is organized as follows: Section 2.1 is devoted to the basic definitions and preliminaries of our model, while Section 2.2 includes the main results. Section 3 is devoted to two applications of the foregoing results to preference aggregation rules for total preorders and arbitrary reflexive relations, respectively. Finally, Section 4 includes a detailed discussion of related literature and some concluding remarks.

2. Strategy-proof aggregation rules in finite median join-semilattices: model and results

2.1. **Definitions and preliminaries.** Let $N = \{1, .., n\}$ denote the finite population of voters, X an arbitrary nonempty *finite* set of alternatives and $\leqslant$ a partial order i.e. a reflexive, transitive and antisymmetric binary relation on X, and $\mathcal{X} = (X, \leqslant)$ the corresponding partially ordered set, or *poset*, on X. We assume that $n \geq 3$ in order to avoid tedious qualifications. The subsets of N are also referred to as *coalitions,* and $(\mathcal{P}(N), \subseteq)$ denotes the partially ordered set of coalitions induced by set-inclusion. An *order filter* of $(\mathcal{P}(N), \subseteq)$ is a set $F \subseteq \mathcal{P}(N)$ of coalitions such that for any $S \in F$ and any $T \subseteq N$, if $S \subseteq T$ then $T \in F$. The *basis* of order filter F is the set of inclusion-minimal elements/coalitions of F, and is denoted by $F^{\min}$.

A *chain* of poset $\mathcal{X} = (X, \leqslant)$ is a set $Y \subseteq X$ such that for any *distinct* $u, v \in Y$ either $u \leqslant v$ or $v \leqslant u$ holds, and its *length* $l(Y)$ is $|Y| - 1$ (where $|Y|$ denote its size). A chain Y of $(X, \leqslant)$ having x as its $\leqslant$-minimum and y as its $\leqslant$-maximum is *maximal* if there is no $z \in X \setminus Y$ such that $x \leqslant z \leqslant y$. For any $x, y \in X$ such that $x < y$ (i.e. $x \leqslant y$ and *not* $y \leqslant x$) the *length* of the order-interval $[x, y] := \{z \in X : x \leqslant z \leqslant y\}$, written $l([x, y])$, is the length of a (maximal) chain of *maximum* length having x as its $\leqslant$-minimum and y as its $\leqslant$-maximum. In particular, $x \in X$ is said to be *covered* by $y \in X$, written $x \ll y$, iff $x < y$ and $[x, y] = \{x, y\}$, namely $l(\{x, y\}) = 1$. The *covering graph* $C(\mathcal{X}) = (X, E^{\ll})$ of $\mathcal{C}$ is the undirected graph having X as vertex-set and $E^{\ll} := \{\{x, y\} \subseteq X : x \ll y \text{ or } y \ll x\}$ as edge-set. A *path* π_{xy} of $C(\mathcal{X})$ connecting two vertices x and y is a maximal chain $\{z_0, ..., z_k\}$ of $\mathcal{X}$ such that $\{z_0, z_k\} = \{x, y\}$ and $z_i \ll z_{i+1}$, for any $i = 1, ..., k-1$, and is of *length* $l(\pi_{xy}) = k$. The set of all paths of $C(\mathcal{X})$ connecting x and y is denoted by Π_{xy}. A *geodesic* from x

to y on $C(\mathcal{X})$ is a path of *minimum length* (or equivalently a *shortest path*) connecting x and y. It can be easily proved (and left to the reader to check) that the *shortest length* function $\delta_{C(\mathcal{X})} : X \times X \to \mathbb{Z}_+$ such that, for any $x, y \in X$, $\delta_{C(\mathcal{X})}(x, y) := l(\pi_{xy})$ (where π_{xy} is a path of minimum length in Π_{xy}) is indeed a *metric* namely for any $x, y, z \in X$: (i) $\delta_{C(\mathcal{X})}(x, y) = 0$ iff $x = y$, (ii) $\delta_{C(\mathcal{X})}(x, y) = \delta_{C(\mathcal{X})}(y, x)$, (iii) $\delta_{C(\mathcal{X})}(x, y) \leq \delta_{C(\mathcal{X})}(x, z) + \delta_{C(\mathcal{X})}(z, y)$.

We denote by $\vee$ and $\wedge$ the *least-upper-bound* and *greatest-lower-bound* binary *partial* operations on X as induced by $\leqslant$, respectively, while for any $Y \subseteq X$, $\vee Y$ and $\wedge Y$ denote the least-upper-bound and greatest-lower-bound of Y (whenever they exist). We also posit -for any $x \in X$- $\uparrow x = \{y \in X : x \leqslant y\}$ i.e. the *(principal) order filter*[1] generated by x. An element $x \in X$ is *meet-irreducible (join-irreducible)* if for any $Y \subseteq X$, $x = \wedge Y$ entails $x \in Y$ ($x = \vee Y$ entails $x \in Y$). Moreover, for any $Y \subseteq X$, $\wedge Y$ ($\vee Y$, respectively) is well-defined if and only if there exists $z \in X$ such that $z \leqslant y$ ($y \leqslant z$, respectively) for all $y \in Y$, namely the elements of Y have a *common lower (upper) bound*. The set of all meet-irreducible elements (join-irreducible elements) of $\mathcal{X} = (X, \leqslant)$ will be denoted by $M_{\mathcal{X}}$ ($J_{\mathcal{X}}$, respectively). Notice that, by construction, for every $x \in X$, $x = \wedge M(x)$ where $M(x) := \{m \in M_{\mathcal{X}} : x \leqslant m\}$ and, dually $x = \vee J(x)$ where $J(x) := \{j \in J_{\mathcal{X}} : j \leqslant x\}$.

The partially ordered set $\mathcal{X} = (X, \leqslant)$ is a (finite) *join-semilattice (meet-semilattice*, respectively) if and only if the least upper bound or join $x \vee y$ (the greatest lower bound or meet $x \wedge y$) is well-defined in X for all $x, y \in X$ so that $\vee : X \times X \to X$ ($\wedge : X \times X \to X$) is a function, and a *lattice* if it is both a join-semilattice and a meet-semilattice.

Notice that a finite join-semilattice $\mathcal{X} = (X, \leqslant)$ has a (unique) universal upper bound or *top element* $\mathbf{1} = \vee X$, and its *co-atoms* are those elements $x \in X$ such that $x \ll \mathbf{1}$: the set of co-atoms of $\mathcal{X} = (X, \leqslant)$ is denoted by $\mathcal{C}_{\mathcal{X}}$. Dually, a finite meet-semilattice $\mathcal{X} = (X, \leqslant)$ has a (unique) universal lower bound or *bottom element* $\mathbf{0} = \wedge X$, and its *atoms* are those elements $x \in X$ such that $\mathbf{0} \ll x$: the set of atoms of $\mathcal{X}$ is denoted by $\mathcal{A}_{\mathcal{X}}$. Notice that a co-atom (atom, respectively) is also a meet-irreducible (join-irreducible, respectively) element. When co-atoms and meet-irreducibles (atoms and join-irreducibles) do in fact *coincide* the join-semilattice (meet-semilattice) is said to be *coatomistic* (*atomistic*, respectively). Let us now introduce the class of finite join-semilattices which is the focus of the present paper.

[1] An *order filter* of a partially ordered set $(X, \leqslant)$ is a set $Y \subseteq X$ such that, for any $y, z \in X$, if $y \leqslant z$ and $y \in Y$ then $z \in Y$.

The (finite) join-semilattice $\mathcal{X} = (X, \leqslant)$ is *median* if it also satisfies the following pair of conditions:

(i) *upper distributivity:* for all $u \in X$, and for all $x, y, z \in X$ such that u is a lower bound of $\{x, y, z\}$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ (or, equivalently, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$) holds i.e. $(\uparrow u, \leqslant_{|\uparrow u})$ -where $\leqslant_{|\uparrow x}$denotes the restriction of $\leqslant$ to $\uparrow u$- is a *distributive lattice*[2] ;

(ii) *co-coronation (or meet-Helly) property:* for all $x, y, z \in X$ if $x \wedge y$, $y \wedge z$ and $x \wedge z$ exist, then $(x \wedge y \wedge z)$ also exists.

Three well-known properties of (finite) upper distributive join-semilattices that will be used below are collected in the following

Claim 1. *(i) Let $m \in M_{\mathcal{X}}$ be a meet-irreducible element of an upper distributive finite join-semilattice $\mathcal{X} = (X, \leqslant)$ and $Y \subseteq X$ such that $\wedge Y$ exists. If $\wedge Y < m$ then there also exists some $y \in Y$ such that $y \leqslant m$ (see e.g. Monjardet (1990));*

*(ii) a finite upper distributive join-semilattice $\mathcal{X} = (X, \leqslant)$ is **graded** i.e. it admits a **rank function** namely a function $r : X \to \mathbb{Z}_+$ such that for any $x, y \in X$ if $x \ll y$ then $r(y) = r(x) + 1$ (see Barbut, Monjardet (1970), Leclerc (1994));*

*(iii) the rank function of a finite upper distributive join-semilattice is a **valuation**, namely for any $x, y \in X$ the following condition holds: if the meet $x \wedge y$ exists then $r(x) + r(y) = r(x \vee y) + r(x \wedge y)$ (Leclerc (1994)).*

Moreover, it is easily checked that if $\mathcal{X} = (X, \leqslant)$ is a median join-semilattice then the partial function $\mu : X^3 \to X$ defined as follows: for all $x, y, z \in X$, $\mu(x, y, z) = (x \vee y) \wedge (y \vee z) \wedge (x \vee z)$

is indeed a *well-defined ternary operation* on X, the *median* of $\mathcal{X}$ which satisfies the following two characteristic properties (see Sholander (1952, 1954):

(μ_1) $\mu(x, x, y) = x$ for all $x, y \in X$

(μ_2) $\mu(\mu(x, y, v), \mu(x, y, w), z) = \mu(\mu(v, w, z), x, y)$ for all $x, y, v, w, z \in X$.

[2]A poset $(Y, \leqslant)$ is a *distributive lattice* iff, for any $x, y, z \in X$, $x \wedge y$ and $x \vee y$ exist, and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (or, equivalently, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$). Moreover, a (distributive) lattice $\mathcal{X}$ is said to be *lower (upper) bounded* if there exists $\perp \in X$ $(\top \in X)$ such that $\perp \leqslant x$ $(x \leqslant \top)$ for all $x \in X$, and *bounded* if it is both lower bounded and upper bounded. A bounded distributive lattice $(X, \leqslant)$ is *Boolean* if for each $x \in X$ there exists a *complement* namely an $x' \in X$ such that $x \vee x' = \top$ and $x \wedge x' = \perp$.

Relying on μ, a ternary (median-induced) **betweenness** relation
$B_\mu = \{(x, z, y) \in X^3 : z = \mu(x, y, z)\}$ is defined on $\mathcal{X}$,
and for any $x, y \in X$,
$I^\mu(x, y) := B_\mu(x, ., y) = \{z \in X : z = \mu(x, y, z)\}$ is the *interval* induced by x and y: therefore, *for any* $x, y, z \in X$, $(x, z, y) \in B_\mu$ (*also written* $B_\mu(x, z, y)$) *if and only if* $z \in I^\mu(x, y)$.

Furthermore, in a (finite) median join-semilattice $\mathcal{X} = (X, \leqslant)$ a *metric* $d_r : X \times X \to \mathbb{Z}_+$ can be defined in a natural way by the following rule[3]: for any $x, y \in X$, $d_r(x, y) = 2r(x \vee y) - r(x) - r(y)$.

Let $\succcurlyeq$ denote a preorder i.e. a reflexive and transitive binary relation on X (we shall denote by $\succ$ and $\sim$ its asymmetric and symmetric components, respectively, by $Top(\succcurlyeq)$ the possibly empty set of its maxima, and by $||$ the set of its *incomparable* ordered pairs i.e. $x||y$ iff neither $x \succcurlyeq y$ nor $y \succcurlyeq x$ hold). Then, $\succcurlyeq$ is said to be **locally unimodal** with respect to betweenness relation B_μ - or B_μ-**lu** - if and only if

U-(i) there exists a *unique maximum* of $\succcurlyeq$ in X, its *top* outcome -denoted $top(\succcurlyeq)$- and

U-(ii) for all $x, y, z \in X$, if $z \in I^\mu(top(\succcurlyeq), y) \smallsetminus \{top(\succcurlyeq)\}$ then not $y \succ z$.

We denote by U_{B_μ} the set of all B_μ-*lu* preorders on X. An N-profile of B_μ-lu preorders is a mapping from N into U_{B_μ}. We denote by $U^N_{B_\mu}$ the set of all N-profiles of B_μ-lu preorders.

Moreover, A set $D \subseteq U^N_{B_\mu}$ of locally unimodal preorders w.r.t. B_μ is **rich** if for all $x, y \in X$ there exists $\succcurlyeq \in D_{\mathcal{X}}$ such that $top(\succcurlyeq) = x$ and $UC(\succeq, y) = I^\mu(x, y)$ (where $UC(\succeq, y) := \{y \in X : x \succcurlyeq y\}$ is the upper contour of $\succcurlyeq$ at y).

An **aggregation rule** for (N, X) is a function $f : X^N \to X$ [4]. An aggregation rule f is **strategy-proof** *on* $U^N_{B_\mu}$ iff for all B_μ-unimodal

[3]A *metric* on X is a real-valued function $\delta : X \times X \to \mathbb{R}_+$ such that for any $x, y, z \in X$:

(i) $\delta(x, y) = 0$ iff $x = y$,

(ii) $\delta(x, y) = \delta(y, x)$

(iii) $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$.

[4]Occasionally, both weaker and stricter versions of aggregation rules will be considered, namely restricted aggregation rules, multi-aggregation rules and constrained aggregation rules.

A *restricted aggregation rule* for (N, X) is a function $f : D \to X$ for some $D \subseteq X^N$. A *multi-aggregation rule* for (N, X) is a function $f : X^N \to \mathcal{P}(X) \setminus \{\emptyset\}$. Notice that a multi-aggregation aggregation rule for (N, X) can also be regarded as an instance of a *restricted aggregation rule* for $(N, \mathcal{P}(X))$.

By contrast, a *constrained aggregation rule* for (N, X) is a function $f : X^N \to C$ for some $C \subseteq X$.

N-profiles $(\succcurlyeq_i)_{i\in N} \in U^N_{B_\mu}$, and for all $i \in N$, $y_i \in X$, and $(x_j)_{j\in N} \in X^N$ such that $x_j = top(\succcurlyeq_j)$ for each $j \in N$, *not* $f((y_i,(x_j)_{j\in N\smallsetminus\{i\}})) \succ_i f((x_j)_{j\in N})$. Finally, an aggregation rule $f : X^N \to X$ is B_μ-**monotonic** iff for all $i \in N$, $y_i \in X$, and $(x_j)_{j\in N} \in X^N$,

$f((x_j)_{j\in N}) \in I^\mu(x_i, f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))$.[5]

Non-trivial strategy-proof aggregation rules should be -at least to some extent- *input-responsive* and *output-unbiased.* A few requirements can be deployed to present several versions and degrees of input-responsiveness and output-unbiasedness of aggregation rules, namely

Inclusiveness: an aggregation rule for (N, X) is **inclusive** if and only if for each voter $i \in N$ there exist $x^N \in X^N$ and $y_i \in X$ such that $f(x^{N\smallsetminus\{i\}}, y_i) \neq f(x^N)$.

Anonymity: an aggregation rule f for (N, X) is **anonymous** if for each $x^N \in X^N$ and each permutation σ of N, $f(x^N) = f(x^{\sigma(N)})$ (where $x^{\sigma(N)} = (x_{\sigma(1)}, ..., x_{\sigma(n)})$).

Idempotence: an aggregation rule f for (N, X) is **idempotent** (or *unanimity-respecting)* if $f(x, ..., x) = x$ for each $x \in X$.

Sovereignty: an aggregation rule f for (N, X) is **sovereign** if for each $y \in X$ there exists $x^N \in X^N$ such that $f(x^N) = y$ i.e. f is an *onto* function.

Neutrality: an aggregation rule f for (N, X) is **neutral** if for each $x^N \in X^N$ and each permutation π of X, $f(\pi(x^N)) = \pi(f(x^N))$ (where $\pi(x^N) = (\pi(x_1), ..., \pi(x_k))$).

Notice that both **Idempotence** and **Neutrality** imply **Sovereignty** **(**but not conversely**),** while **Anonymity** and **Sovereignty** jointly imply **Inclusiveness** (but not conversely). However, it is easily checked that *if Strategy-proofness holds,* **Sovereignty** and **Idempotence** are in fact equivalent.

In particular, let $\mathcal{X} = (X, \leqslant)$ be a finite *join-semilattice* and $M_\mathcal{X}$ the set of its meet-irreducible elements, and for any $x^N \in X^N$, and any $m \in M_\mathcal{X}$, posit $N_m(x^N) := \{i \in N : x_i \leqslant m\}$. Then, the following properties of an aggregation rule can also be introduced:

$M_\mathcal{X}$**-Independence:** an aggregation rule $f : X^N \to X$ is $M_\mathcal{X}$**-independent** if and only if for all $x_N, y_N \in X^N$ and all $m \in M_\mathcal{X}$: if $N_m(x_N) = N_m(y_N)$ then $f(x_N) \leqslant m$ if and only if $f(y_N) \leqslant m$.

[5] B_μ-monotonicity (or, equivalently, $\mathcal{I}^\mu$-monotonicity) of f amounts to requiring all of its projections f_i to be *gate maps* to the image of f (see van de Vel (1993), p.98 for a definition of gate maps). The introduction of B_μ-monotonic functions in a strategic social choice setting is essentially due to Danilov (1994).

Isotony: an aggregation rule $f : X^N \to X$ is **Isotonic** if $f(x_N) \leqslant f(x'_N)$ for all $x_N, x'_N \in X^N$ such that $x_N \leqslant x'_N$ (i.e. $x_i \leqslant x'_i$ for each $i \in N$).

It can be easily shown (see Monjardet (1990)) that the *conjunction* of $M_{\mathcal{X}}$**-Independence** and **Isotony** is equivalent to the following condition:

Monotonic $M_{\mathcal{X}}$**-Independence:** An aggregation rule $f : X^N \to X$ is **monotonically** $M_{\mathcal{X}}$**-independent** if and only if for all $x_N, y_N \in X^N$ and all $m \in M_{\mathcal{X}}$: if $N_m(x_N) \subseteq N_m(y_N)$ then $f(x_N) \leqslant m$ implies $f(y_N) \leqslant m$.[6]

2.2. **Main results.** We are now ready to state the main result of this paper concerning strategy-proofness of aggregation rules on rich domains of locally unimodal profiles in median join-semilattices[7].

Theorem 1. *Let* $\mathcal{X} = (X, \leqslant)$ *be a finite median join-semilattice,* B_μ *its median-induced betweenness, and* $f : X^N \to X$ *an aggregation rule for* (N, X)*. Then, the following statements are equivalent:*

(i) f *is strategy-proof on* D^N *for any rich domain* $D \subseteq U_{B_\mu}$ *of locally unimodal preorders w.r.t.* B_μ *on* X*;*

(ii) f *is* B_μ*-monotonic;*

(iii) f *is monotonically* $M_{\mathcal{X}}$*-independent.*

Proof. (i)$\Longrightarrow$(ii) By contraposition. Let us assume that $f : X^N \to X$ is *not* B_μ-monotonic: thus, there exist $i \in N$, $x'_i \in X$ and $x_N = (x_i)_{i \in N} \in X^N$ such that $f(x_N) \notin [x_i, f(x'_i, x_{N \smallsetminus \{i\}})]$. Then, consider a preorder $\succcurlyeq^*$on X defined as follows: $x_i = top(\succcurlyeq^*)$ and for all $y, z \in X \smallsetminus \{x_i\}$, $y \succcurlyeq^* z$ iff (a) $\{y, z\} \subseteq [x_i, f(x'_i, x_{N \smallsetminus \{i\}})] \smallsetminus \{x_i\}$ or (b) $y \in [x_i, f(x'_i, x_{N \smallsetminus \{i\}})] \smallsetminus \{x_i\}$ and $z \notin [x_i, f(x'_i, x_{N \smallsetminus \{i\}})]$ or (c) $y \nparallel z$, $y \notin [x_i, f(x'_i, x_{N \smallsetminus \{i\}})]$ and $z \notin [x_i, f(x'_i, x_{N \smallsetminus \{i\}})]$. Clearly, by construction $\succcurlyeq^*$consists of three indifference classes with $\{x_i\}$, $[x_i, f(x'_i, x_{N \smallsetminus \{i\}})] \smallsetminus \{x_i\}$ and $X \smallsetminus [x_i, f(x'_i, x_{N \smallsetminus \{i\}})]$ as top, medium and bottom indifference classes, respectively. Now, observe that $\succcurlyeq^* \in U_{B_\mu}$(which is by definition a rich locally unimodal domain w.r.t. B_μ). To check that such a statement holds true, take any $y, z, v \in X$ such that $y \neq z$ and $v \in [y, z]$

[6]The notions of $J_{\mathcal{X}}$-Independence and Monotonic $J_{\mathcal{X}}$-Independence are defined similarly by dualization for a finite median inf-semilattice $\mathcal{X} = (X, \leqslant)$ as follows: for all $x_N, y_N \in X^N$ and all $j \in J_{\mathcal{X}}$, if

$N_j(x_N) := \{i \in N : j \leqslant x_i\} \subseteq$

$\subseteq N_j(y_N) := \{i \in N : j \leqslant y_i\}$

then $j \leqslant f(x_N)$ implies $j \leqslant f(y_N)$.

[7]A similar result holds for finite median meet-semilattices, and can be easily established by dualization of the relevant arguments.

i.e. $\mu(y,v,z)=v$ (if $y=z$ then $v=y=z$ and there is in fact nothing to prove). If $\{y,z\}\subseteq[x_i,f(x_i',x_{N\smallsetminus\{i\}})]$ then by definition $\mu(x_i,f(x_i',x_{N\smallsetminus\{i\}}),y)=y$ and $\mu(x_i,f(x_i',x_{N\smallsetminus\{i\}}),z)=z$. Thus, by property (μ_2) of μ,

$\mu(\mu(x_i,f(x_i',x_{N\smallsetminus\{i\}}),y),\mu(x_i,f(x_i',x_{N\smallsetminus\{i\}}),z),v)=$
$=\mu(\mu(y,z,v),x_i,f(x_i',x_{N\smallsetminus\{i\}})$, whence
$\mu(\mu(x_i,f(x_i',x_{N\smallsetminus\{i\}}),y),\mu(x_i,f(x_i',x_{N\smallsetminus\{i\}}),z),v)=$
$=\mu(y,z,v)=v$ implies
$\mu(\mu(y,z,v),x_i,f(x_i',x_{N\smallsetminus\{i\}})=\mu(y,z,v)=v$ i.e. $v\in[x_i,f(x_i',x_{N\smallsetminus\{i\}})]$.

Clearly, $\{y,z\}\neq\{x_i\}$ since $y\neq z$. Now, assume without loss of generality that $y\neq x_i$: thus $v\succcurlyeq^* y$ by definition of $\succcurlyeq^*$. If on the contrary $\{y,z\}\cap(X\smallsetminus[x_i,f(x_i',x_{N\smallsetminus\{i\}})])\neq\varnothing$ then clearly by definition of $\succcurlyeq^*$there exists $w\in\{y,z\}$ such that $v\succcurlyeq^* w$. Thus, $\succcurlyeq^*\in U_{B_\mu}$ as claimed. Also, by assumption $f(x_N)\in X\smallsetminus[x_i,f(x_i',x_{N\smallsetminus\{i\}})]$ whence by construction $f(x_i',x_{N\smallsetminus\{i\}})\succ^* f(x_N)$. But then, f is *not* strategy-proof on $U_{B_\mu}^N$.

(ii)$\Longrightarrow$(i) Conversely, let f be B_μ-monotonic. Now, consider any $\succcurlyeq=(\succcurlyeq_j)_{j\in N}\in U_{B_\mu}^N$ and any $i\in N$. By definition of B_μ-monotonicity, $f(top(\succcurlyeq_i),x_{N\smallsetminus\{i\}})\in[top(\succcurlyeq_i),f(x_i,x_{N\smallsetminus\{i\}})]$ for all $x_{N\smallsetminus\{i\}}\in X^{N\smallsetminus\{i\}}$ and $x_i\in X$. But then, since clearly $top(\succcurlyeq_i)\succcurlyeq_i f(top(\succcurlyeq_i),x_{N\smallsetminus\{i\}})$, either $f(top(\succcurlyeq_i),x_{N\smallsetminus\{i\}})=top(\succcurlyeq_i)$ or *not* $f(x_i,x_{N\smallsetminus\{i\}})\succ_i f(top(\succcurlyeq_i),x_{N\smallsetminus\{i\}})$ by local unimodality of $\succcurlyeq_i$w.r.t. B_μ. Hence, *not* $f(x_i,x_{N\smallsetminus\{i\}})\succ_i f(top(\succcurlyeq_i),x_{N\smallsetminus\{i\}})$ in any case. It follows that f is indeed strategy-proof on $U_{B_\mu}^N$.

(ii)$\Rightarrow$(iii) Suppose that f is B_μ-monotonic. Hence, for all $i\in N$, $y_i\in X$, and $(x_j)_{j\in N}\in X^N$, $f((x_j)_{j\in N})\in I^\mu(x_i,\ f(y_i,(x_j)_{j\in N\setminus\{i\}})$ i.e. $f((x_j)_{j\in N})=\mu(x_i,f((x_j)_{j\in N}),f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))$. Therefore, for any meet-irreducible element $m\in M_{\mathcal{X}}$, $f((x_j)_{j\in N})\leqslant m$ if and only if $\mu(x_i,f((x_j)_{j\in N}),f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))=(x_i\vee f((x_j)_{j\in N}))\wedge(f((x_j)_{j\in N})\vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))\wedge(x_i\vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))\leqslant m$.

It follows that if $f((x_j)_{j\in N})\leqslant m$ then [$x_i\leqslant m$ or $f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})\leqslant m$]. Indeed, suppose that $f((x_j)_{j\in N})\leqslant m$

yet [$x_i\nleqslant m$ and $f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})\nleqslant m$]. Then, $(x_i\vee f((x_j)_{j\in N}))\nleqslant m$, $f((x_j)_{j\in N})\vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})\nleqslant m$, and $x_i\vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})\nleqslant m$. Therefore, since $\mathcal{X}$ is upper distributive, $(x_i\vee f((x_j)_{j\in N}))\wedge(f((x_j)_{j\in N})\vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))\nleqslant m$ whence, by upper distributivity again, $(x_i\vee f((x_j)_{j\in N}))\wedge(f((x_j)_{j\in N})\vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))\wedge(x_i\vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))\nleqslant m$

i.e. $\mu(x_i,f((x_j)_{j\in N}),f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))=f((x_j)_{j\in N})\nleqslant m$, a contradiction.

Now, suppose that $m \in M_{\mathcal{X}}$, $f(x_N) \leqslant m$ and $N_m(x_N) \subseteq N_m(y_N)$ for some $x_N := (x_j)_{j\in N}$, $y_N := (y_j)_{j\in N} \in X^N$: we need to establish the claim that $f(y_N) \leqslant m$ as well.

By B_μ-monotonicity of f, $x_i \leqslant m$ or $f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \leqslant m$ for any $i \in N$. Thus, if $x_i \leqslant m$ then also $y_i \leqslant m$, by assumption. Hence, $f((x_j)_{j\in N}) \leqslant m$ and B_μ-monotonicity of f entail $f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \leqslant m$: indeed, by B_μ-monotonicity $f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \in I^\mu(y_i,\ f((x_j)_{j\in N}))$ i.e.

$f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) = (y_i \vee f(y_i,(x_j)_{j\in N\setminus\{i\}})) \wedge (f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \vee f((x_j)_{j\in N})) \wedge (y_i \vee f((x_j)_{j\in N})) \leqslant$

$\leqslant (y_i \vee f((x_j)_{j\in N})) \leqslant m$. It follows that $f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \leqslant m$ in any case.

But then, from B_μ-monotonicity of f and $f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \leqslant m$, it similarly follows that $x_{i+1} \leqslant m$ or $f((y_i,y_{i+1},(x_j)_{j\in N\smallsetminus\{i,i+1\}})) \leqslant m$. Now, $x_{i+1} \leqslant m$ entails $y_{i+1} \leqslant m$ as well hence $f(y_i,(x_h)_{h\in N\smallsetminus\{i\}}) \leqslant m$ and B_μ-monotonicity jointly imply $f((y_i,y_{i+1},(x_j)_{j\in N\smallsetminus\{i,i+1\}})) \leqslant m$, by the same argument previously employed. Repeating the argument, we eventually obtain $f((y_i)_{i\in N}) \leqslant m$, which implies that f is indeed *monotonically $M_{\mathcal{X}}$-independent* as required.

(iii) $\Longrightarrow$(ii) Suppose that f is monotonically $M_{\mathcal{X}}$-independent but *not* B_μ-monotonic. Thus, there exist $i \in N$, $(x_j)_{j\in N} \in X^N$, $y_i \in X$ such that

$f((x_j)_{j\in N}) \neq \mu(x_i, f((x_j)_{j\in N}), f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))$

i.e. there must exist $m \in M_{\mathcal{X}}$ such that $f((x_j)_{j\in N}) \leqslant m$ but $(x_i \vee f((x_j)_{j\in N})) \wedge (f((x_j)_{j\in N}) \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \wedge (x_i \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})) \nleqslant m$ or $(x_i \vee f((x_j)_{j\in N})) \wedge (f((x_j)_{j\in N}) \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \wedge (x_i \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})) \leqslant m$ but $f((x_j)_{j\in N}) \nleqslant m$.

Thus, suppose that $f((x_j)_{j\in N}) \leqslant m$ and $(x_i \vee f((x_j)_{j\in N})) \wedge (f((x_j)_{j\in N}) \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \wedge (x_i \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})) \nleqslant m$.

Then, it must be the case that $x_i \nleqslant m$ and $f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \nleqslant m$ whence by construction $N_m((x_j)_{j\in N}) \subseteq N_m((y_i,(x_j)_{j\in N\smallsetminus\{i\}}))$ and therefore $f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \leqslant m$ by monotonic $M_{\mathcal{X}}$-independence, a contradiction.

Next, suppose that

$(x_i \vee f((x_j)_{j\in N})) \wedge (f((x_j)_{j\in N}) \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \wedge (x_i \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})) \leqslant m$

and $f((x_j)_{j\in N}) \nleqslant m$.

Since, by upper distributivity of $\mathcal{X}$, it must be the case that either $(x_i \vee f((x_j)_{j\in N})) \leqslant m$ or $(f((x_j)_{j\in N}) \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \leqslant m$ or else $(x_i \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})) \leqslant m$, it follows that $(x_i \vee f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})) \leqslant m$ hence in particular both $x_i \leqslant m$ and $f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \leqslant m$. Thus,

$N_j((y_i,(x_j)_{j\in N\smallsetminus\{i\}}))\subseteq N_j((x_j)_{j\in N})$ and $f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})\leqslant m$ whence, by monotonic $M_{\mathcal{X}}$-independence, $f((x_j)_{j\in N})\leqslant m$, a contradiction again, and the thesis is established. □

A similar argument is used for the case of *total* preorders on (not necessarily finite) bounded *distributive lattices* in Savaglio and Vannucci (2019), and in Vannucci (2019). It should also be emphasized here that, obviously, (finite) distributive lattices are a prominent special subclass of (finite) median join-semilattices.

Corollary 1. *Let $\mathcal{X}=(X,\leqslant)$ be a finite median join-semilattice, B_μ its median-induced betweenness, and $f:X^N\rightarrow X$ an aggregation rule. Then, the following statements are equivalent:*

(i) f is strategy-proof on D^N for every rich domain $D\subseteq U_{B_\mu}$ of locally unimodal preorders on w.r.t. B_μ on X;

(ii) for each $m\in M_{\mathcal{X}}$ there exists an order filter F_m of $(\mathcal{P}(N),\subseteq)$ such that

$f(x_N)=f_{\{F_m:m\in M_{\mathcal{X}}\}}(x_N):=\bigwedge\{m\in M_{\mathcal{X}}:N_m(x_N)\in F_m\}$ for all $x_N\in X^N$.

Proof. Immediate from Theorem 1 and dualization of Proposition 1.4 of Monjardet (1990). In particular, each order filter F_m consists of the *locally m-winning coalitions for f,* namely for every $m\in M_{\mathcal{X}}$

$$F_m:=\left\{\begin{array}{c} T\subseteq N:\text{there exists } x_N\in X^N \text{ such that}\\ \{i\in N:x_i\leqslant m\}=T \text{ and } f(x_N)\leqslant m\end{array}\right\}.$$ □

It should be emphasized that the class of aggregation rules $f_{\{F_m:m\in M_{\mathcal{X}}\}}$ identified by Corollary 1 is in principle very comprehensive indeed. More specifically, Corollary (ii) allows a broad description of such rules as those returning the *strictest consensus among the admissible alternatives actually sponsored by the agents of the relevant coalitions* (as specified by the order filters F_m). In particular, the class of aggregation rules thus characterized encompasses a lot of suitably 'inclusive' and/or 'unbiased' rules, including the following:

- *Quorum system aggregation rules*, namely functions $f_{\{F_m:m\in M_{\mathcal{X}}\}}$ such that every order filter F_m is *transversal* i.e. $S\cap T\neq\varnothing$ for all $S,T\in F_m$.
- *Inclusive aggregation rules,* namely functions $f_{\{F_m:m\in M_{\mathcal{X}}\}}$ such that $\bigcup_{m\in M_{\mathcal{X}}} F_m^{\min}=N$.

- *Collegial aggregation rules,* namely functions $f_{\{F_m:m\in M_{\mathcal{X}}\}}$ such that for some $m \in M_{\mathcal{X}}$, there exists a *non-empty* $S_m \subseteq N$ with $F_m \subseteq \{T \subseteq N : S_m \subseteq T\}$.
- *Outcome-biased aggregation rules*, namely functions $f_{\{F_m:m\in M_{\mathcal{X}}\}}$ where $F_m = \varnothing$ for some $m \in M_{\mathcal{X}}$.
- *Neutral aggregation rules*, namely functions $f_{\{F_m:m\in M_{\mathcal{X}}\}}$ where $F_m = F_{m'}$ whenever $m \wedge m'$ exists.
- *Quota aggregation rules*, namely anonymous aggregation rules i.e. functions $f_{\{F_m:m\in M_{\mathcal{X}}\}}$ such that for each $m \in M_{\mathcal{X}}$ there exists an integer $q_{[m]} \leq |N|$ with $F_m = \{T \subseteq N : q_{[m]} \leq |T|\}$. In particular, a quota aggregation rule is also *neutral* iff $q_{[m]} = q_{[m']}$ whenever $m \wedge m'$ exists.

A prominent instance of a neutral quota aggregation rule is *co-majority* as defined below.

Definition 1. *(Co-majority rule) Let $\mathcal{X} = (X, \leqslant)$ be a finite median join-semilattice, and N a finite set. Then, the co-majority rule $f^{\partial maj}$ for (N, X) is defined as follows: for all $x_N \in X^N$,*

$$f^{\partial maj}(x_N) := \bigwedge_{S\in\mathcal{W}^{maj}} (\bigvee_{i\in S} x_i)$$

where $W^{maj} = \left\{S \subseteq N : |S| \geq \lfloor \frac{|N|+2}{2} \rfloor \right\}$.

It is easily seen, and left to the reader to check, that the co-majority rule is in particular a positive instance of an *idempotent, inclusive* and *transversal* aggregation rule.

As a further corollary of Theorem 1 and Corollary 1 we obtain a new characterization of the co-majority rule via strategy-proofness, anonymity as defined above and the following well-known general property for aggregation rules, namely

Bi-Idempotence: for any $x_N \in X^N$ and $y, z \in X$, if $x_i \in \{y, z\}$ for all $i \in N$, then $f(x^N) \in \{y, z\}$.

Clearly, Bi-Idempotence amounts to a local requirement combining 'decisiveness' (the ability to select a single outcome) and 'faithfulness' (the ability to select the outcome among the proposals actually advanced) both under perfect binary polarization and under perfect agreement.

Thus, we have the following characterization result of the co-majority rule.

Proposition 1. *Let $\mathcal{X} = (X, \leqslant)$ be a finite median join-semilattice, B_μ its median betweenness relation, $D \subseteq U_{B_\mu}$ a rich domain of locally unimodal preorders with respect to B_μ. Then, an aggregation rule $f : X^N \rightarrow X$ satisfies Anonymity, Bi-Idempotence and is strategy-proof on D^N with $|N|$ odd if and only if f is the co-majority rule $f^{\partial maj}$.*

Proof. Immediate from Theorem 1 above and a straightforward dualization of Corollary 7.4 of Monjardet (1990). □

In order to appreciate the actual content of the co-majority rule and its specialization to the case of total preorders to be discussed in the next section, some further properties of median join-semilattices are to be introduced and discussed here.

Claim 2. *(Barthélemy (1978), Monjardet (1981), Leclerc (1994)). Let $\mathcal{X} = (X, \leqslant)$ be a finite upper distributive join-semilattice, 1 its top element, $C(\mathcal{X})$ its covering graph, and r its normalized rank function defined as follows: for any $x \in X$, $r(x) := r(1) - l([x, 1])$. Then,*
(i) the function $d_r : X \times X \rightarrow \mathbb{Z}_+$ such that for any $x, y \in X$
$d_r(x, y) := 2r(x \vee y) - r(x) - r(y)$
is a metric on X;
(ii) $d_r = \delta_{C(\mathcal{X})}$;
(iii) $\delta_{C(\mathcal{X})}(x, y) = \delta_{C(\mathcal{X})}(x, z) + \delta_{C(\mathcal{X})}(z, y)$ for any $x, y, z \in X$ such that $z \in \pi_{xy}$ for some geodesics π_{xy} from x to y on $C(\mathcal{X})$.

Proof. See Barthélemy (1978) (Proposition 1), Monjardet (1981) (Theorem 8), and Leclerc (1994) (Theorem 3.1). □

As a consequence, for any nonnegative integer $n \in \mathbb{N}$ a nonempty *metric median set* $\mathbf{m}(x_1, ..., x_n)$ can be defined on any finite family -or *profile*- of n elements x_i, $i = 1, ..., n$ of a finite *median* join-semilattice $\mathcal{X} = (X, \leqslant)$ with rank function r and covering graph $C(\mathcal{X})$ as follows:

$\mathbf{m}(x_1, ..., x_n) := \arg\min_{y \in X} \sum_{i=1}^{n} d(y, x_i)$, where $d = d_r = \delta_{C(\mathcal{X})}$ as defined above.

Thus, a *metric median function* $\mathbf{m} : \bigcup_{n \in \mathbb{N}} X^n \rightarrow \mathcal{P}(X)$ (where $\mathcal{P}(X)$ denotes the power set of X), and all of its restrictions $\mathbf{m}_{(n)} : X^n \rightarrow \mathcal{P}(X)$ to a fixed $n \in \mathbb{N}$, are well-defined and *nonempty-valued*. In particular, if n is *odd* then it is well-known and easily proved that $\mathbf{m}_{(n)}$ is *single-valued,* hence it can also be regarded as an n-ary algebraic *operation* on X, written $\widehat{\mathbf{m}}_{(n)} : X^n \rightarrow X$ (see e.g. Bandelt, Barthélemy

(1984), Monjardet, Raderanirina (2004), Hudry, Leclerc, Monjardet, Barthélemy (2009)).

The following well-known key result clarifies the tight connection (indeed, the equivalence) between the co-majority aggregation rule and the foregoing restrictions of the metric median function on a finite median join-semilattice $\mathcal{X} = (X, \leqslant)$.

Claim 3. *(Bandelt, Barthélemy (1984)) Let $\mathcal{X} = (X, \leqslant)$ be a finite median join-semilattice and* $\mathbf{m}$ *its metric median function. Then, for any $n \in \mathbb{N}$, and any $(x_1, ..., x_n) \in X^n$*

$$f^{\partial maj}(x_N) := \bigwedge_{S \in \mathcal{W}^{maj}} (\bigvee_{i \in S} x_i) \in \mathbf{m}_{(n)}(x_1, ..., x_n).$$

Moreover, if n is odd then $f^{\partial maj}(x_N) :=$

$$= \bigwedge_{S \in \mathcal{W}^{maj}} (\bigvee_{i \in S} x_i) = \widehat{\mathbf{m}}_{(n)}(x_1, ..., x_n).$$

Proof. Immediate, by Proposition 5 and dualization of Corollaries 1 and 2 of Bandelt, Barthélemy (1984). □

Thus, the co-majority rule is essentially the same as a *metric median rule*, namely a rule that selects a metric median.

Finally, some further facts concerning betweenness relations in finite median join-semilattices are worth mentioning here in order to appreciate the naturalness and robustness of the betweenness relation involved in our main characterization Theorem.

Generally speaking, at least *three* distinct betweenness relations can be defined in a natural way on any finite *graded* join-semilattice (see e.g. Sholander (1952, 1954), Avann (1961), Van de Vel (1993)), namely:

median betweenness B_μ (for all $x, y, z \in X$, $B_\mu(x, y, z)$ iff $\mu(x, y, z) = y$ where is the possibly *partial* median operation as defined above),

interval betweenness B_I (for all $x, y, z \in X$, $B_I(x, y, z)$ iff $y \leqslant x \vee z$ and $x \leqslant y$ or $z \leqslant y$),

metric betweenness B_d (for all $x, y, z \in X$, $B_d(x, y, z)$ iff $d(x, y) + d(y, z) = d(x, z)$, with $d = d_r = \delta_{C(\mathcal{X})}$ the interval-length-based metric as defined above).

Now, it turns out that if a finite join-semilattice is *median* then the relationships among B_μ,B_I and B_d is very tight, as made precise by the following claim.

Claim 4. *(Sholander (1952, 1954), Avann (1961), Barbut, Monjardet (1970), Leclerc (1994)). Let* $\mathcal{X} = (X, \leqslant)$ *be a finite median join-semilattice. Then,* $B_I \subseteq B_\mu = B_d$*. Moreover, if* $\mathcal{X} = (X, \leqslant)$ *is in particular a distributive lattice, then*

(i) $B_I(x, y, z)$ *holds if and only if* $x \wedge z \leqslant y \leqslant x \vee z$*;*

(ii) $B_I = B_\mu = B_d$*;*

(iii) $d_r(x, y) = r(x \vee y) - r(x \wedge y) = |(M(x) \setminus (M(y)) \cup (M(y) \setminus M(x))|$ *(where* $M(z) := \{m \in M_{\mathcal{X}} : z \leqslant m\}$*).*

It is thus confirmed, in particular, that local unimodality (the notion of single-peakedness introduced above and used in Theorem 1) rests on a very natural and robust notion of betweenness on the underlying poset $\mathcal{X} = (X, \leqslant)$, which is in turn tightly anchored to an 'intrinsic' *metric* of $\mathcal{X}$ itself. It should also be emphasized here that if $\mathcal{X}$ is in particular a distributive lattice of binary relations on a finite set A (e.g. the lattice of all *reflexive* binary relations on A to be considered below), then point (iii) of the previous Claim establishes that d_r is precisely the so-called *Kemeny distance* for binary relations as defined below (see Kemeny (1959)).

Kemeny distance on binary relations. Let A be a finite set and $(B_A, \subseteq)$ the poset of all binary relations on A [8]. Then the *Kemeny distance* on $(B_A, \subseteq)$ is the function $d_K : B_A \rightarrow \mathbb{Z}_+$ defined as follows: for any $R, R' \in B_A$,

$$d_K(R, R') := | \{(x, y) \in A \times A : xRy \text{ and } \textit{not } yR'x\} \cup \cup \{(x, y) \in A \times A : xR'y \text{ and } \textit{not } yRx\} |$$ [9].

We are now ready to consider a most significant application of the previous results that involves *strategy-proof aggregation of preferences* including (Arrowian) *social welfare functions* and their strategy-proofness properties*:* the next section is entirely devoted to that topic.

3. Applications to strategy-proof preference aggregation

The major examples of finite median join-semilattices we are going to consider involve the set of all total preorders on a finite set. Indeed, there are at least two natural but subtly distinct ways of relating the

[8] Observe that $(B_A^r, \subseteq)$ is indeed a distributive lattice since it is obviously closed with respect to both intersection $\cap$ and union $\cup$.

[9] Thus, the Kemeny distance is just a specialization of the set-theoretic *symmetric difference metric* to binary relations (see e.g. Barbut, Monjardet (1970)).

latter set of total preorders to a median join-semilattice. Let us start from the first, and more straightforward one.

Example 1. The join-semilattice of total preorders on a finite set.

Let A be a nonempty finite set of alternative social states, $\mathcal{R}_A^T$ the set of all total preorders (i.e. reflexive, transitive and connected binary relations) on A. Let us define the join of two total preorders on A as the *transitive closure* $\overline{\cup}$ of their set-theoretic union. Then, by construction, $\mathcal{X}' := (\mathcal{R}_A^T, \overline{\cup})$ is a join-semilattice, and satisfies both *upper distributivity* (by Claim (P.1) of Janowitz (1984)), and *co-coronation* (by Claims (P.3) and (P.5) of Janowitz (1984)). It follows that $(\mathcal{R}_A^T, \overline{\cup})$ thus defined is indeed a *median join-semilattice* (whose median ternary operation is denoted here μ'), and its meet-irreducibles are the *total preorders* $R_{A_1A_2} \in \mathcal{R}_A^T$ *having just two (non-empty) indifference classes* A_1, A_2 such that (i) (A_1, A_2) is a two-block ordered partition of A, written $(A_1, A_2) \in \Pi_A^{(2)}$, namely $A_1 \cup A_2 = A$, $A_1 \cap A_2 = \emptyset$ and (ii) $[xR_{A_1A_2}y$ and *not* $yR_{A_1A_2}x]$ if and only if $x \in A_1$ and $y \in A_2$. Such total preorders $R_{A_1A_2}$ with $(A_1, A_2) \in \Pi_A^{(2)}$ are also the *co-atoms of* $(\mathcal{R}_A^T, \overline{\cup})$[10].

Thus, a most interesting application of our main result involves *aggregation rules for preference profiles* consisting of total preorders on the set of alternative outcomes, including *social welfare functions* in the classic Arrowian sense[11]. Such an application is made precise by the following definitions and propositions.

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[10]Hence, the join-semilattice of total preorders is in particular *co-atomistic* (i.e. every element which is not the maximum of the semilattice is the least upper bound -or inductively extended join- of a family of meet-irreducible elements).

[11]Let N, A be two (finite) sets and $\mathcal{R}_A^T$ the set of all total preorders on A. An (*Arrowian) social welfare function* for (N, A) is a function

$f : (\mathcal{R}_A^T)^N \rightarrow \mathcal{R}_A$,

namely a function specifying an unique total preference preorder on A for an *arbitrary* profile of n total preference preorders on A.

In the rest of this paper Arrowian social welfare functions will be often referred to simply as *'social welfare functions'.*

Occasionally, and somewhat confusingly, the very same label is also used to refer to (what we shall rather denote as) *Arrowian strict social welfare functions*

$f : (\mathcal{L}_A)^N \rightarrow \mathcal{L}_A$ where $\mathcal{L}_A$ is the set of all *linear orders* (i.e. *antisymmetric* total preorders on A).

By contrast, a *Bergson-Samuelson social welfare function* for (N, A) is a function $f : \{r^N\} \rightarrow \mathcal{R}_A^T$, with $r^N \in \mathcal{R}_A^T$.

Proposition 2. *Let A be a nonempty finite set of alternative social states, $\mathcal{R}_A^T$ the set of all total preorders (i.e. reflexive, transitive and connected binary relations) on A, $\mathcal{X}' = (\mathcal{R}_A^T, \overline{\cup}))$ the join-semilattice of total preorders on A, μ' its median ternary operation and $B_{\mu'}$ the corresponding betweenness as previously defined, and $f : (\mathcal{R}_A^T)^N \to \mathcal{R}_A^T$ an aggregation rule for $(N, \mathcal{R}_A^T)$. Then, the following statements are equivalent:*

(i) f is strategy-proof on D^N for every rich domain $D \subseteq U_{B_{\mu'}}$ of locally unimodal preorders w.r.t. $B_{\mu'}$ on $\mathcal{R}_A^T$;

(ii) for each $R_{A_1A_2} \in M_{\mathcal{X}'}$ there exists an order filter $F_{A_1A_2}$ of $(\mathcal{P}(N), \subseteq)$ such that

$f(R_N) = f_{\left\{F_{A_1A_2}:(A_1,A_2)\in\Pi_A^{(2)}\right\}}(x_N) :=$

$= \bigwedge \{R_{A_1A_2} \in M_{\mathcal{X}'} : \{i \in N : R_i \subseteq R_{A_1A_2}\} \in F_{A_1A_2}\}$ *for all* $R_N \in (\mathcal{R}_A^T)^N$.

Proof. Immediate, from Theorem 1 and Example 1. □

Notice that, as a consequence of the prevous characterization result, there exist a large class of 'classical' Arrowian social welfare functions on (N, A) which are *inclusive* and *idempotent* (or unanimity-respecting) as well as *strategy-proof* on an arbitrary rich domains of locally unimodal preorders with respect to the betweenness relation $B_{\mu'}$ of the median join-semilattice $(\mathcal{R}_A^T, \overline{\cup})$ of total preorders on A. Such a large class includes aggregation rules which are respectively *neither* anonymous nor neutral, *just* anonymous, *just* neutral, or *both* anonymous and neutral. To see this, consider the following list of examples:

- *Inclusive quorum systems* , namely functions $f_{\left\{F_{R_{A_1A_2}}:R_{A_1A_2}\in M_{\mathcal{X}'}\right\}}$ such that every order filter $F_{R_{A_1A_2}}$ is *transversal* i.e. $S \cap T \neq \varnothing$ for all $S, T \in F_{R_{A_1A_2}}$ and $\bigcup_{R_{A_1A_2}\in M_{\mathcal{X}}} F^{\min}_{R_{A_1A_2}} = N$ (observe that such a class includes any rule such that for every $R_{A_1A_2} \in M_{\mathcal{X}'}$, $F_{R_{A_1A_2}}$ is *simple-majority collegial* i.e. there exists a *minimal* simple majority coalition $S_{A_1A_2} \subseteq N$, $|S_{A_1A_2}| = \left\lfloor \frac{|N|+2}{2} \right\rfloor$ with $F_{R_{A_1A_2}} = \{T \subseteq N : S_{A_1A_2} \subseteq T\}$). Generally speaking, inclusive quorum systems need not be anonymous or neutral.
- *Outcome-biased aggregation rules*, namely functions $f_{\left\{F_{R_{A_1A_2}}:R_{A_1A_2}\in M_{\mathcal{X}'}\right\}}$ where $F_{R_{A_1A_2}} = \varnothing$ for some $R_{A_1A_2} \in M_{\mathcal{X}'}$ (observe that they include the subclass of those aggregation

rules such that for some *total preorder* $\overline{R} \in \mathcal{R}_A^T$, including possibly a *linear order,* $F_{R_{A_1 A_2}} = \varnothing$ for every $R_{A_1A_2} \in M_{\mathcal{X}'}$ such that $\overline{R} \subseteq R_{A_1A_2}$).

- *Neutral aggregation rules*, namely functions $f_{\left\{F_{R_{A_1A_2}}: R_{A_1A_2} \in M_{\mathcal{X}'}\right\}}$ where $F_{R_{A_1A_2}} = F_{R_{A_1'A_2'}m'}$ whenever $R_{A_1A_2} \wedge R_{A_1'A_2'}$ exists.
- *Quota aggregation rules*, i.e. functions $f_{\left\{F_{R_{A_1A_2}}: R_{A_1A_2} \in M_{\mathcal{X}'}\right\}}$ such that for each $R_{A_1A_2} \in M_{\mathcal{X}'}$ there exists an integer $q_{[R_{A_1A_2}]} \leq |N|$ with $F_m = \left\{T \subseteq N : q_{[R_{A_1A_2}]} \leq |T|\right\}$ (such rules are clearly anonymous, but not necessarily neutral: they are of course neutral as well if, furthermore, $F_{R_{A_1A_2}} = F_{R_{A_1'A_2'}}$ whenever $R_{A_1A_2} \wedge R_{A_1'A_2'}$ exists).

A remarkable family of *anonymous* but typically *not* neutral aggregation rules for $(N, \mathcal{R}_A^T)$ is that of *Condorcet-Kemeny rules*, as defined below (see also Young, Levenglick (1978), Young (1995)).

Definition *(Generalized Condorcet-Kemeny aggregation rules)* Let $\mathcal{X}' = (\mathcal{R}_A^T, \overline{\cup}))$ be the join-semilattice of total preorders on finite set A as defined above, $C(\mathcal{X}')$ its covering graph, $\delta_{C(\mathcal{X}')}$ the shortest-path metric on $C(\mathcal{X}')$, $\mathcal{L}_A \subseteq \mathcal{R}_A^T$ the set of linear orders on A, N a finite set, and $\preceq$ a linear order on $\mathcal{R}_A^T$.

The *generalized Condorcet-Kemeny aggregation rule* for $(N, \mathcal{R}_A^T)$ induced by $\preceq$ is the function $f_{\preceq}^{CK^*} : (\mathcal{R}_A^T)^N \to \mathcal{R}_A^T$ defined as follows: for all $R_N \in (\mathcal{R}_A^T)^N$,

$$f_{\preceq}^{CK^*}(R_N) := \min\nolimits_{\preceq} \left\{ \begin{array}{c} R \in \mathcal{R}_A^T : \sum_{i \in N} \delta_{C(\mathcal{X}')}(R, R_i) \leq \sum_{i \in N} \delta_{C(\mathcal{X}')}(R', R_i) \\ \text{for all } R' \in \mathcal{R}_A^T \end{array} \right\}.$$

In particular, the (strict) *Condorcet-Kemeny aggregation rule* for $(N, \mathcal{R}_A^T)$ induced by $\preceq$ is the function $f_{\preceq}^{CK} : (\mathcal{R}_A^T)^N \to \mathcal{L}_A$ defined as follows: for all $R_N \in (\mathcal{R}_A^T)^N$,

$$f_{\preceq}^{CK}(R_N) := \min\nolimits_{\preceq} \left\{ \begin{array}{c} R \in \mathcal{L}_A : \sum_{i \in N} \delta_{C(\mathcal{X}')}(R, R_i) \leq \sum_{i \in N} \delta_{C(\mathcal{X}')}(R', R_i) \\ \text{for all } R' \in \mathcal{L}_A \end{array} \right\}.$$

Notice that a (strict) Condorcet-Kemeny rule amounts to a *constrained* generalized Condorcet-Kemeny rule. It should also be emphasized that generalized Condorcet-Kemeny aggregation rules require a prefixed linear order $\preceq$ as a tie-breaker device whenever the *remoteness function* $\sum_{i \in N} \delta_{C(\mathcal{X}')}(., R_i)$ of a profile R_N admits several distinct minima: that is *the only role* of $\preceq$ in $f_{\preceq}^{CK}$ and $f_{\preceq}^{CK^*}$, and the source of the typical failure of Condorcet-Kemeny rules to satisfy Neutrality. It follows that, to the extent that uniqueness of minima of the remoteness function is warranted, the outcome of Condorcet-Kemeny rules is unaffected by the choice of $\preceq$ and Neutrality is restored. That is precisely the case when the size n of the set of agents N is odd, as implied by the following characterization result:

Proposition 3. *Let $\mathcal{R}_A^T$ be the set of all total preorders on a finite set A, $\mathcal{X}' = (\mathcal{R}_A^T, \overline{\cup}))$ the join-semilattice of total preorders on A, μ' its median ternary operation and $B_{\mu'}$ the corresponding betweenness relation as previously defined, N a finite set such that $|N|$ is an odd number, and $f : (\mathcal{R}_A^T)^N \to \mathcal{R}_A^T$ an aggregation rule for $(N, \mathcal{R}_A^T)$. Then, the following statements are equivalent:*

(i) f satisfies Anonymity and Bi-Idempotence, and is strategy-proof on D^N for every rich domain $D \subseteq U_{B_{\mu'}}$ of locally unimodal preorders w.r.t . $B_{\mu'}$ on $\mathcal{R}_A^T$;

(ii) $f = \widehat{f}^{\partial maj}$ where $\widehat{f}^{\partial maj} : (\mathcal{R}_A^T)^N \to \mathcal{R}_A^T$ denotes the co-majority aggregation rule for $(N, \mathcal{R}_A^T)$;

(iii) $f = f_{\preceq}^{CK} = f_{\preceq'}^{CK}$ i.e. the generalized Condorcet-Kemeny aggregation rule for $(N, \mathcal{R}_A^T)$ for any pair of linear orders $\preceq$, $\preceq'$ on $\mathcal{R}_A^T$.

Proof. Immediate from Theorem 1, Proposition 1, Claim 1, and Example 1 above. □

Thus, in particular, when the size of N is odd the generalized Condorcet-Kemeny rule for $(N, \mathcal{R}_A^T)$ is precisely the same as the co-majority rule, and can be characterized as the unique aggregation rule for $(N, \mathcal{R}_A^T)$ (or, in other terms, the unique *Arrowian social welfare function)* which is Anonymous, Bi-Idempotent and strategy-proof on $U_{B_{\mu'}}$(and any of its rich subdomains).[12]

[12]Is should be noticed that the requirement that $n = |N|$ be odd is not at all as restrictive as it might seem at first sight. In fact, for n even our aggregation rule f for $(N, \mathcal{R}_A^T)$ might be embedded in a natural way into a more comprehensive aggregation rule $\widetilde{f}$ for $(N, \mathcal{R}_A^T \times \mathbb{Z})$ (where $\mathbb{Z}$ denotes the set of integer numbers) as supplemented with the natural projection from $\mathbb{Z}$ to the finite additive group $\mathbb{Z}_n$ of integers modulo n. Such an aggregation rule implements a pseudo-random 'anonymous' selection of a 'president' in N to the effect of producing an artificially but fairly augmented 'electorate' of *odd* size. Furthermore, a similar construct obtained by replacing $\mathbb{Z}_n$ with $\mathbb{Z}_k$ (where $k := |A|$) results in a further aggregation rule $\overrightarrow{f}$ for $(N, \mathcal{R}_A^T \times \mathbb{Z})$ which implements a pseudorandom 'neutral' choice of *one* linear order among those consistent with the total preorder selected by f at

Moreover, notice that (for an odd n) $\widehat{f}^{\partial maj} \equiv f_{\preceq}^{CK^*}$satisfies a weak version of the so-called **Condorcet principle**, namely for every

$(R_i)_{i\in N} \in \mathcal{R}_A^T$ and $x \in A$, *if* x is a *Condorcet winner* -that is

$\{i \in N : xR_iy$ and *not* $yR_ix\} \in \mathcal{W}^{maj}$ for every $y \in A \setminus \{x\}$- *then* $x \in Top(\widehat{f}^{\partial maj}((R_i)_{i\in N}))$

(where for any $R \in \mathcal{R}_A^T$, $Top(R) := \{x \in A : xRy$ for all $y \in A\}$).

To check this, suppose x is indeed a Condorcet winner, yet $x \notin Top(\widehat{f}^{\partial maj}((R_i)_{i\in N}))$. Thus, there exist $y \in X \setminus \{x\}$ and a meet-irreducible $R_{[y][x]}$ of the join-semilattice $(\mathcal{R}_A^T, \overline{\cup})$ (i.e. a two-indifference-class total preorder having y among its maxima and x among its minima) such that $Top(\widehat{f}^{\partial maj}((R_i)_{i\in N})) \subseteq R_{[y][x]}$. But then, upper distributivity of $(\mathcal{R}_A^T, \overline{\cup})$ entails that $\overline{\bigcup_{i\in T}} R_i \subseteq R_{[y][x]}$ for some $T \in \mathcal{W}^{maj}$ whence $R_i \subseteq R_{[y][x]}$ for each $i \in T \in \mathcal{W}^{maj}$, a contradiction.

It is easily checked that $\widehat{f}^{\partial maj}$ also satisfies Neutrality if $n := |N|$ is odd . It follows that for any odd n there exists an Arrowian social welfare function on the *full* domain of total preorders on an arbitrary finite set which is anonymous, neutral, idempotent (because Bi-Idempotence clearly implies Idempotence), satisfies a monotonic independence property w.r.t. the meet-irreducible total preorders (which are the co-atoms of the join-semilattice $(\mathcal{R}_A^T, \subseteq)$ i.e. the total preorders having just two indifference classes) *and is strategy-proof* on any rich locally unimodal preference domain on $\mathcal{R}_A^T$. Therefore, $\widehat{f}^{\partial maj}$ is in particular a *social welfare function that satisfies all the properties required by Arrow's (Im)Possibility Theorem except for* the *Independence of Irrelevant Alternatives (IIA) condition* [13]. What is then the relationship between $M_{\mathcal{X}}$-Independence ($M_{\mathcal{X}}$-I) and IIA? It is quite clear that under Idempotence $M_{\mathcal{X}}$-I is definitely *weaker* than IIA because, as a consequence of Proposition 1, the former is consistent with Anonymity and Neutrality of an (Arrowian) unanimity-respecting social welfare function while the latter is not. Indeed, as established by Hansson (1969), IIA in combination with Anonymity and Neutrality provides a characterization of the *constant* social welfare function having the *universal indifference* relation $A \times A$ as its *unique* value (hence in particular the

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any profile. Such an aggregation rule $\overrightarrow{f}$ is in fact *constrained* (actually an $\mathcal{L}_A^T$-constrained one), since its values are constrained to lie in $\mathcal{L}_A^T \subseteq \mathcal{R}_A^T$.

[13]Recall that Arrow's IIA (in binary form) is a condition on social welfare functions $f : (\mathcal{R}_A^T)^N \to \mathcal{R}_A^T$ defined as follows: for every x,$y \in A$ and any

R_N , $R'_N \in (\mathcal{R}_A^T)^N$ such that xR_iy if and only if xR'_iy for each $i \in N$, $xf(R_N)y$ entails $xf((R'_N)y$.

former combination of properties is inconsistent with Idempotence). In other terms, strenghtening $M_{\mathcal{X}}$-Independence to IIA is *just impossible* for unanimity-respecting, anonymous and neutral Arrowian social welfare functions.

As it turns out, reconciling unanimity-respecting and strategy-proof preference aggregation to IIA is however possible by moving away from the domain of total preorders, hence from Arrowian social welfare functions, towards some more comprehensive preference domain. This observation brings us to our second leading example, to which we now turn[14].

Example 2. The lattice of reflexive binary relations on a finite set.

Clearly enough, any distributive lattice $(X, \vee, \wedge)$ also provides an example of a median join-semilattice.

In particular, let A be a nonempty finite set of alternative social states, B_A^r the set of all reflexive binary relations on A, $(B_A^r, \subseteq)$ the set-inclusion poset on B_A^r. Let us then define the join $\vee$ and meet $\wedge$ of two reflexive binary relations on A as their set-theoretic union $\cup$ and intersection $\cap$, respectively. Hence, $\mathcal{X}'' := (B_A^r, \cup, \cap)$ is indeed, by construction, a (bounded) *distributive lattice*. It follows that $\cup$-closedness of B_A^r and both upper-distributivity and co-coronation trivially hold in $\mathcal{X}''$ i.e. $(B_A^r, \cup)$ is in particular a *median join-semilattice* whose median μ'' is precisely the median of the distributive lattice $(B_A^r, \cup, \cap)$. Namely, for any $R_1, R_2, R_3 \in B_A^r$,

$\mu''(R_1, R_2, R_3) = (R_1 \cup R_2) \cap (R_2 \cup R_3) \cap (R_3 \cup R_1) = (R_1 \cap R_2) \cup (R_2 \cap R_3) \cup (R_3 \cap R_1)$.

Moreover, it can be easily shown (and left to the reader to check) that

$M_{\mathcal{X}''} = \mathcal{C}_{\mathcal{X}''} = \{A^2 \setminus \{(a, b)\} : a, b \in A, a \neq b\}$, and
$J_{\mathcal{X}''} = \mathcal{A}_{\mathcal{X}''} = \{\Delta_A \cup \{(a, b)\} : a, b \in A, a \neq b\}$
where $\Delta_A := \{(a, a) : a \in A\}$.

Hence, $\mathcal{X}''$ is in particular a *co-atomistic and atomistic* lattice.

It should be emphasized that the set of all total preorders on A is clearly a *subset* but *not* a sub-join semilattice of the join-semilattice reduct $(B_A^r, \cup)$ of the lattice $(B_A^r, \cup, \cap)$, since the union of two total preorders may *not* be transitive (to see this, consider e.g. $A = \{a, b, c, d\}$, and the linear orders, $R_1 := abcd$, $R_2 := dbca$ (written according to

[14]A *third* relevant example is the median join-semilattice $\mathcal{R}_A^T \times \mathcal{P}(A)$, which is particularly convenient when it comes to addressing squarely agenda-manipulation issues. Such semilattice will be discussed in some detail elsewhere.

the usual 'decreasing' notation). Now, $R_1 \cup R_2 = \{(x,x) : x \in A\} \cup \{(a,b),(b,a),(a,c),(c,a),(a,d),(d,a),(b,c),(b,d),(d,b),(c,d),(d,c)\}$
which is not transitive since $\{(c,a),(a,b)\} \subseteq R_1 \cup R_2$ but $(c,b) \notin R_1 \cup R_2$).

Let us now introduce the strenghtening of *Monotonic $M_{\mathcal{X}}$-Independence* which results from substituting IIA for $M_{\mathcal{X}}$-Independence.

Definition 2. ***Monotonic IIA:*** *Let $(X,\leqslant)$ be a partially ordered set. Then, an aggregation rule $f : X^N \to X$ is* ***monotonically IIA*** *if for all $x_N, y_N \in X^N$ and all $z \in X$: if $N_m(x_N) \subseteq N_m(y_N)$ then $f(x_N) \leqslant z$ implies $f(y_N) \leqslant z$.*[15]

We are now ready to show that when the relevant join-semilattice is (the join-reduct of) $\mathcal{X}'' := (\mathcal{B}^r_A, \cup, \cap)$ we can rely on the full force of a counterpart of Theorem 1 for bounded distributive lattices (see e.g. Savaglio, Vannucci (2019)) to obtain the following result.

Proposition 4. *Let A be a nonempty finite set of alternative social states, $\mathcal{B}^r_A$ the set of all reflexive binary relations on A,*
$\mathcal{X}'' := (\mathcal{B}^r_A, \cup, \cap)$ the (bounded) distributive lattice induced on $\mathcal{B}^r_A$ by $\cup$ and $\cap$, μ'' its median ternary operation and $B_{\mu''}$ the corresponding betweenness as previously defined, and $f : (\mathcal{B}^r_A)^N \to \mathcal{B}^r_A$ an aggregation rule for $(N, \mathcal{B}^r_A)$.
Then, the following statements are equivalent:
(i) f is strategy-proof on D^N for every rich domain $D \subseteq U_{B_{\mu''}}$ of locally unimodal preorders w.r.t. $B_{\mu''}$ on $\mathcal{B}^r_A$;
(ii) f is $B_{\mu''}$-monotonic;
(iii) f is monotonically $M_{\mathcal{X}''}$-independent;
(iv) f is monotonically $J_{\mathcal{X}''}$-independent;
(v) f is monotonically IIA;
(vi) there exists an order filter $\mathcal{F}$ of $(\mathcal{P}(N), \subseteq)$ and a family $\{R_S \in \mathcal{B}^r_A : S \in \mathcal{F}\}$ of reflexive relations on A such that

$$f(R_N) = \bigcap_{S \in \mathcal{F}} ((\cup_{i \in S} R_i) \cup R_S) \text{ for all } R_N \in (\mathcal{B}^r_A)^N;$$

(vii) there exists an order filter $\mathcal{F}$ of $(\mathcal{P}(N), \subseteq)$ and a family $\{R_S \in \mathcal{B}^r_A : S \in \mathcal{F}\}$ of reflexive relations on A such that

$$f(R_N) = \bigcup_{S \in \mathcal{F}} ((\cap_{i \in S} R_i) \cap R_S) \text{ for all } R_N \in (\mathcal{B}^r_A)^N.$$

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[15]It should be remarked here that Monotonic IIA is indeed equivalent to the conjunction of IIA and Isotony as defined above (see e.g. Monjardet (1990)).

Proof. (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iii): Immediate, from Theorem 1 and Example 2 above.

(ii)$\Longleftrightarrow$(vi)$\Longleftrightarrow$(vii): It follows immediately from Theorem 1 of Savaglio, Vannucci (2019).

(ii)$\Longleftrightarrow$(iv): It follows immediately from Theorem 1 of Vannucci (2019).

(iii)$\Longleftrightarrow$(iv): It follows from the following

Lemma: Let $\mathcal{X}^* := (X, \leqslant)$ be a finite distributive lattice, N a finite set and $f : X^N \longrightarrow X$ an aggregation rule for (N, X). Then, f is $M_{\mathcal{X}^*}$-independent if and only if it is also $J_{\mathcal{X}^*}$-independent.

Proof. It is well-known that if $\mathcal{X}^* := (X, \leqslant)$ is a finite distributive lattice then $(J_{\mathcal{X}^*}, \leqslant)$ and $(M_{\mathcal{X}^*}, \leqslant)$ are order-isomorphic i.e. there is a bijection $\phi : J_{\mathcal{X}^*} \longrightarrow M_{\mathcal{X}^*}$ such that, for any $j, j' \in J_{\mathcal{X}^*}$, $j \leqslant j'$ if and only if $\phi(j) \leqslant \phi(j')$ (see e.g. Davey, Priestley (1990), p. 178): in particular, for any $j \in J_{\mathcal{X}^*}$,

$m_j = \phi(j) := \bigvee(X \setminus \uparrow j)$ with $\downarrow m_j = X \setminus \uparrow j$

(and dually, for any $m \in M_{\mathcal{X}^*}$, $j_m = \phi^{-1}(m) := \bigwedge(X \setminus \downarrow m)$ with $\uparrow j_m = X \setminus \downarrow m$.

Indeed, $\phi(j_m) = \bigvee(X \setminus \uparrow j_m) = \bigvee(X \setminus (X \setminus \downarrow m)) = \bigvee(\downarrow m) = m)$.

Next, observe that for any $j \in J_{\mathcal{X}^*}$, $i \in N$ and $x_N \in X^N$, $j \leqslant x_i$ implies $x_i \in \uparrow j$ whence $x_i \notin \downarrow m_j$, namely $x_i \nleqslant m_j$ and conversely, $x_i \nleqslant m_j$ implies $x_i \in \uparrow j$. Thus, $N_j(x_N) = N \setminus N_{m_j}(x_N)$. It follows that, for any $x_N, x'_N \in X^N$, $N_j(x_N) = N_j(x'_N)$ if and only if $N_{m_j}(x_N) = N_{m_j}(x'_N)$. Now, suppose that $f : X^N \longrightarrow X$ is monotonically $M_{\mathcal{X}^*}$-independent, and suppose that for some $j \in J_{\mathcal{X}^*}$, and $x_N, x'_N \in X^N$: $N_j(x_N) = N_j(x'_N)$ and $j \leqslant f(x_N)$. Then, $N_{m_j}(x_N) = N_{m_j}(x'_N)$ and $f(x_N) \nleqslant m_j$ hence $f(x'_N) \nleqslant m_j$, by Monotonic $M_{\mathcal{X}^*}$-Independence. As a consequence $j \leqslant f(x'_N)$, by the previous observation. It follows that f is monotonically $J_{\mathcal{X}^*}$-independent as well. A dual argument involving $j_m \in J_{\mathcal{X}^*}$ for an arbitrary $m \in M_{\mathcal{X}^*}$establishes the converse entailment from Monotonic $J_{\mathcal{X}^*}$-Independence to Monotonic $M_{\mathcal{X}^*}$-Independence, as required.

(iv)$\Longleftrightarrow$(v): It follows immediately from our previous observation that $J_{\mathcal{X}''} = \mathcal{A}_{\mathcal{X}''} = \{\Delta_A \cup \{(a,b)\} : a, b \in A, a \neq b\}$ (see Example 2 above), which implies the equivalence of $J_{\mathcal{X}''}$-Independence and IIA (hence of Monotonic $J_{\mathcal{X}''}$-Independence and Monotonic IIA) for $\mathcal{X}'' := (B^r_A, \cup, \cap)$. □

It goes without saying that the strategy-proof aggregation rules for $(N, \mathcal{B}^r_A)$ characterized above comprise counterparts to inclusive quorum systems, quota rules and all the other aggregation rules for total preorders mentioned above. The co-majority rule $f_r^{\partial maj} : (\mathcal{B}_A^r)^N \rightarrow \mathcal{B}_A^r$ is defined by the identity

$f_r^{\partial maj}(R_N) = \bigcap_{S \in W^{maj}} (\cup_{i \in S} R_i)$ for each $R_N \in (\mathcal{B}_A^r)^N$, which is obtained from the general formula under statement (vi) of the previous proposition by setting $\mathcal{F} = W^{maj} := \left\{ S \subseteq N : |S| \geq \lfloor \frac{|N|+2}{2} \rfloor \right\}$ and $R_S = \Delta_A$ for each $S \in W^{maj}$.

But new possibilities arise here: to begin with, a version of the *majority rule* $f_r^{maj} : (\mathcal{B}_A^r)^N \rightarrow \mathcal{B}_A^r$ is now well-defined by the identity

$f_r^{maj}(R_N) = \bigcup_{S \in W^{maj}} (\cap_{i \in S} R_i)$ for each $R_N \in (\mathcal{B}_A^r)^N$,

which is obtained from the general formula under statement (vii) of the previous proposition by setting $\mathcal{F} = W^{maj}$ and $R_S = A \times A$ for each $S \in W^{maj}$.

Of course, the outputs of f_r^{maj}(and $f_r^{\partial maj}$, or for that matter of any idempotent aggregation rule for $(N, \mathcal{B}_A^r)$) may well be nontransitive or even *intransitive* (i.e. include cycles with asymmetric components): to see this, just consider a profile consisting of identical *cyclic* reflexive relations. But then, it is easily seen (and left to the reader to check) that the aggregation rules for $(N, \mathcal{B}_A^r)$ resulting from idempotent ones by just *removing cycles* from their outputs through a *minimal* number of pair-deletions are also $B_{\mu''}$-monotonic (though, of course, not idempotent but rather *weakly idempotent* in the following sense: for any profile $R_N \in (\mathcal{B}_A^r)^N$ such that $R_i = R_j = R$ for all $i, j \in N$, $f(R_N) \subseteq R$). Thus, here is a new (sub)class of interesting strategy-proof aggregation rules for $(N, \mathcal{B}_A^r)$ whose output for any profile of total preorders is indeed a total preorder (let us call them *minimal monotonic retracts* just for ease of reference).

Furthermore, for an odd-sized N the majority rule for $(N, \mathcal{B}_A^r)$ turns out to coincide with the co-majority rule. This is made precise by the following proposition.

Proposition 5. *Let A be a nonempty finite set of alternative social states, $\mathcal{B}_A^r$ the set of all reflexive binary relations on A, $\mathcal{X}'' := (\mathcal{B}_A^r, \cup, \cap)$ the (bounded) distributive lattice induced on $\mathcal{B}_A^r$ by $\cup$ and $\cap$, μ'' its median ternary operation and $B_{\mu''}$ the corresponding betweenness relation, N a finite set such that $|N|$ is an odd number, and $f : (\mathcal{B}_A^r)^N \rightarrow \mathcal{B}_A^r$ an aggregation rule for $(N, \mathcal{B}_A^r)$. Then, the following statements are equivalent:*

(i) f satisfies Anonymity and Bi-Idempotence, and is strategy-proof on D^N for every rich domain $D \subseteq U_{B_{\mu''}}$ of locally unimodal preorders w.r.t . $B_{\mu''}$ on $\mathcal{B}^r_A$;

(ii) $f = f^{maj}_r = f^{\partial maj}_r$;

(iii) $f = f^{CK^r}_{\preceq} = f^{CK^r}_{\preceq'}$ i.e. the generalized Condorcet-Kemeny aggregation rule for $(N, \mathcal{B}^r_A)$ for any pair of linear orders $\preceq, \preceq'$ on $\mathcal{B}^r_A$.

Proof. Immediate from Corollary 1 and Example 2. □

Thus, when N has an odd size, generalized Condorcet-Kemeny aggregation rules for $(N, \mathcal{B}^r_A)$ and $(N, \mathcal{R}^T_A)$ are amenable to the same sort of simple characterization via Anonymity, Bi-Idempotence and Strategy-Proofness on certain rich single-peaked domains.

4. Related literature and concluding remarks

The present work focusses on strategy-proof aggregation rules in *finite median join-semilattices* and their applications to several preference aggregation rules, including as a special prominent case (Arrowian) social welfare functions, namely aggregation rules taking arbitrary profiles of total preference preorders as inputs and returning a total preference preorder as output[16]. Now, addressing strategy-proofness issues for such preference aggregation rules requires a suitable specification of the agents' preferences on outcomes, namely their *preferences on preferences.* It is well-known, in view of the Gibbard-Satterthwaite 'impossibility theorem', that (i) some *domain-restriction on the foregoing 'preferences on preferences'* is required in order to open up the possibility to design interesting and non-dictatorial strategy-proof preference-aggregation rules, and (ii) some form of *single-peakedness* is a most natural and plausible domain-restriction to that effect. But single-peakedness notions typically rely in turn on an underlying ternary *betweenness relation* defined on the 'preference space' which is supposedly shared by the relevant agents and thus should presumably be 'naturally' embedded in that space. Therefore, we are immediately confronted with a list of key issues to address, namely:

[16]It is worth emphasizing here that our usage of the term 'Arrowian social welfare function' -while arguably sound and well-grounded- is by no means widely established. Sometimes that term is also used to denote aggregation rules for profiles of linear orders, possibly with the additional conditions of Idempotence and the Arrowian 'Independence of Irrelevant Alternatives' requirement (see e.g. Sethuraman, Chung Piaw, Vohra (2003)).

- What sort of preference relations are to be aggregated? Arbitrary *reflexive binary relations*, *total preorders*, *linear orders*, or others? (Of course, the answer has to be related to the general structure of the outcome space to focus on).
- Are the preference profiles to be aggregated of an arbitrary but *fixed finite size* (fixed population approach), or of *every possible finite size* (variable population approach)?
- What *type of aggregation protocol* are we to consider? Namely, given a profile of preference relations of the prescribed type as input, what kind of object is the output required to be? A single preference relation of the prescribed type (*aggregation* with no qualifier, namely *exact or pure aggregation*), one or more preference relations of the prescribed type (*multi-aggregation*), a single preference relation belonging to a class which includes but does not reduce to the prescribed type for the input *((domain-)restricted aggregation)*, or a single preference relation of the same type as that prescribed for inputs but enjoying some *additional requirements ((codomain-)constrained aggregation)?*
- What sort of *single-peakedness* property for the relevant 'preferences on preferences' are we to focus on, or equivalently, what is the most natural/plausible notion of *betweenness* on the basic 'preference space' to refer to?

This paper relies on a definite choice of focus for each one of the foregoing issues, namely:

(a) the basic preference domain should *include all the total preorders*;

(b) the preference profiles to be aggregated are of some *fixed size*;

(c) the type of aggregation protocol to focus on is (pure) *aggregation* (and possibly *constrained aggregation)*;

(d) the *betweenness relation* to be used in order to define single-peaked 'preferences on preferences' should be *the one 'naturally' dictated by the underlying basic preference domain* of the aggregation rule under consideration.

In this section we shall use the characteristic features described by points (a)-(b)-(c)-(d) above as a convenient benchmark in order to locate the present paper and its marginal contribution within the non-negligible body of related literature.

To begin with, the study of aggregation rules for ordered sets, semi-lattices, and lattices was pioneered by Monjardet and his co-workers, who provide characterizations of several classes of such rules mostly within a fixed population setting but also, occasionally, within a variable population framework (see e.g. Barthélemy, Monjardet (1981), Bandelt, Barthélemy (1984), Monjardet (1990), Barthélemy, Janowitz (1991), Leclerc (1994), Monjardet, Raderanirina (2004), Hudry et al. (2009)). In particular, characterizations of the simple majority and co-majority rules (sometimes also denoted as *'median' rules*) are established in several latticial and semilatticial settings both as aggregation rules within a fixed population framework (see e.g. Monjardet (1990)) and as multi-aggregation rules within a variable population framework (see e.g. Barthélemy, Janowitz (1991), Monjardet, Raderanirina (2004)).

Concerning the special case of preference aggregation, an early characterization of (a version of) the Condorcet-Kemeny rule, regarded as a *multi-aggregation rule* for *linear orders* in a *variable population* setting is due to Young, Levenglick (1978). Indeed, Young and Levenglick prove that the Condorcet-Kemeny multi-aggregation rule is in fact the unique function $f : \bigcup_{n \in \mathbb{N}} (\mathcal{L}_A)^n \longrightarrow \mathcal{P}(\mathcal{L}_A) \setminus \{\emptyset\}$ that satisfies the following three properties: *neutrality* (i.e. it is invariant with respect to changes of labels of the elements of A), a version of the *Condorcet principle* (namely, for any profile $R_N \in (\mathcal{L}_A)^n$: (i) if x has a strict majority against y at R_N then y cannot be the immediate predecessor of x in any 'social' preference order $R \in f(R_N)$, and (ii) if individual preferences between x and y are equally split at R_N then if x is the immediate predecessor of y according to *some* 'social' preference order $R \in f(R_N)$, then there also exists *another* $R' \in f(R_N)$ such that y is the immediate predecessor of x according to R'), and '*consistency*' across committees/electorates (i.e. for any pair of profiles R_N, R_M such that $N \cap M = \emptyset$, if $f(R_N) \cap f(R_M) \neq \emptyset$ then $f((R_N, R_M)) = f(R_N) \cap f(R_M)$).[17]

In a similar vein, but in a much more general setting and building partly on Barthélemy, Janowitz (1991), McMorris, Mulder, Powers (2000) establishes a further elegant characterization of the *median function* as a *multi-aggregation rule*

[17]In subsequent work (see e.g. Young (1995)), it is emphasized that at any profile R_N of linear orders on a finite A the linear orders selected by the Condorcet-Kemeny rules can also be regarded as the *maximum likelihood* rankings according to the evidence provided by R_N. It should be noted that Young's argument is quite general and also applies to wider classes of preference relations on A including the set of all total preorders $\mathcal{R}_A^T$ and the set of all reflexive relations $\mathcal{B}_A^r$.

$f : \bigcup_{n \in \mathbb{N}} X^n \longrightarrow (\mathcal{P}(X) \setminus \{\varnothing\})$ for a median meet-semilattice $(X, \leqslant)$ in a *variable population* framework, using suitably generalized counterparts of both part (ii) of the Condorcet principle (labelled as $\frac{1}{2}$-*Condorcet property*) and *'consistency'* across populations/electorates as presented above, and a very mild *'faithfulness'* condition simply requiring $f((x)) = \{x\}$ for each x in X.

The present paper obviously owes much to that most remarkable body of literature. Notice, however, that the contributions mentioned above *do not consider at all strategy-proofness properties of aggregation rules* (or, for that matter, nonmanipulability properties of *any* sort).

The issue of *strategy-proofness* for preference aggregation rules has been indeed addressed in the previous literature, but never -to the best of the authors' knowledge- with respect to the 'full' domain of all total preorders. Under the heading 'social welfare functions', Bossert and Storcken (1992) study in fact aggregation rules for *linear* orders on a finite set (hence what we refer to as *strict social welfare functions*) and their *coalitional strategy-proofness* properties with respect to topped metric total preference preorders (on the set of linear orders) as induced by a suitably 'renormalized' version of the Kemeny distance to be further discussed below. They prove an impossibility theorem for those coalitionally strategy-proof and sovereign *strict* social welfare functions that also satisfy a certain condition of independence from extrema.

Working within a *variable population* framework, Bossert and Sprumont (2014) offer several possibility results concerning *restricted* strategy-proof aggregation rules (mapping profiles of *linear orders* on a finite set A into *total preorders* on A) which are strategy-proof on the domain of topped preferences (on the set of total preorders) that are single-peaked with respect to the median betweenness of the distributive lattice of reflexive binary relations on A (which amounts to an outcome space $\mathcal{B}_A^r$which is far more comprehensive than the 'small' domain-base $\mathcal{L}_A$ or even the larger codomain $\mathcal{R}_A^T$ of the aggregation rule[18]). That paper identifies some (variable-population) strategy-proof *restricted* aggregation rules on $\mathcal{R}_A^T$ including (strict) Condorcet-Kemeny rules, a class of variable-population counterparts of our *monotonic retracts of the majority relation* as introduced above, and a family of rules denoted as *status-quo rules* that are related to the class of *outcome-biased rules*

[18]Thus, in a sense, the median-induced betweenness relation under consideration (and the resulting single-peakedness property) is *not* the one 'naturally' dictated by the codomain $\mathcal{R}_A^T$ (let alone the strictly smaller domain-base $\mathcal{L}_A$) of the aggegation rule.

mentioned above as one family of examples covered by Proposition 2. An (implicit) characterization of such monotonic majority-retracts is also provided, and the family of status-quo rules is explicitly characterized (but the strict Condorcet-Kemeny rules are not). Thus, the present paper provides *extensions of the fixed-population counterparts* of such strategy-proof *restricted* aggregation rules to strategy-proof *exact* aggregation rules for *total preorders*, and a *unified* joint characterization of all of them (see in particular Corollary 1, Propositions 2 and Proposition 3 above), as well as a specific characterization of generalized Condorcet-Kemeny rules for the case of *odd-dimensional* domains. Notice, however, that the notion of betweenness underlying the relevant notion of single-peakedness for 'preferences on preferences' that guarantees the strategy-proofness of such *exact* rules in the present paper is in fact a most 'natural' one, namely the median betweenness which is characteristic of their domain-base $\mathcal{R}_A^T$ (but is *not* well-defined on its subdomain $\mathcal{L}_A$).

The issue of strategy-proof aggregation in *arbitrary* (possibly infinite) join-semilattices is addressed in Bonifacio, Massó (2020) within a *fixed population* framework. To be sure, that work focuses in fact on so-called 'simple rules', namely anonymous and unanimity-respecting *social choice functions* with the *'tops-only'-property*[19]. But then, such 'simple rules' are essentially equivalent to *anonymous* and *idempotent* aggregation rules which are endowed with an *explicitly pre-defined domain of preference profiles of total preorders*. In particular, the Authors consider a restriction on total topped preference preorders they denote (join-)'*semilattice-single-peakedness*'[20] which results in a maximal domain that is consistent with the existence of strategy-proof 'simple rules'. Then, they proceed to characterize *the subclass* of anonymous and idempotent strategy-proof aggregation rules, establishing that they are precisely the *'supremum' rule* $f^{\vee}$ and a family of '*generalized quota-supremum*' rules[21]. It should also be noticed that such a comparatively

[19]A social choice function for (N, A) is a function $f : \mathcal{D}^N \rightarrow A$ where $\mathcal{D} \subseteq \mathcal{R}_A^T$: it satisfies the *tops-only property* if $f(R_N) = f(R'_N)$ whenever $t(R_i) = t(R'_i)$ for each $i \in N$, and $|t(R_i)| = |t(R'_i)| = 1$ for all $i \in N$ (with $t(R_i) := \{x \in A : xR_i y \text{ for all } y \in A\}$).

[20]The notion of semilattice-single-peakedness (SSP) for total preorders on a join-semilattice $(X, \leqslant)$ was first introduced in Chatterji, Massó (2018). A total preorder R on X is SSP in $(X, \leqslant)$ if and only if : (i) R has a unique maximum element x^* in X; (ii) yRz for each $y, z \in X$ such that $x^* \leqslant y \leqslant z$;

(iii) $(x^* \vee u)Ru$ for each $u \in X$ such that $x \nleqslant u$.

[21]The *'supremum'* (or *join n-projection)* rule $f^{\vee}$ for (N, X) is defined as follows: $f^{\vee}(x_N) := \vee_{i \in N} x_i$. A *generalized quota-supremum* rule returns a certain prefixed alternative x^* if x^* reaches a prespecified quota, and $\vee_{i \in N} x_i$ otherwise.

weak notion of semilattice-single-peakedness is admittedly consistent with the notion of single-peakedness induced by metric-betweenness according to the shortest-path-metric on the covering graph of the semilattice[22].

However, semilattice-single-peakedness is clearly bound to relinquish any connection not only to a median-induced betweenness if the relevant semilattice is *not* median, but also to the most *natural* rank-based metric betweenness if the semilattice also happens to be *not even graded*[23]. Therefore, in the latter case there is no natural metric to ground the claim that a certain type of single-peakedness describes a sort of 'preferences on preferences' that are induced in a 'natural' -hence plausibly shared- way by the *actual* basic preferences of agents.

It is also worth mentioning here that, in any case, strategy-proofness only concerns *strategic manipulation* of a preference-aggregation process, namely manipulation of the outcome of a certain game by means of an *appropriate choice of strategy* in the available strategy-set(s). In other terms, a *given game* is implicitly being taken for granted, including of course the *population* of its players and the set of its possible alternative outcomes, or its *agenda.* But then, *manipulation of the agenda* (or, for that matter, of the relevant population of players itself) can also be considered: notice, however, that from a game-theoretic perspective, that is a kind of *structural (as opposed to strategic) manipulation* since it amounts to a change of the *game itself.*

Such a broader perspective on manipulation issues in preference-aggregation is apparent (if mostly implicit) in Sato (2015)[24]. Indeed, Sato's contribution relies on a *fixed population* framework and is mainly focussed on *strict social welfare functions* as defined on some *connected*

[22]That is so because (if the join-semilattice $(X, \leqslant)$ is *discrete* i.e. it has no bounded infinite chain) for any pair of elements $x, y \in X$ which are *not* $\leqslant$-comparable the join $x \vee y$ must lie on a shortest path from x to y of the covering graph of the semilattice.

[23]Lattices (hence, of course, semilattices) which are not graded are quite common: in the present context, the lattice of *partial preorders* is perhaps the most obvious example (see e.g. Barbut, Monjardet (1970)).

[24]This is also, arguably, Arrow's own perspective on manipulation issues (see Arrow (1963)), except that he overtly renounces to address *strategic* manipulation issues, while acknowledging their substantial import (see e.g. Arrow (1963), chpt. 1). By contrast, *agenda-manipulation* issues play a key role in the arguments offered by Arrow to support his own proposal of the Independence of Irrelevant Alternatives (IIA) condition for social welfare functions (a more detailed discussion of the relationship of IIA to agenda manipulation will be provided elsewhere).

domain of linear orders over a finite set A[25]. However, it also considers the family of *social choice functions* which are induced by any such strict social welfare function on the subsets of A through maximization -at each preference profile- of the 'social' linear order selected at that profile (as restricted to the relevant subset of A). In that connection, *four* notions of *nonmanipulability* for strict social welfare functions are considered, with the primary aim to address issues of *strategic* manipulation[26]. Then, relying on a *renormalized and 'contracted'* version $\widehat{d}_K$ [27] of the *Kemeny distance* as defined previously, Sato introduces a 'continuity-type' condition for strict social welfare functions called *Bounded Response*. A strict social welfare function f satisfies Bounded Response if $\widehat{d}_K(f(R_N), f(R'_N)) \leq 1$ whenever two preference profiles R_N, R'_N are the same except for the preference of a single agent i, and R_i and R'_i are *adjacent* (i.e. R'_i is obtained from R_i by permuting the R_i-ranks of *a single* pair of alternatives with *consecutive* R_i-ranks)[28]. In a similar vein, a very mild *Adjacency-Restricted Monotonicity* condition for strict social welfare functions is considered, requiring that for any such pair of 'adjacent' profiles R_N, R'_N and any $x, y \in A$, if $[yR_ix$, xR'_iy and $xf(R_N)y]$ then $xf(R'_N)y$ as well. The main result established by Sato (2015) is *the equivalence* of the following statements for a strict social welfare function f on a connected domain of linear orders on A: (1) f satisfies Bounded Response and *at least one* of the four nonmanipulability conditions mentioned above; (2) f satisfies Bounded

[25]A *connected* domain of linear orders over A is a set $\mathcal{D} \subseteq \mathcal{L}_A$ such that for any $R, R' \in \mathcal{D}$ there exists a finite family $\{R_1, ..., R_k\} \subseteq \mathcal{D}$ such that (i) $R_1 = R$; (ii) $R_k = R'$; (iii) for every $i = 1, ..., k-1$, R_i and R_{i+1} can be mutually obtained by reversing the respective ranks of two adjacent (or consecutive) elements of A that are 'adjacent' (i.e. consecutive) according to the other. Thus, a (strict) social welfare function on a connected domain is a function $f : \mathcal{D}^N \longrightarrow \mathcal{L}_A$ (clearly, it is also *restricted* for $(N, \mathcal{L}_A)$ if $\mathcal{D} \neq \mathcal{L}_A$). Observe that $\mathcal{L}_A$ itself is of course a connected domain.

[26]One of them is akin to the notion of strategy-proofness for aggregation rules proposed by Bossert, Sprumont (2014) as discussed above, and another one relies on the 'renormalized' Kemeny distance for linear orders. By contrast, the last two nonmanipulability notions invoke the induced maximizing choices on A, and on its subsets, respectively (and are also most suitable to address certain agenda-manipulation issues).

[27]Namely, the 'halved' Kemeny distance for linear orders. That is essentially the distance between rankings due to Kendall, given by the minimal number of transpositions of adjacents elements that is necessary to obtain one linear order starting from another one (see e.g. Kendall (1955)).

[28]It is worth recalling here that 1 is the minimum positive value of both d_K and $\widehat{d}_K$.

Response and *each one* of the four nonmanipulability conditions mentioned above; (3) f satisfies Adjacency-Restricted Monotonicity and the Arrowian *Independence of Irrelevant Alternatives (IIA)* condition.

As a corollary of that result (and of arguments from standard proofs of the Arrowian 'impossibility' theorem for strict social welfare functions) a new characterization of *dictatorial* strict social welfare functions in terms of Bounded Response, one of the four equivalent nonmanipulability conditions mentioned above, and Sovereignty (or Ontoness)[29] is established. Furthermore, the set $\mathcal{D}^{sp(Q)}$of linear orders on A which are *single-peaked* with respect to some *fixed linear order* Q on A can also be shown to be a *connected* domain, and the strict social welfare function f^{wmaj}induced by the method of '(weak) majority decision'[30] clearly satisfies both Adjacency-Restricted Monotonicity and IIA. Hence, it immediately follows that $\widetilde{f}^{wmaj} : \mathcal{D}^{sp(Q)} \rightarrow \mathcal{L}_A$ is a (restricted) strict social welfare function which satisfies both Bounded Response and all of the four nonmanipulability conditions mentioned above (hence, in particular, the strategy-proofness properties implied by the first two conditions from that list).

Thus, at least when applied to *strict* social welfare functions, the combination of Bounded Response and standard nonmanipulability conditions (including, more specifically, strategy-proofness requirements) tends apparently to reproduce a well-known pattern. Namely, 'impossibility' theorems on the *full* domain of linear orders, and some 'possibility' results on suitably *restricted domains* of linear orders (to the effect that e.g. several versions of the simple majority rule provide well-defined and strategy-proof *restricted strict social welfare functions* on certain single-peaked domains of linear orders[31]).

It should be noticed that the co-majoritarian 'median' rules $\widehat{f}^{\partial maj}$, $f_r^{\partial maj}$ (or, equivalently, generalized Condorcet-Kemeny rules $f_{\preceq}^{CK}$, $f_{\preceq}^{CK^r}$), and the other strategy-proof rules characterized in Theorem 1 of the present paper, are in fact *well-defined on the full domain of linear orders* on A: however, only their strict versions induce *strict social welfare functions* because otherwise their *values* at certain profiles of linear orders on A need *not* be themselves linear orders on A. Furthermore,

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[29]A strict social welfare function f is *sovereign* if for any $L \in \mathcal{L}_A$ there exists $R_N \in \mathcal{L}_A^N$ such that $f(R_N) = L$.

[30]Namely, for any $R_N \in (\mathcal{L}_A)^N$, and $x, y \in A$, $x f^{wmaj}(R_N) y$ if and only if $|N_x(R_N)| \geq |N_y(R_N)|$.

[31]The significant body of literature devoted to the elaboration of such two related themes is extensively reviewed in the fourth chapter of Gaertner (2001).

the Kemeny distance is well-defined on their respective characteristic domain-components and codomains ($\mathcal{R}_A^T$ and $\mathcal{B}_A^r$, respectively).

Therefore, the definition of Bounded Response can be easily reformulated for aggregation rules $f : (\mathcal{R}_A^T)^N \longrightarrow \mathcal{R}_A^T$ and $f : (\mathcal{B}_A^r)^N \longrightarrow \mathcal{B}_A^r$ with respect to the Kemeny distance d_K (as opposed to its 'halved' version $\widehat{d_K}$), and the same holds -incidentally- for Adjacency-Restricted Monotonicity. But such an extended notion of Bounded Response would be clearly *distinct* from the original one proposed by Sato and considered above.

By contrast, the existence issue for strategy-proof social welfare functions as aggregation rules on the *full domain of total preorders or even larger sets of reflexive binary relations* on a finite set has never been addressed explicitly in previously published work, as mentioned above. The results of the present work imply that, at least for an *odd-sized* population of agents, even anonymous and neutral social welfare functions on the *full domain of total preference preorders* on a finite set *do exist*, and are indeed *strategy-proof* on suitably defined single-peaked domains of 'preferences on preferences' (i.e. arbitrary rich locally unimodal domains).

Arguably, such social welfare functions may also be regarded as a *positive* solution to a suitably reformulated version of the classic Arrowian preference aggregation problem. Namely, the focus is restricted to *strategic* as opposed to *structural* manipulation, and the Arrowian Independence condition IIA is accordingly replaced with a most 'natural' and milder independence requirement tightly related to the intrinsic order-theoretic structure of $\mathcal{R}_A^T$ [32]. In other words, we have here a *first* explicit escape route from Arrow's 'impossibility' theorem on preference aggregation, which relies on retention of the 'transitivity plus totality' format requirement for preference relations as combined

[32]Indeed, consider $\widehat{f}^{\partial maj}$ for (N, A) with $|N| = |A| = 3$, and profiles R_N, R'_N of linear orders with (under the usual permutation-based notation for linear orders, and square-bracket notation to denote indifference):

$R_1 = R'_1 = xyz$

$R_2 = R'_2 = yzx$

$R_3 = zxy$, $R'_3 = xzy$.

Note that R_3 and R'_3 are *adjacent*.

Nevertheless, as it is easily checked,

$\widehat{f}^{\partial maj}(R_N) = [xyz]$, while

$\widehat{f}^{\partial maj}(R'_N) = x[yz]$.

It is then immediately seen that the co-majority rule (which clearly satisfies $M_{\mathcal{X}}$-Independence with respect to $\mathcal{X} = (\mathcal{R}_A^T, \overline{\cup})$) does *not* satisfy IIA with respect to $\mathcal{X}$. In fact, $R_i | \{x, y\} = R'_i | \{x, y\}$ for every $i \in N$. Yet, $yf(R_N)x$ while *not* $yf(R'_N)x$.

with a considerable *weakening* of IIA that relies on the (semi)latticial structure of the set of total preorders[33]. That sort of weakening, however, requires in fact a *fixed agenda* setting. From a mechanism-design perspective, it amounts to a sort of *divide-and-conquer* approach to collective choice problems: the agenda-formation process and the underlying protocols (if any) are to be taken for granted, and thereby ignored[34].

Furthermore, our results on strategy-proof aggregation rules for arbitrary reflexive preference relations suggest a *second* escape route from Arrow's 'impossibility' theorem, which relies instead on retaining the *full force of IIA* to ensure both strategy-proofness and (arguably) immunity to some sort of agenda-manipulation, while renouncing *entirely* any substantial requirement on the structure of preference-information outputs *and* inputs.

Remarkably, if perhaps unsurprisingly, such escape routes from Arrow's 'impossibility' result involve aggregation rules that share an allowance for 'universal indifference', namely for a global stalemate as a possible output. Notice, however, that in both cases *constrained* aggregation rules which invariably provide *linear orders* as their outputs can be designed by augmenting them with a (pseudo-)random component as previously mentioned (see Note 11 above). Moreover, versions of

[33]Such a weakening is totally unrelated to other sorts of weakenings of IIA previously proposed in the literature including several versions of *Positionalist Independence,* as introduced and discussed by Hansson (1973) with no reference whatsoever to nonmanipulability issues. The strongest of them, labelled as Strong Positionalist Independence (SPI) by Hansson himself, requires invariance of aggregate preference between any two alternatives x, y for any pair of preference profiles whose restrictions to $\{x, y\}$ are identical *only if for every agent/voter the respective closed preference intervals having x and y as their extrema are also identical.* SPI has been recently rediscovered -and relabeled as Modified IIA- by Maskin (2020). Maskin motivates it in terms of resistance to certain sorts of 'vote splitting' effects, hence broadly speaking with reference to manipulation issues, including strategic manipulation. Notice, however, that what is at stake in that proposal is strategy-proofness of the '*maximizing' social choice function* induced by a certain social welfare function (as opposed to strategy-proofness of the social welfare function itself).

[34]That is not meant to imply that disentangling structural and strategic manipulation is always easy or indeed possible in actual practice. For instance, if alternative outcomes are candidates for an appointment or in a political election then strategic candidacy is virtually always possible. But strategic candidacy may be regarded precisely as a structural manipulation of the aggregation rule which is channelled through a forced change of *available* strategies. Anyway, it is worth noticing that under social choice functions which admit Condorcet winners at any profile of their domain *and select them,* no agent can benefit by giving up her own candidacy (see Dutta, Jackson, Le Breton (2001), Proposition 1).

each one of them have been repeatedly evoked in the previous social choice-theoretic literature[35], but have scarcely if ever been considered together. What is arguably novel here, in that respect, is that the foregoing classical 'escape routes' are explicitly connected to *both* strategic and/or structural manipulation issues, and given a most explicit formulation within a *common* general framework.

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