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Stability and determinants of the public debt-to-GDP ratio: a Stock-Flow Consistent investigation

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Abstract

This paper aims to outline the stability conditions and the determinants of the public debt-to-GDP ratio within a theoretical framework representing the main features of a monetary economy of production. To this end, we develop two macro – Stock Flow Consistent (SFC) models that, unlike traditional ones that are studied through simulations, are solved analytically. In detail, we firstly derive such conditions from a SFC model of a dynamic version of the traditional income-expenditure scheme with endogenous public debt service and only fiat money. Secondly, we extend the model to include investments and bank loans, thus considering both fiat and private money creation. Thereby, we develop an analytically solvable SFC model based on the Supermultiplier approach. Our main findings outline that:

i) The steady-state value of the public debt-to-GDP ratio is determined by the saving rate, the growth rate of primary public spending, the tax rate, the capital intensity of the production process and the interest rate. Given these values, there exists a "natural" level of the public debt-to-GDP ratio towards which the system converges in the long-run. In particular, the public debt-to-GDP ratio depends positively on the saving rate and negatively on the tax rate and growth rate of autonomous spending, while the interest rate has a non-linear effect. This result calls into question the idea of imposing exogenously given thresholds for targeting budgetary rules independently from the very specific features of each economic system;

ii) The necessary condition for the stability of the public debt-tp-GDP ratio is the absence of fiscal rules jointly to no full-hoarding of income from interest on public bonds. It becomes sufficient when one of the following is fulfilled: the growth rate of primary public expenditure or the interest rate or the propensity to consume out-of-wealth is higher than zero.

Finally, we highlight that permanent expansions in the level of public expenditure have only a transitory effect on the public debt-to-GDP ratio, in the long-run its value goes back to the level determined by the above-mentioned parameters. The only fiscal manoeuvres that Government has at its disposal to lower the ratio are: a persistent increase in the growth rate of public spending or an increase in the tax rate.

Keywords: Public debt-to-GDP ratio, Stability, Fiscal policy, Stock-Flow Consistent models.

JEL classification: E12, E17, E42, E43, E52, E62

1. Introduction

In the last decades, a long-standing discussion about the relationship between public debt and GDP growth has characterized the economic and policy debate. The common argument across mainstream literature is that high levels of public debt would hamper economic growth and austerity policies are pinpointed as the measures needed to guarantee the sustainability of public finances boosting contemporarily GDP growth.

In this regard, the debate has focused, among other things, upon the apparently unescapable recommendation to lower the public debt-to-GDP ratio. Such indicator has been extensively used as a benchmark for public debt sustainability and, as in the famous parameters of the 1992 Maastricht Treaty and the 2012 Fiscal Compact, it has become a specific objective for economic policy agenda. In this sense, the Eurozone austerity agenda has been implemented and justified in order to respect the thresholds fixed in such treaties.

Mainstream authors claim the existence of a non-linear relationship between the public debt-to-GDP ratio and GDP growth: above a certain threshold, the increase in public debt would depress economic growth (Reinhart and Rogoff, 2010a; Abbas, 2010; Cane et al., 2010; Cecchetti et al, 2011; Egert, 2015). However, as pointed out by the same authors, on the empirical ground there is no consensus about the value and significance of this threshold, as well as on the causal relationship (Panizza and Presbiero 2013; Gomez and Sosvilla-Rivero 2015). On the theoretical side, within the neoclassical approach, there is no clear justification for the existence of such a kind of correlation or non-linear relationship (Krugman, 2021). In general, a theoretical scheme outlining the factors determining the levels of public debt-to-GDP ratio is lacking, neither mechanisms linking the level of public debt-to-GDP ratio and GDP growth are detectable. Better to say, it is the public expenditure that, depressing private savings, can crowd out private investment and capital accumulation, but not the public debt-to-GDP ratio per se. That is, the negative effect would be still there although public spending is completely financed through fiscal revenues. Given such mechanisms, the common conclusion in mainstream theory to reduce the public debt-to-GDP ratio is that, once a primary balance budget is imposed, the interest rate has to be lower than the growth rate of the economy.

However, a growing literature on fiscal multipliers is pointing out that expansion in public spending can have a persistent and positive effect on GDP (Blanchard and Perotti, 2002; Galí et al., 2007; Ramey, 2011a; 2016; Auerbach and Gorodnichenko, 2012, 2017; Caldara and Kamps, 2017; Ramey and Zubairy, 2018; Fatás and Summers, 2018; Gechert et al.; 2019, Deleidi et al. 2020). This has relevant implications for the relationship between public spending, GDP growth and public debt sustainability. Indeed, the positive values of estimated multipliers (Gechert, 2015; Ramey, 2016; 2019) have led to questioning the idea that consolidation policies may reduce the public debt-to-GDP ratio (Fatás and Summers, 2018).

Within the post-Keynesian approach, several works based on the notion of “functional finance” (Lerner, 1943) have focused on the role of fiscal policies and Government deficit in stabilizing the economy and stock-flow ratios (Arestis and Sawyer, 2003; Fontana, 2009, Hein and Stockhammer 2010)[1].

In particular, the Stock-Flow Consistent (SFC) approach has been demonstrated to be particularly fruitful in highlighting the liability/asset relations across sectors forming the macroeconomy and the determinants of stock and flow ratios (see Caverzasi and Godin 2013; 2015; Nikiforos and Zezza, 2017 for a revision). This approach gives particular relevance to the monetary nature of savings characterizing capitalist economies and their ex-post determination with respect to the exogenous injection of purchasing power into the system, such as investments and public expenditure. The key features of the SFC approach can be summarized as follows (Fontana et al. 2020): (a) the consideration of both flow and stock variables and the related macro-accounting consistency constraints; (b) unlike real assets, financial assets held by an agent or sector have an accounting counterpart in the liability side of the balance sheets of other agents or sectors; (c) The explicit formalization of a sequential process where all agents/sectors have to realize their transition in monetary terms; (d) the use of dynamic macroeconomic models, where endogenous variables move forward non-ergodically in historical time; (e) each financial stock is associated with its own flow, meaning that the former is continuously fueled by (and, in turn, fuels) the latter. This is coherent with the ‘quadruple accounting’ principle, according to which any economic transaction requires at least four recorded entries for the accounting matrices to balance out (Copeland 1949; Godley and Lavoie 2007).

Along this approach, Lavoie and Godley (2007) have shown that, within a stock-flow consistent model of the traditional income-expenditure scheme, the stability of the public debt-to-GDP ratio is granted when the propensity to consume out-of-wealth is higher than zero. Ryo and Skott (2013) find that public debt stability can be ensured by active fiscal instruments or if the tax rate on interests accrued on public bonds is sufficiently high. Hein (2018), adopting a neo-Kaleckian approach, analyses the implication for the stability of the public debt-to-capital ratio in a model where the long-run growth rate is led by autonomous government expenditure. They find two stability conditions: i) both the propensity to consume out-of-wealth of rentiers and animal spirits have not to be too strong relative to the autonomous growth rate of public expenditure; ii) the interest rate has not to be higher than the autonomous growth rate of public spending.

In general, within the post-Keynesian approach, there is no necessary causal relationship that goes from the public debt-to-GDP ratio to the rate of growth. Conversely, this literature highlights the causal relation that goes from the growth rate of GDP to the public debt-to-GDP ratio pointing out the central role played by the autonomous component in determining the long-run growth rate. In particular, the persistent effect on GDP produced by expansion in autonomous components such as public spending has been extensively investigated within Supermultiplier literature (Girardi and Priboni, 2016; Freitas, 2020; Deleidi and Mazzuccato, 2019; Deleidi et al., 2020, Skott et al. 2020).

[1] The general post-Keynesian argument sustains that, in the short run, because the multiplier is higher than one, an increase in the public expenditure would reduce the public deb-to-GDP ratio. Anyway, this kind of static analysis, abstracting from the relationship between the intertemporal dynamics with which the multiplier is expressed and the accounting period of the public debt-to-GDP ratio, cannot be used to study long-run positions.

However, although the power of SFC approach in explaining the monetary nature of debt and savings, in contrast with the real one of savings, loans and debt characterizing the neoclassical approach, the factors explaining the long-run dynamic of the public debt-to-GDP ratio has been not investigated in much detail. In particular, a general depiction of the determinants and stability condition of the public debt-to-GDP ratio within the SFC approach is missing.

The aim of this paper is to fill this gap, deriving these conditions within a theoretical framework representing the main features of a monetary economy of production and where the long-run income is guided by the autonomous components of demand. We derive the analytical steady-state values of public debt- and private wealth-to-GDP ratios for both SFC multiplier and supermultiplier models. To our knowledge, a similar attempt has been pursued only in Godley and Lavoie (2007). However, this model does not consider capital accumulation, bank money, public debt service and interest rates on the stock of savings and debts.

In particular, we develop two different SFC models: (i) a dynamic model of the traditional income-expenditure scheme with only fiat money and endogenous public debt service; (ii) a dynamic model reproducing the long-run interaction between the multiplier and the accelerator with both bank and fiat money (Supermultiplier model). All the analytical results can be verified through model simulation, and vice versa using the R-code in Appendix D[2].

Our main results can be summarized as follows: i) the public debt-to-GDP ratio is determined by the saving rate, the growth rate of public spending, the tax rate, the capital intensity of the production process and the interest rate. Given these values, there exists a "natural" value of the public debt-to-GDP ratio ingrained in the economic system. In detail, the public debt-to-GDP ratio depends positively on the saving rate and negatively on the tax rate and growth rate of autonomous spending, while the interest rate has a non-linear effect. The imposition of any fiscal rule which is inconsistent with the "natural" level undermines the stability of the public debt-to-GDP ratio; ii) The necessary condition of stability is the absence of fiscal rules jointly to no hoarding of income from interest on government debt. This condition becomes sufficient when one of the following is fulfilled: the growth rate of primary public expenditure or the interest rate or the propensity to consume out-of-wealth is higher than zero; iii) Permanent expansions in the level of public expenditure have only a transitory effect on the public debt-to-GDP ratio, in the long-run its value goes back to the level determined by above-mentioned parameters. Only a permanent increase in the growth rate of primary public spending can reduce the long-run level of the public debt-to-GDP ratio.

The rest of the paper is organized as follows: Section 2 presents a theoretical overview comparing the different analyses coming from the General Equilibrium approach and Post-Keynesian theories on the nature of savings and loans. Section 3 analyses the stability conditions and the determinants of the public debt-to-GDP ratio within a dynamic stock-flow consistent model of the traditional income-expenditure scheme. Section 4 derives the stability conditions and the determinants of the public debt-to-GDP ratio within an

[2] R-codes to reproduce all the simulations and model results are available at https://github.com/LorenzoDiDomenico/Stability and determinants of the public debt-to-GDP ratio: a Stock-Flow Consistent investigation.. All the analytical derivations behind the results presented in the paper can be sent upon request.

analytically solvable SFC model based on the Supermultiplier approach. Section 5 and 6 analyze respectively the impact of expansionary fiscal policies and the introduction of fiscal rules. Section 7 concludes.

2. Theoretical overview

Within neoclassical literature, an “unprecise” high level of public debt is supposed to be a cause of low growth rates, both in the short- and long-run via several channels: high public debt can adversely affect capital accumulation and growth via higher long-term interest rates (Gale and Orzag, 2003; Baldacci and Kumar, 2010), higher future distortionary taxation (Barro, 1979; Dotsey, 1994), lower future public infrastructure spending (Aizenmann et al., 2007), higher inflation (Sargent and Wallace, 1981; Barro, 1995; Cochrane, 2011), and greater uncertainty about prospects and policies. In more extreme cases of a debt crisis, by triggering a banking or currency crisis, these effects can be magnified (Burnside et al., 2001; Hemming et al., 2003).

However, on the empirical ground of analysis, there is still no consensus on the significance and causal relations between these two variables and models provide ambiguous results. “Growth in Time of Debt” represents the reference contribution for this line of thought. In these articles, Reinhart and Rogoff (2010a, 2010b) assert that “whereas the link between growth and debt seems relatively weak at normal debt levels, median growth rates for countries with public debt over roughly 90 per cent of GDP are about one per cent lower than otherwise” (RR 2010a p. 573). This work has provided significant support for the austerity agenda that has been ascendant in Europe and USA since 2010. According to the authors, the two papers formed the basis for testimony before the Senate Budget Committee and represent the only evidence cited in the “Paul Ryan Budget” on the consequence of high public debt on economic growth.

Starting from this seminal contribution a large strand of literature has investigated this relation attempting to identify the possible non-linearities and the threshold beyond which public debt harms GDP growth. For instance, Cane et al. (2010) find a similar non-linear effect on growth above 77% of GDP, lower levels of public debt contribute to increase investment and get faster economic growth. In Cecchetti et al. (2011) the threshold is fixed at around 85% of GDP. Abbas and Chrisitnes (2010) using panel data of low-income and emerging countries describe a positive contribution to economic growth when domestic debt presents moderate levels, but when it represents more than 35 % of bank deposits it has a negative impact. Egert (2015) shows that the threshold can be lower than 90%, even between 20% and 60% of GDP depending on data frequency, time and country dimension and other assumptions. Baum et al (2013), studying the short-term impact of debt in 12 Euro area economies, detect a positive effect below 67% and a negative one above 95%. Minea and Parent (2002) find the same kind of non-linearity but with different thresholds.

Others authors tried to include the possibility of a reverse causality running from growth to debt. But also around this issue, there is currently no consensus. Ferreira (2009), analysing 20 OECD countries, founded a bidirectional causal relationship between public debt and growth. Pasunte-Avjovin and Sanso-Navarro (2015), using a panel bootstrap Granger causality test, show that government debt does not cause real GDP growth.

Gomez and Sosvilla-Rivero (2015) confirm that, considering the whole sample period 1980-2013, there is no negative causation between public debt and GDP growth. They find an inverse Granger-causality only from 2007 to 2009 above a debt threshold that goes from 56% to 103%. Ramos-Herrera and Sosvilla- Rivero, using the Word Bank's classification for income, initially indicate that the countries which present the lowest public debts are characterized by the highest economic growth. Nevertheless, this result change when they analyse the countries by income level. In this regard, they conclude that debt overhang effects cannot be related to a specific debt threshold and that the relationship between public debt and growth is complex.

Finally, as admitted by Panizza and Presbiero (2013), there is not a robust result because small changes in data or the econometric methods yield different results concerning the causal relationship. In this regard, also the econometric work of Reinhart and Rogoff (2010a and 2010b) has been proved to be misleading. Hendon et al. (2013), replicating their exercise founded coding errors, selective exclusion of available data, and unconventional weighting of summary statistics. They demonstrated that once properly calculated, the average real GDP growth rate for countries carrying a public debt-to-GDP ratio of over 90% is actually 2.2 per cent, not -0.1 per cent. That is, the average GDP growth at a public debt-to-GDP ratio over 90% is not different than when the debt-to-GDP ratio is lower.

On the theoretical side, it is tricky to find an explanation of the impact of the high level of public debt on GDP growth. In the Ramsey-Cass-Koopmans model (Ramsey, 1928; Cass, 1965; Koopmans, 1965) Ricardian equivalence holds and government bonds do not represent net wealth for households, and the public debt is neutral to long-run output[3]. In the Blanchard Overlapping Generation model (1985), instead, the public debt slightly crowds out physical capital and reduces long-run output. In this model the structure of the economy is the same as in the RCK model, the only difference is the finite horizon of households. Anyway, as estimated by Dombi and Dadak, in this model a percentage point change in the public debt-to-GDP ratio reduces the steady-state per-capita output only by 0.008-0.032 per cent (Dombi and Dedak, 2019). Thus, also in this model, fiscal policies result be neutral. As demonstrated by Evans (1991) these insignificant results depend on the fact that the Ricardian equivalence is still a good approximation of the model.

Laxton and Symansky (1997) and Faruqee (2003) show that once some relevant life-cycle aspects of household behaviour are included in the model the "burden" of public debt can be considerable in the Blanchard model as well[4]. However, the value of the output-loss with respect to an increase in the public debt-to-GDP ratio is very sensitive to the parameter calibration: a decrease in the intertemporal elasticity of substitution brings the impact of public debt on GDP growth to zero (Faruqee and Laxton, 2000).

In general, to have a robust and significant crowding-out effect in the neoclassical model, it is needed that the saving ratio is constant and exogenous, as in the Solow model (Solow, 1956). This is particularly valid in the endogenous growth model (Mankiw et al., 1992), where the saving ratio affects also the long-run growth

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[3] Barro (1974) shows that in a neoclassical word, if intergenerational links prevail, government bonds do not represent net wealth for the households and the Ricardian equivalence holds.

[4] Laxton and Symansky (1997) introduce wages following a hump-shaped life-cycle pattern, while Faruque (2003) includes death probability which increases with the age.

of technological progress. Dedak and Dombi (2018) show that in a human capital augmented Solow model a one percentage point increase in the debt-to-GDP ratio reduces the long-run output by 0.167 per cent[5].

Ultimately, in neoclassical models, the possibility of having a harmful effect of public debt on GDP depends on whether the Ricardian equivalence holds or not. The absence of intergenerational linkages or an exogenous saving rate implies that the Ricardian equivalence does not hold and an increase in public expenditure financed by deficit or taxes causes a decrease in the average saving ratio.

The general idea is that given the full-employment output automatically reached by free market forces, the higher the level of public expenditure, the higher will be the share of total output that will be consumed and not saved[6]. Because savings determine the path of capital accumulation, an increase in public expenditure depresses the long-run GDP growth. If the Ricardian equivalence holds (as in the RCK model with an infinite horizon), the negative effect on aggregate savings caused by an increase in public expenditure is offset by an increase in the saving rate of households, thus the long-run accumulation process is "preserved". Conversely in a model where the intergenerational link is absent, the future tax burden of the present deficit financing will fall to some extent on new generations, thus households do not modify their consumption plan[7]. For the same reason, in models where the household saving rate is exogenous, an increase in public debt affect negatively the aggregate saving ratio.

Anyway, it is worth noticing that in these models the impact of the public debt on GDP growth can be assessed only in terms of variations or via exercises in comparative statistics. In this sense, there is no way to determine a certain threshold beyond which the public debt-to-GDP ratio is harmful to economic growth.

As also argued by Krugman (2021), there is no theoretical reason in neoclassical models for having the non-linearity depicted by some empirical-mainstream works: "[…] There isn't a growth cliff at a debt ratio of 90 per cent. Nothing in standard macro suggested either of these things should be true. All that supported them was bad statistical analysis". In this respect, the common assertion "*high-level of public debt*" appears to have no sound fundamentals, and the adjective "high" is lacking of any kind of quantification methodology.

In this regard, it is important to clarify that in the above-mentioned models, it is not the level of public debt per se which is detrimental to capital accumulation, but it is public expenditure. Indeed the same mechanisms would apply in the case public expenditure is completely financed by taxes (thus, without any changes in the accumulation of public debt). In this context, the public debt-to-GDP ratio is just a reflex of the share of public expenditure on total output. Along this line, according to the well-known crowding-out effect, an increase in public spending, reducing the amount of loanable funds, would cause an increase in interest rates and (therefore) a reduction in private investments. Thus, the only detectable connection between the public debt-to-GDP ratio and economic growth should pass through the negative relationship between public spending and

[5]This effect strongly depends on the value of the saving rate and the growth rate of population. The effect falls with an increase in the saving rate.

[6] These models assume that the output absorbed by public sector is totally consumed (no public investments). In Ausher (2000), debt is used to finance productive public capital, in this case public debt has a positive impact on the growth rate.

[7] In the Blanchard model, the effect is negligible because the endogenous change in the saving rate in response to public expenditure is still operational although without a total displacement.

accumulation assuming that the public deficit would be financed by issuing public bonds. However, also approaching this reasoning, within the theory, the explanation for a non-linear relationship is still missing.

Within neoclassical literature, there are other attempts to connect the level of public debt and economic growth through interest rate variations, and without passing through the learnable found theory. For instance, Alesina and Perotti (1997) and Alesina and Ardagna (2010) argue that a reduction in public spending would produce a reduction in interest rates also through a change in expectations. A credible decrease in public spending would lead to a decrease in the expected debt-to-GDP ratio and will thus lower the probability of sovereign default. A reduced risk premium will be required and, in turn, lower long-term interest rates and corresponding higher asset prices will emerge (Ardagna, 2004). Consequently, both the wealth effect, generated by an increase in the asset price and the corresponding decrease in the interest rates would stimulate private investments.

In short, it seems to be absent any theoretical argument for the neoclassical research branch to empirically look for a threshold for the public debt-to-GDP ratio above which the public debt is detrimental to economic growth. Consistently with such a theoretical approach, the empirical analysis should be limited only to the impact of public expenditure on accumulation.

However, although there is neither empirical nor theoretical justification even within the neoclassical strand, the idea that high levels of public debt-to-GDP ratios are detrimental to economic growth is widely held among institutions, economists and policymakers. In this regard, a common recommendation to ensure sustained long-run growth is to stabilize or reduce the public debt-to-GDP ratio imposing public budget constraints. At this point, if a primary balance budget is imposed, the stability of the public debt-to-GDP ratio is granted if the output growth rate is equal to the interest (Diamond, 1965; Giavazzi and Pagano; 1990; Barrett, 2018; Mehrotra and Sergeyes, 2020; Blanchard 2019). This result comes from simple arithmetic of the public debt-to-GDP ratio:

$$d_t = \frac{1 + r_t}{1 + g_t} d_{t-1} + x_t \tag{1}$$

Where x_t is the primary deficit-to-GDP ratio, d_t is the public debt-to-GDP ratio, r_t is the interest rate on public bonds and g_t is the nominal GDP growth rate. Following this arithmetic, by imposing d_t equal to d_{t-1}, the condition that stabilizes the debt-to-GDP ratio can be derived:

$$x = \left(\frac{g - r}{1 + g}\right) d \approx (g - r)d \tag{2}$$

If $x_t = 0$, the growth rate of public debt is equal to the interest rate and the debt-to-GDP ratio has an explosive trend if $r > g$. Thereby, the mainstream policy recommendation to stabilize or reduce d^* is that imposing a primary balance budget constraint, the growth rate of the economy has to be equal to or higher than the interest rate (see Aspromourgos et al. 2009 for a review of the literature on the subject).

In general, this interpretation of the arithmetic of the public debt-to-GDP ratio derives from the hypothesis of an absence of interdependence between primary deficit (thus public spending), debt service and GDP growth. This is conceivable in a framework where real output and real savings are independent of public spending and deficit and where, under flexible prices and wages, the market reaches the full-employment output. In this case expansions of public expenditure cannot modify the output and public deficit, least of all, the private wealth. Supply factors determine the amount of output and private wealth.

Our analysis wants to outline how the relationships between GDP growth and public debt-to-GDP ratio and interpretation of the arithmetics on public debt-to-GDP stabilization changes when we move from a neoclassical world to a framework that takes rigorously into consideration the monetary nature of the production process (Keynes, 1933; Dillard, 1980; Lavoie, 1984; Hahn, 1989; Wray, 1999; Graziani, 2003; Lucarelli and Passarella, 2012; Fontana and Realfonso, 2005; 2017).

Through this differentiation, we wish to emphasize the differences that arise from a different understanding of the nature of savings, in real terms on the one hand and monetary terms on the other. In a monetary economy of production, monetary savings can exit only ex-post with respect to public spending and investment decisions, symmetrically to the bookkeeping of the aggregate debt generated when financing such components. In this context, as the principle of effective demand determines output, the stability conditions cannot be determined without considering the co-movement between interest rate, public spending and primary deficit, through the variation of output and fiscal revenues.

To his extent, we will show that, in absence of public budget constraints, the stability condition is fulfilled by an endogenous adjustment of output growth rate and primary deficit/surplus so that the primary deficit becomes equal to $x = \left(\frac{g-r}{1+g}\right) d.$ This implies that the condition of $r > g$ jointly with a primary balance budget does not have *per se* any relevance for the determination of the public debt-to-GDP ratio in the long term. Indeed, g represents an adjusting variable and cannot be taken as independent with respect to the assumption on the primary public deficit. Conversely, as it will be shown, the imposition of the public budget constraint that is not consistent with the natural public debt-to-GDP ratio of the economy can produce further expansions in the public debt-to-GDP ratio or even explosive trajectories.

2.1 Monetary and real savings: implications for the concept of debt

The shift in the formalization of contemporaneous economies as *contractual-barter* economies, typically portrayed by Walrasian or General Equilibrium (GE) approaches (Walras, 1874; Mckenzie, 1989; Malinvaud, 1953; Arrow and Debreu, 1959; Bliss, 1975; Shelvon and Whalley, 1992), to a monetary economy of production (Keynes, 1933; Graziani, 2003; Fontana and Realfonso, 2003) drastically twists the genesis of savings and debts. Along these lines, our analysis is grounded on the understanding that the existence of savings in capitalist economies is closely linked to the monetary sphere, giving a core relevance to the interdependence between monetary savings, money and debt creation. Such feature results central in this work to fully capture the macro-relationships between the financing of autonomous components, investments, public debt and private savings.

In neoclassic theory, savings are intended as the share of real output that has not been consumed and they directly correspond to the output produced by firms, while they do not need any ex-ante debt creation to exist. GE positions exist without any necessity to introduce debt or money creation. In this context, following the typical mechanism determining the equilibrium prices and quantities of GE, the "agreement" between workers and capitalists on how much to produce and its real term distribution is sufficient to start the production and to generate real output, real savings or the volume of capital goods. Agents exchange their endowments (labor and capital) with firms and receive back the output in real terms. Firms just act as intermediate boxes transforming inputs into outputs and not a genuine process of sales is operating.

Such real process of exchange ideally follows in the meta-time the matching of demand and supply curves (Gallegati et al. 2012). Based on their preferences, agents construct their supply curve according to the quantity of real output they can obtain by supplying a certain amount of factor. At the same time, based on marginal productivity, firms set factor demand. The tatonnement process makes all agents' plans mutually consistent (how much they will exchange with firms) before the production process starts and input-output exchanges are intended as simultaneously carried out in real terms (Gaffeo et al. 2007).

Money is not strictly required to perform such process of production and exchange, it can be attached secondarily through a money market that works as the market of real goods (as in the IS-LM model). As exposed by Hahn (1973b), models based on Walrasian, Arrow-Debreu or time-0 auction have no role for money, credit, liquidity, banks or central bank. Walrasian auctioneer reduces the model to a perfect barter and the Arrow Debreu extension to a world of complete markets with Arrow securities eliminates uncertainty, replacing *the facto* all the functions of money. In a centralized economy, characterized by the instantaneous definition of intertemporal quantities exchanged in equilibrium, the representative firm does not act with any temporal discontinuity between the phases in which it *"takes"* and *"gives"* to the other agents. Theoretical exchanges of all quantities are collapsed in one single instant and exchanges can be settled in real terms. The Walrasian auction happens outside of real-time. All offers to buy and sell are nullified until the auctioneer has found the equilibrium price vector. That is, real-time does not begin, and trade does not take place, until the Walrasian auctioneer has solved the system of optimal conditions.

Unsurprisingly, the natural role of banks could be exclusively that of intermediaries between the real savings of agents. According to the *loanable found theory (*Ohlin et al. 1937), current investment possibilities rely on the ex-ante generated amount of real savings, while the interest rate equalizes demand and supply[8]. The core assumption in GE models is an all-purpose good (APG), which can be used interchangeably as a consumption good, an investment good, and a financial asset (see Mankiw, 2019, p. 520). With the APG as

[8] It is worth noticing that the dissociation between saving and investment decisions is theoretically possible only assuming that the output is homogeneous and can take contemporaneously the form of consumption and capital (corn economy). Otherwise, households should somehow decide to receive their income in terms of capital goods and, among other problematic mechanisms, could not be possible to not consider a saving decision as already incorporated in the existence of the investment goods.

the only financial asset, the saving of households provides the supply of “funds” on the capital market. If the households decide not to consume the APG, they make it available as a financial asset that can be supplied on the financial market and used as an investment good (Bofinger, 2019).

According to this thesis, money does not influence the phenomenon of credit, agents lend and borrow in real terms the share of non-consumed output. Economic phenomena such as savings can be defined independently from money. Because in real economies, credit comes in monetary terms, monetary values can be obtained simply by multiplying the volume of real variables by a monetary price.

In this respect, although after the financial crises some aspects of the financial sector have been incorporated into second- and later-generation DSGE models (Benes at al. 2014; Gertler and Kiyotaki, 2015; Christiano et al, 2015), they remain (fundamentally) barter economies where inputs and outputs are simultaneously exchanged in real terms and the introduction of money is not required by construction (Borio and Disyatat, 2010; Rogers, 2018). Conversely, as warned by Hahn’s problem (Hann, 1965), the attachment of money to the moneyless perfect barter GE brings paradoxical results converting money and monetary institutions into frictions in the model, contra the principle that monetary exchange is more efficient than barter (Rogers, 2019).

Against this vision, post-Keynesian approaches typically propose to analyze a market-based economy as a monetary economy of production where money is regarded as the fuel (and not merely the lubricant) of the economic engine (Passarella and Sawyer, 2015). According to this approach, the formalization of a contemporary monetary economy of production has to take explicitly into account that agents have to use money to acquire inputs, consumption goods, inputs, pay wages and distribute dividends (Fontana and Realfonso 2005; Ronchon and Seccareccia, 2013). Since firms' costs and debt are in monetary terms and capitalists need money to buy consumption goods, firms also have to realize sales and profits in monetary terms. All these steps have to be explicitly modelled by adopting a sequential scheme (Graiziani, 2003). Any lending/borrowing process is not based on scarce or real resources that have been previously produced. Banks are not simply financial intermediaries but they are creating liquidity ex nihilo and injecting money into the economy when financing firms’ production decisions. Credit and money connect the intertemporal dynamic of production and sales: firms anticipate production costs expecting to recover a higher amount of money through future revenues. Since firms have to realize profits and sales in monetary terms, any exogenous injection of purchasing power (or autonomous component of demand) affects the level of sales and production through the multiplier effect.

In particular, the exogenous injections of purchasing power into the system trigger the dynamic of production over time while monetary savings are determined ex-post with respect to spending decisions (investments and autonomous component of demand). Money is endogenous, it enters into circulation when autonomous components of demand or production decisions are financed, generating a corresponding variation in aggregate debt. In particular, as clearly emerge from SFC macro-accountability which represents the work-house of such approach, aggregate savings are equal to aggregate debt so much so that the former could not exist without the latter. Thereby, financial wealth accumulation (or saving accumulation) is symmetrical to the accumulation of public and private debt. In this approach, savings are a monetary phenomenon since they cannot be defined independently from money (Bertocco and Kalajzić, 2022).

Conversely to a real-exchange economy, in this framework, there is no pre-agreement among agents about the level of production and its future distribution among participants in the production process. No market clearing mechanism which revises prices to equalize demand and supply is operational. Money demand cannot be derived based on agents' necessity to exchange the predetermined equilibrium quantities, nor the stock of money can be determined by an external authority that fixes the supply. On the contrary, the stock of money is endogenous, it enters into circulation within the process of production through autonomous spending and investment decisions, and it is residual with respect to households' savings and portfolio decisions.

In a nutshell, the shift from a contractual-real exchange economy to a monetary economy implies that, while in the former investments cannot exist without any ex-ante saving formation, in the latter monetary savings cannot exist without any ex-ante investment decision (or any exogenous injection of purchasing power financed by debt creation) as well as deposits cannot exist without any ex-ante loan creation.

For these reasons, neoclassical arguments descending from the theory of loanable funds (I.e. an increase in the public deficit would produce an increase in interest rate through an increase in the demand for money) results be theoretically deducible only in a framework depicting a *contractual-barter economy* where each individual can produce autonomously, or in agreement with other agents endowed with different factors, a real commodity, directly consuming a part of it and saving the rest. Thereby, everyone can independently create real savings without anything else happening in other parts of the economic system as a whole.

However, the transposition of this logic to the case of a monetary economy raises several criticalities. Here, not only do firms or individuals not produce money to be saved or 'consumed', but for a sector to hold positive net wealth, one or more sectors must have negative wealth. In this sense, in a closed monetary economy, the private sector can hold positive net savings only in the presence of a symmetric stock of public debt. This follows, indirectly, from the fact that money creation and its injection are accounted as debt creation (be it bank money or fiat money).

The contrasting proprieties descending from the framework adopted to represent a market-based economy lead to opposite visions about the nature of monetary stock and flows of the economy and mechanisms affecting macroeconomic variables such as the public debt-to-GDP ratios. The main issue arising from representing a market-based economy as a monetary economy of production is that private savings can determined only ex-post to public spending and investments, and can exists only through a correspondant generation of public and private debt.

In this paper, we adopt the SFC approach to formalize the main features of a monetary economy of production and the interrelationship between the financial and real sides of the economy. Thhoutline the determinants of the public debt-to-GDP ratio.

3. Determinants and stability of the Public debt-to-GDP ratio: the SFC - income/expenditure model

In this section, we analyse the determinants and stability conditions of the public debt-to-GDP ratio by developing a SFC model that considers only public expenditure as an exogenous component of demand (or exogenous injections of purchasing power). Thus, only fiat money is considered in the model, public spending is originally financed by CB overdraft which acts as the lender of last resort. Thereby, we develop a dynamic version of the income-expenditure model including endogenous public debt service and propensity to consume out-of-wealth higher than zero.

In detail, we perform our analysis assuming no fiscal rules and considering that, given an exogenous pattern of directly public expenditure, Government pays the endogenously determined public debt service. The following system represents the Stock-Flow Consistent model of the income-expenditure scheme in dynamic terms:

$$\begin{cases} Y_t = C_t + \bar{G} + rB_{t-1}^h \\ C_t = \left[Y_{t-1}^P + rB_{t-2}^h(1-\alpha)\right](1-\theta)c_1 + S_{t-1}c_2 \\ B_t = B_{t-1} + \bar{G} - \theta Y_t + B_{t-1}^h r \\ S_t = S_{t-1}(1-c_2) + Y_{t-1}(1-\theta)(1-c_1) \\ B_t^h = \beta_B S_t \\ M_t = S_t - B_t^h \\ B_t^{CB} = B_t - B_t^h \\ H_t = M_t \\ Y_t^P = C_t + \bar{G} \end{cases} \quad (3)$$

where B_t is the public debt, S_t is the stock of household savings, B_t^h and B_t^{CB} are, respectively, the stock of public bonds held by households and CB, M_t is the amount of deposits, H_t is the stock of reserves held by commercial bank at CB, r is the interest rate on public debt, θ is the average tax rate, c_1 is the propensity to consume out-of-income, c_2 is the propensity to consume out-ot-wealth, α is the share of income generated by the interest accrued on public bonds and hoarded by households, $\bar{G}$ is the primary public expenditure (it is constant over time), Y_t is the gross income including financial yelds and Y_t^P is the gross income from production. We are assuming that the interest rate on deposits is zero.

Production depends on consumption demand and public spending, consumption depends on the disposable income distributed at the end of the previous period and the stock of wealth. Each period represents one production cycle. Public expenditure sets in motion the economic system, it represents the exogenous injection of purchasing power realized in each period that triggers the income-expenditure sequence realized over the following periods. I.e, in the first period, Government demands a certain amount of goods from the production sector, which satisfies this demand and gets paid by the Government. At the end of the period, the production sector will distribute the correspondent income to households, this amount will constitute the base for consumption in the next period. In the second period, the demand will correspond to public spending plus the consumption financed by the income distributed at the end of the previous period etc. As a consequence, the long-run income is the result of the overlaps of income-expenditure sequences triggered by the public expenditures that have been realized in each period. Each income-expenditure sequence corresponds to the *ping-pong* realized between the production sector and households over the periods through income payment, induced consumption and firms' revenues.

The Government initially finance its spending through CB overdraft, while its net debt with CB will be determined by the combination of fiscal revenues generated by such spending, fiscal multipliers induced by previous spending and the public bonds demand of households. Specularly, the stock of households deposits corresponds to the stock of public bonds held by CB. We are assuming that commercial bank do not buy public bonds, while it keeps the deposits as reserves at the CB.Tables 1a and 1b report respectively the balance sheet and transaction matrix of the economy.

Assets	Households	C-Sector	Government	Bank	BC	$\sum$
Check deposits[9]	$+M_c$			$-M_c$		0
Time deposits	$+M$			$-M$		
HPM	0			$+H$	$-H$	0
Public bonds	$+B_h$		$-B$		$+B_{cb}$	0
Net wealth	$-V_h$		$+V_b$			0
$\sum$	0	0	0	0	0	0

Table 1a: Balance sheet

[9] The distinction between "check" and "time" deposits is necessary for the sequential approach: income distributed at the end of the period (from which the consumption demand in the following period is generated) appears in the form of "check" deposits, while "time" deposits are the share of income saved and held in the form of deposits. Unlike check deposits, time deposits accrue interests in each period.

	Households	C-Sector	Government	Bank		Central Bank		$\sum$
				Current	Capital	Current	Capital	
Consumption	$-C$	$+C$	0					0
Gov. spending	0	$+G$	$-G$					0
Income	$+Y$	$-Y$						0
Taxes	$-T$		$+T$					0
Profit CB			$+f_{CB}$			$-f_{CB}$		0
Int. Bond	$+r_b B_{t-1}^h$		$-r_b B_{t-1}$			$+r_b B_{t-1}^{cb}$		0
Int. deposits	$+r_m M_{t-1}$			$-r_m M_{t-1}$				
Int. reserves				$+r_m H_{t-1}$		$-r_m H_{t-1}$		
Δ Check deposits	$-\Delta M^c$				$+\Delta M^c$			0
Δ Time deposits	$-\Delta M$				$+\Delta M$			0
Δ Bonds	$-\Delta B_h$		$+\Delta B$				$-\Delta B_{bc}$	0
Δ H					$-\Delta H$		$+\Delta H$	0
$\sum$	0	0	0	0	0	0	0	0

Table 1b: Transaction matrix

The system can be rewritten in a reduced form of a system of second-order difference equations with three dynamic variables and three equations (and considering no hoarding on public bonds interests: $\alpha = 0$) :

$$\begin{cases} B_t = B_{t-1} + \bar{G} - \theta Y_t + S_{t-1} r \\ Y_t = Y_{t-1}\, c_1\, (1-\theta) + S_{t-1}(c_2 + r) + \bar{G} \\ S_t = S_{t-1}(1-c_2) + Y_{t-1}(1-\theta)(1-c_1) \end{cases} \tag{4}$$

The redundant equation is:

$$S_t = B_t - Y_t(1-\theta) = B_t - M_t^c, \tag{5}$$

$$M_t^c = B_t^{CB}. \tag{5.1}$$

For the sake of simplicity, we are considering that households hold all their savings in the form of public bonds. However, as shown in Appendix A, in a framework without private money and equities, the interest rate on the amount of deposits is indirectly paid by the Government. As a consequence, the actual interest rate paid on public debt results to be a weighted average of the interest rate on deposits and public bonds. The weight depends on the share of household wealth that is held in terms of public bonds. Thus, such an assumption can be easily relaxed[10].

Equation (4.1) states the accountancy of the public debt in each period. If households hold all their wealth in terms of public bonds, the actual interest paid by Government on its debt is equal to the interest accrued by the stock of household savings. The share of interests paid to the CB is only a clearing entry since it comes back through the distribution of profits to the Government. Equation (4.2) expresses the dynamic of income and production: in each period the demand for goods depends on the net income distributed to households at the end of the last period and the stock of wealth plus public spending. The income in each period depends on the production (that is equal to demand) and the interest accrued on the stock of wealth. Equation (4.3) describes the dynamic of the stock of savings in each period. Each period corresponds to a production cycle, while incomes are paid ex-post, at the end of the period. The change in savings corresponds to the difference between the disposable income and the consumption of the period. The redundant equation expresses the relationship between assets and liabilities of the economy. Because we are considering a closed economy without investments and bank loans, and public spending represents the only exogenous injection of purchasing power in the system, private savings equals public debt which is the only financial asset/liability of the economy.

However, although households hold all their wealth in the form of public bonds, there is always a frictional amount of public bonds held by CB that is equal to check deposits (Appendix B reports a simulation explaining the reason behind this). For the same reason, the public debt is slightly higher than the stock of saving and it differs by current disposal income. Since income is paid at the end of the period, the income flow generated by public spending through a correspondent amount of deficit figure as savings, it appears as check deposits (M_t^c). As the redundant equation states, the sum of check deposits and stock of savings equals the public debt, while check deposits are equal to the amount of public bonds held by CB. Unlike time deposits or public bonds, check deposits do not accrue interests.

Starting from the system (4), we study the endogenously determined long-run growth rate of stock and flow variables to analyse the stability conditions and the determinant of the public debt-ot-GDP ratio (all analytical results can be reproduced and verified through the R codes in Appendix B).

Rewriting equation (4.2), we can get the stock of savings in function of the levels of income:

$$S_{t-1} = \frac{Y_t - Y_{t-1}\, c_1\, (1-\theta) - \bar{G}}{c_2 + r}\ , \tag{6}$$

and substituting equation (6) in equation (4.3), after some manipulations, we get the second-order difference equation which describes the intertemporal dynamic of income:

$$Y_{t+1} - Y_t\, [c_1\, (1-\theta) + 1 - c_2] - Y_{t-1}(1-\theta)[c_2 + r(1-c_1) - c_1] = \bar{G}c_2. \tag{7}$$

[10] Equivalently, system (4) can be interpreted as a system where the interest rate on deposits and reserves is equal to that on public bonds.

The solution is:

$$Y_t = \frac{2^{-1-t}\bar{G}[\mathrm{c}_2\mathrm{a}(-2^{1+t} + (b-a)^t + (b+a)^t) - d((b-a)^t - (b+a)^t)]}{(-\theta c_2 + (-1+c_1)(-1+\theta)r)a}\ , \tag{8}$$

where:

$$a = \sqrt{\left(c_2 - 1 - c_1(1-\theta)\right)^2 + 4(1-\theta)\left(c_2 - c_1 + r(1-c_1)\right)}, \tag{9}$$

$$b = 1 + c_1(1-\theta) - c_2, \tag{10}$$

$$d = c_2(1 + c_1(-1+\theta) - 2\theta + c_2) + 2(-1+c_1)(-1+\theta)r, \tag{11}$$

Given the value of public spending over time, equation (8) describes the level of income in each period. Taking the limit of (8) for t that goes to infinity we can study the long-run convergence of the growth rate of GDP:

$$\lim_{t\to\infty} \frac{2^{-1-t}\bar{G}[\mathrm{c}_2\ \mathrm{a}\ (-2^{1+t} + (b-a)^t + (b+a)^t) - d((b-a)^t - (b+a)^t)]}{(-\theta c_2 + (-1+c_1)(-1+\theta)r)a},$$

$$\begin{cases} 0 < \log(b+a) > \log(2) \\ \log(b+a) < \log(b-a) \\ 0 < \log(b-a) > \log(2) \end{cases} \rightarrow \begin{cases} b+a > 0 \\ b+a > 1 \\ b+a < 2 \\ b+a < C \\ b-a < 2 \\ b-a > 0 \\ b-a > 1 \end{cases} \tag{12}$$

Depending on the value of c_1, c_2, θ and r, two different regimes emerge: stationary state with zero-growth and a steady-growth regime determined by the growth rate of public debt service. Since $0 < c_1, c_2 r < 1$, conditions 12.1, 12.2, 12.4, 12.5, 12.6, 12.7 are always verified, thus condition 12.3 determines the two regimes:

$$c_1(1-\theta) - c_2 + \sqrt{(-1 + c_1(\theta-1) + c_2)^2 + 4(\theta-1)(c_1 - c_2 + (-1+c_1)r)} < 1. \tag{13}$$

If the previous condition is fulfilled, the long-run growth rate is zero and income converges to a stationary value:

$$g_Y \approx 0$$

$$Y^* = \lim_{t\to\infty} Y_t = \frac{\mathrm{c}_2\bar{G}}{\theta(c_2 + r - c_1 r) - r(1-c_1)} \tag{14}$$

Conversely, if $c_1(1-\theta) - c_2 + \sqrt{(-1 + c_1(\theta-1) + c_2)^2 + 4(\theta-1)(c_1 - c_2 + (-1+c_1)r)} \geq 1$, the interest-led regime emerges. That is, for low values of the propensities to consume (c_1 and c_2) and/or high interest rate, the steady-state growth rate of the economy is positive and converges to the following value:

$$g^{*} = g_{Y}^{*} = \lim_{t\to\infty}\frac{Y_t}{Y_{t-1}} - 1 = \frac{2[\mathrm{c}_2\mathrm{a}(-2^{1+t} + (b-a)^t + (b+a)^t) - d((b-a)^t - (b+a)^t)]}{\mathrm{c}_2\mathrm{a}(-2^t + (b-a)^{t-1} + (b+a)^{t-1}) - d((b-a)^{t-1} - (b+a)^{t-1})} - 1 = \frac{a+b}{2} - 1 =$$

$$\frac{\sqrt{\left(c_2 - 1 - c_1(1-\theta)\right)^2 + 4(\theta - 1)\left(c_1 - c_2 - r(1 - c_1)\right)} + 1 + c_1(1-\theta) - c_2}{2} - 1 \qquad (15)$$

In the first regime, the long-run growth rate of public debt service is zero, and in the second one is positive. The financing of the directly public spending ($\bar{G}$) necessary implies a variation in the stock of debt and, thus, an impact on the public debt service. Latter depends on the path of private saving accumulation: if the combination of parameters is such that the accumulation of savings proceeds in the long-run, the growth rate of public debt service is positive.

Once the public debt service needed to maintain the exogenously determined level of primary public expenditure is included in the model, another source of autonomous expenditure comes into play. The public debt service figures as a semi-autonomous component of demand. Namely, it is a component of demand which is independent of current income but whose pattern is endogenous. Differently from primary public expenditure (which can be assumed as exogenous), this component is endogenously determined by the saving rate, the interest rate and the tax rate (the saving and the tax rate determines the public deficit). Depending on the value of these parameters, though the interest rate accrued on savings, the economy can present a positive growth rate ultimately due to a positive growth rate of the stock of public debt and public debt service.

If the growth rate of primary public expenditure is positive (g_G), the GDP growth rate will converge to the highest values between the endogenously determined growth of public debt service $g_{G_B}^{*}$ and g_G. Figure 1 summarizes graphically the relations between the growth rate of income, the interest rate, and the propensity to consume out-of-income (c_1) and out-of-wealth (c_2) when $g_G = 0$.

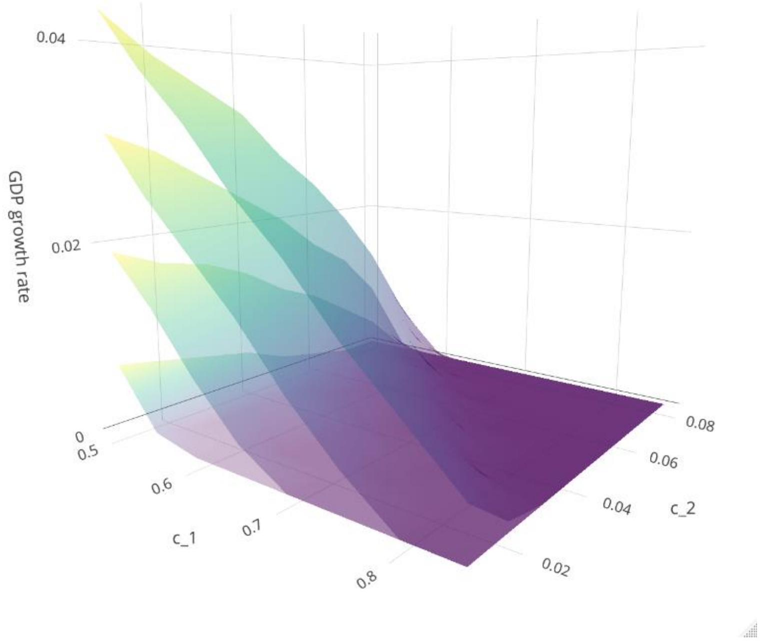

Figure 1: Steady-state growth rates for different combinations of the propensity to consume out of income (x-axis) and wealth (y-axis). Each surface represents increasing levels of the interest rate (the transparency of surfaces decreases as the interest rate increases).

The graph presents the values of c_1 and c_2, respectively on the x-axis and y-axis. The z-axis shows the output growth rate. Each surface represents the relationship between the propensity to consume and the GDP growth rate in correspondence with different levels of the interest rate. When the surfaces are flat (dark purple zone in the graph), namely for high values of the propensity to consume (that is, $b + a < 2$), the economy reaches a steady-state where households achieve the desired wealth-to-income ratio: public deficit and accumulation of savings (and public debt) exhaust and the income reaches a stationary state. In the other zones, for low values of c_1 and c_2($b + a > 2$), the growth rates are positive and are positively affected by the interest rate. Here, the steady-state is characterized by a positive accumulation of savings, public debt and, hence, a positive growth rate of public debt service. Households' saving decisions, though accrued interest rate, have an expansive effect on the same income preventing the achievement of the target wealth-to-income ratio[11].

We can verify that the partial derivative of g with respect to r is positive[12]:

$$\frac{\partial g}{\partial r} = \frac{(1-\theta)(1-c_1)}{\sqrt{(\mathrm{c2} - \mathrm{c1}(1-\theta) - 1)^2 - 4(\mathrm{c1} - \mathrm{c2} - (1-\mathrm{c1})r)(1-\theta)}} > 0. \tag{16}$$

[11] This dynamic characterizes, by construction, the scenario in which $c_2 = 0$. In this case, if the interest rate is higher than zero, the accumulation of savings never runs out and, with that, the accumulation of public debt. In this case, the output growth rate endogenously determined is always positive.
[12] This expression is valid only to describe the steady-growth scenario.

The interests accrued on the public debt become part of the income and consumption of households. Thus, public debt service, adding new purchasing power into the system, triggers a multiplier effect, like any other autonomous component of demand. Thereby, the inclusion of endogenous debt service implies that a part of the output growth rate is endogenously determined by household saving decisions and interest rates that, in turn, determine the growth rate of public debt service. Public debt service, exactly like all autonomous components, contributes to the determination of the growth rate of output.

Since the current deficit is a function of the multiplier effect, the higher the savings rate, the higher the deficit and the accumulation of public debt and the growth rate of interest expenditure. Of course, this relationship does not mean that a higher saving ratio is preferable to fostering economic growth. Conversely, at lower levels of the saving rate, the same growth rate of the total public expenditure can be reached by an equal expansion in the growth rate of primary public expenditure. In this case, the output growth rate would be the same but, due to a higher value of the multiplier, the value of the public debt-to-GDP ratio would be lower. The last mechanism will be clear as the section proceeds.

To check the stability of the public debt-GDP-ratio, we have to verify that the growth rate of the stock of savings and public debt converges to the growth rate of income. Substituting $Y_{t-1} = \frac{S_t - S_{t-1}(1-c_2)}{(1-\theta)(1-c_1)}$ in equation (4.2), we get the second-order difference equation which describes the intertemporal dynamic of the stock of debt:

$$S_{t+1} - S_t[(1-c_2) + c_1\,(1-\theta)] + S_{t-1}(1-\theta)[c_1 - c_2 - r(1-c_1)] = \bar{G}(1-\theta)(1-c_1). \qquad (17)$$

The solution describing the value of the stock of savings in each period is:

$$S_t = \frac{2^{-1-t}\left((b-\mathrm{a})^t\left(\left(2(1+c-d)+b(-2+d)\right)+\mathrm{a}d\right)-(b+\mathrm{a})^t\left(\left(2(1+c-d)+b(-2+d)\right)-\mathrm{a}d\right)-2^{1+t}\mathrm{a}d\right)\bar{G}}{\mathrm{a}(-1+b-c)}, \qquad (18)$$

where:

$$a = \sqrt{(1+c_1(1-\theta)-c_2)^2 - 4(1-\theta)[c_1-c_2-r(1-c_1)]}, \qquad (19)$$

$$b = 1 + c_1(1-\theta) - c_2, \qquad (20)$$

$$c = (1-\theta)[c_1 - c_2 - r(1-c_1)]\,, \qquad (21)$$

$$d = (1-\theta)(1-c_1), \qquad (22)$$

In the long run, when $b + a > 2$, the growth rate of saving is:

$g_S^* =$

$$= \lim_{t\to\infty} \frac{(b-\mathrm{a})^t\left[\left(2(1+c-d)+b(-2+d)\right)+\mathrm{a}d\right]-(b+\mathrm{a})^t\left[\left(2(1+c-d)+b(-2+d)\right)-\mathrm{a}d\right]-2^{1+t}\mathrm{a}d}{2\left\{(b-\mathrm{a})^{t-1}\left[\left(2(1+c-d)+b(-2+d)\right)+\mathrm{a}d\right]-(b+\mathrm{a})^{t-1}\left[\left(2(1+c-d)+b(-2+d)\right)-\mathrm{a}d\right]-2^t\mathrm{a}d\right\}}$$

$- 1$

$$= \frac{a+b}{2} - 1. \quad (23)$$

The growth rate of savings converges to the growth rate of income and the wealth-to-GDP ratio and public debt-to-GDP ratio stabilize at the following values:

$$g_S^* = g_Y^* = g,$$

$$\frac{S^*}{Y^*} = \frac{(1-c_1)(1-\theta)}{c_2+g^*}, \quad (24)$$

$$\frac{B^*}{Y^*} = (1-\theta)\frac{1+g+c_2-c_1}{c_2+g^*}, \quad (25)$$

In this scenario, the stability condition $x = \left(\frac{g-r}{1+g}\right)d$ is fulfilled through an endogenous adjustment of the growth rate of GDP and primary deficit:

$$x_t = \left(\frac{g-r}{1+g}\right)d_t. \quad (26)$$

Conversely, in the stationary-state scenario, the stock of savings converges to the following value:

$$S^* = \frac{(1-c_1)(1-\theta)\bar{G}}{c_2\theta - r(1-\theta)(1-c_1)}, \quad (27)$$

and the stationary values of the private wealth-to-GDP and public debt-to-GDP ratios are:

$$\frac{S^*}{Y^*} = \frac{(1-c_1)(1-\theta)}{c_2}, \quad (28)$$

$$\frac{B^*}{Y^*} = (1-\theta)\frac{1+c_2-c_1}{c_2}. \quad (29)$$

In this scenario, the stability condition is fulfilled through an endogenous adjustment of the primary surplus that converges to the same value of public debt service:

$$x_t = d_{t-1}r. \quad (30)$$

As a consequence, total public deficit is equal to zero and there is no accumulation of savings.

These demonstrations show that, when $\alpha \neq 1$ and exogenously given thresholds for targeting budgetary rules are absent, the sufficient condition for the stabilization of the public debt-to-GDP ratio is the following:

$$c_2 > 0 \lor r > 0 \lor g_G > 0, \quad (30)$$

namely, d^* stabilizes if at least one among the propensity to consume out-of-wealth, the interest rate and the growth rate of primary public expenditure is higher than zero. In a nutshell, given $\alpha \neq 1$, the only scenario in which the public debt-to-GDP ratio explodes is when the interest rate, the growth rate of primary public spending and the propensity to consume out-of-wealth are zero. When $r = 0$, $c_2 = 0$ and $g_G = 0$, the system converges to a stationary income:

$$Y_t = \frac{\bar{G}\,\{[2c_1(1-\theta)]^t - 2^t\}}{2^t[c_1(1-\theta)-1]} = \frac{\bar{G}\,\{2^t[c_1^t(1-\theta)^t - 1]\}}{2^t[c_1(1-\theta)-1]}. \tag{30}$$

For t that goes to infinite:

$$g_Y = \lim_{t\to\infty} \frac{c_1^t(1-\theta)^t - 1}{c_1^{t-1}(1-\theta)^{t-1} - 1} - 1 = 0\,, \tag{31}$$

$$\lim_{t\to\infty} \frac{\bar{G}\,[c^t(1-\theta)^t - 1]}{c(1-\theta)-1} \approx \frac{\bar{G}}{1-c_1(1-\theta)} \qquad Hp\colon\ 0 < c_1, \theta < 1\ , \tag{32}$$

It is worth noticing that equation (8) is a generalized form of the traditional formulation of the “equilibrium” income determined in the traditional income/expenditure model. In this scenario, the debt-to-GDP ratio explodes as income reaches a stationary level and the accumulation of savings proceeds:

$$\frac{B}{Y} \approx \infty.$$

Indeed, a constant public expenditure over time entails a stationary level of income, while in each period the accumulation of savings proceeds and, symmetrically, of public debt.

The second scenario in which the stability of public debt-to-GDP ratio is not ensured is when there is full-hoarding ($\alpha = 1$) on the income generated by the interest on the public debt. In this case, public spending never flows back to the real economy via consumption demand and public debt-to-GDP ratio explodes. In this case, the public debt-to-GDP ratio stabilizes only if the growth rate of primary public spending is positive.

Table 1 resumes the steady-state value of macroeconomic variables for different combinations of r and c_2, and considering $\alpha = 0$ (no hoarding on the income generated by interests accrued on public bonds). Both the values of $\frac{B^*}{Y^*}$ and $\frac{B^*}{Y^{P*}}$ are reported, the first stands for the ratio between public debt and total income (income from production plus financial yields), the second stands for the ratio between public debt and income for production (GDP).

	$\boldsymbol{c_2 > 0, r = 0}$ $\boldsymbol{\alpha = 0}$	$\boldsymbol{c_2 \geq 0, r > 0}$ $\boldsymbol{\alpha = 0}$		$\boldsymbol{c_2 = 0, r = 0}$ $\boldsymbol{\alpha = 0}$
		$e(13) < 1$	$e(13) \geq 1$	
$\boldsymbol{g}$	0	0	$\frac{1}{2}(a+b-2)$	0

$\boldsymbol{Y^*}$	$\frac{\bar{G}}{\theta}$	$\frac{c_2\bar{G}}{\theta c_2 + r(\theta + c_1(1-\theta) - 1)}$	$Growth$	$\frac{\bar{G}}{1 - c_1(1-\theta)}$
$\frac{\boldsymbol{S^*}}{\boldsymbol{Y^*}}$	$\frac{(1-c_1)(1-\theta)}{c_2}$	$\frac{(1-c_1)(1-\theta)}{c_2}$	$\frac{(1-c_1)(1-\theta)}{c_2+g}$	$Explosive$
$\frac{\boldsymbol{S^*}}{\boldsymbol{Yp^*}}$	$\frac{(1-\theta)(1-c_1)}{c_2 - r(1-\theta)(1-c_1)}$	$\frac{(1-\theta)(1-c_1)}{c_2 - r(1-\theta)(1-c_1)}$	$\frac{(1+g)(1-\theta)(1-c_1)}{(1+g)(g+c_2) - r(1-\theta)(1-c_1)}$	$Explosive$
$\frac{\boldsymbol{B^*}}{\boldsymbol{Y^*}}$	$\frac{(1-\theta)(1+c_2-c_1)}{c_2}$	$\frac{(1-\theta)(1+c_2-c_1)}{c_2}$	$\frac{(1-\theta)(1+c_2-c_1+g)}{c_2+g}$	$Explosive$
$\frac{\boldsymbol{B^*}}{\boldsymbol{Yp^*}}$	$\frac{(1-\theta)(1-c_1+c_2)}{c_2 - r(1-\theta)(1-c_1)}$	$\frac{(1-\theta)(1-c_1+c_2)}{c_2 - r(1-\theta)(1-c_1)}$	$\frac{(1+g)(1-\theta)(1+g-c_1+c_2)}{(1+g)(g+c_2) - r(1-\theta)(1-c_1)}$	$Explosive$

Table 1: Determinants and steady-state values of the public debt-to-GDP ratio

As summarized in Table 1, the public debt-to-GDP ratio is a positive function of the saving rate of households and a negative function of the tax rate, propensity to consume out-of-income and out-of-wealth. Through the multiplier, the propensity to consume effects positively both the level of GDP and the dynamic of fiscal revenues: the higher the propensity to consume, the higher the amount of money that flows back to the Government for each unit of pubic spending (and lower is the public deficit). Parallelly, given the level of primary public spending, the higher multiplier implies a higher level of GDP.

The interest rate has a non-linear effect. For the combinations of c_1 and c_2 that give rise to stationary-state of income, the increase in the interest rate raises the public debt-to-GDP ratio[13]. For the combinations of c_1 and c_2 that give rise to a steady growth, the effect of an increase in the interest rate depends on the effect of the growth rate. Computing the partial derivatives of d^* and the growth rate with respect to each parameter we can asses the above-mentioned relations. Since all parameters enter, directly, in the determination of d^* and, indirectly, in the determination of g^*, we have to compare the values of the derivatives of the compound function to assess the ultimate relation between each parameter and the public debt-to-GDP ratio. The general expression of the public deb-to-GDP ratio is:

$$d^* = \frac{B^*}{Yp^*} = \frac{(1+g)(1-\theta)(1+g-c_1+c_2)}{(1+g)(g+c_2) - r(1-\theta)(1-c_1)} \tag{33}$$

The partial derivates of d^* with respect to g,r, θ, c_1 and c_2 are:

$$\frac{\partial\partial d^*}{\partial g} = -\frac{(1-\theta)(1-c_1)\Big((1+g)\big(1+g+2r(1-\theta)\big) + r(1-\theta)(c_2-c_1)\Big)}{\big((1+g)(g+c_2) - r(1-\theta)(1-c_1)\big)^2} < 0 \tag{34}$$

$$\frac{\partial\partial d^*}{\partial c_1} = -\frac{(1+g)(1-\theta)\big(1+g+r(1-\theta)\big)(g+c_2)}{\big((1+g)(1+c_2) - r(1-\theta)(1-c_1)\big)^2} < 0 \tag{35}$$

$$\frac{\partial\partial d^*}{\partial c_2} = -\frac{(1+g)(1-\theta)\big(1+g+r(1-\theta)\big)(1-c_1)}{\big((1+g)(g+c_2) - r(1-\theta)(1-c_1)\big)^2} < 0 \tag{36}$$

[13] This is valid within the range of variation of the interest rate taken into consideration. Indeed the condition about the stationary state or steady state with positive growth depends also on the value of the interest rate.

$$\frac{\partial\partial d^*}{\partial r} = \frac{(1+g)(1-\theta)^2(1-c_1)(1+g-c_1+c_2)}{\left((1+g)(g+c_2)-r(1-\theta)(1-c_1)\right)^2} > 0 \quad (37)$$

While all partial derivatives are negative, the partial derivative with respect to r is positive. The partial derivatives of g^* with respect to r, θ, c_1 and c_2 are:

$$\frac{\partial g^*}{\partial c_1} = \frac{1}{2}\left(1 + \frac{4(1+r)(-1+\theta) + 2\left(-1+\mathrm{c2}-\mathrm{c1}(1-\theta)\right)(-1+\theta)}{2\sqrt{\left(-1+\mathrm{c2}-\mathrm{c1}(1-\theta)\right)^2 + 4(\mathrm{c1}-\mathrm{c2}-(1-\mathrm{c1})r)(-1+\theta)}} - \theta\right) < 0 \quad (38)$$

$$\frac{\partial g^*}{\partial c_2} = \frac{1}{2}\left(-1 + \frac{2\left(-1+\mathrm{c2}-\mathrm{c1}(1-\theta)\right) - 4(-1+\theta)}{2\sqrt{\left(-1+\mathrm{c2}-\mathrm{c1}(1-\theta)\right)^2 + 4(\mathrm{c1}-\mathrm{c2}-(1-\mathrm{c1})r)(-1+\theta)}}\right) < 0 \quad (39)$$

$$\frac{\partial g^*}{\partial \theta} = \frac{1}{2}\left(-\mathrm{c1} + \frac{4(\mathrm{c1}-\mathrm{c2}-(1-\mathrm{c1})r) + 2\mathrm{c1}\left(-1+\mathrm{c2}-\mathrm{c1}(1-\theta)\right)}{2\sqrt{\left(-1+\mathrm{c2}-\mathrm{c1}(1-\theta)\right)^2 + 4(\mathrm{c1}-\mathrm{c2}-(1-\mathrm{c1})r)(-1+\theta)}}\right) < 0 \quad (40)$$

$$\frac{\partial g^*}{\partial r} = \frac{(1-\theta)(1-c_1)}{\sqrt{(\mathrm{c2}-\mathrm{c1}(1-\theta)-1)^2 - 4(\mathrm{c1}-\mathrm{c2}-(1-\mathrm{c1})r)(1-\theta)}} > 0 \quad (41)$$

Each one enters with an opposite sign in the determination of g and d^*. By multiplying each partial derivative with respect to g and the partial derivative of d^* with respect to g^*, it is possible to compute the indirect effect of such parameters on d^*, through the growth channel. Comparing these values with the partial derivatives of d^* with respect to r, θ, c_1 and c_2 (direct channel) we can compute the net effect on the public debt-to-GDP ratio:

$$\frac{\partial g^*}{\partial c_1}\frac{\partial d^*}{\partial g} < \frac{\partial\partial d^*}{\partial c_1} \rightarrow \frac{\partial d^*}{\partial c_1} < 0 \quad (42)$$

$$\frac{\partial g^*}{\partial c_2}\frac{\partial d^*}{\partial g} < \frac{\partial\partial d^*}{\partial c_2} \rightarrow \frac{\partial d^*}{\partial c_2} < 0 \quad (43)$$

$$\frac{\partial g^*}{\partial \theta}\frac{\partial\partial d^*}{\partial g} < \frac{\partial\partial d^*}{\partial \theta} \rightarrow \frac{\partial d^*}{\partial \theta} < 0 \quad (44)$$

$$\frac{\partial g^*}{\partial r}\frac{\partial\partial d^*}{\partial g} >< \frac{\partial\partial d^*}{\partial r} \rightarrow \frac{\partial d^*}{\partial r} \geq< 0 \quad (45)$$

As derivatives show, in the steady growth scenario, the public debt-to-GDP ratio is negatively affected by the propensities to consume ($c_1$and c_2) and the tax rate. The effect of the interest rate is non-linear.

The partial derivative of the compound function of public debt-to-GDP ratio with respect to r is concave and depends on the effect of the interest rate on the growth rate. The thresholds over which the sign of the effect of the interest rate change depends, in turn, on the propensities to consume and the tax rate. When the propensity to consume rises, the threshold lowers (See Figure 2).

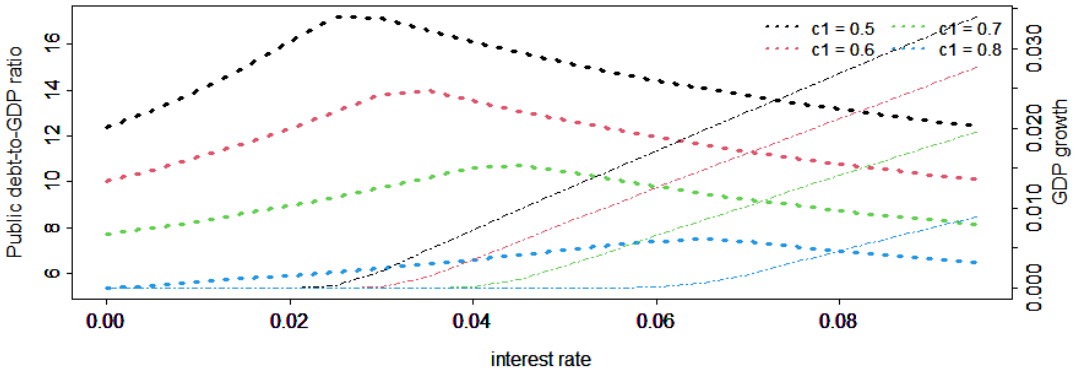

Figure 2: Relation between interest rate, GDP growth (thinnest lines) and public debt-to-GDP ratio for different values of the propensity to consume

For the combination of parameters that give rise to a stationary state, the same relations apply except for r. Here, an increase in the interest rate always produces an increase in the public debt-to-GDP ratio. In this scenario, the public debt-to-GDP ratio is:

$$d^* = \frac{(1-\theta)(1-c_1+c_2)}{c_2 - r(1-\theta)(1-c_1)} \tag{43}$$

The partial derivatives are:

$$\frac{\partial d^*}{\partial c_1} = -\frac{(1-\theta)\big(1+r(1-\theta)\big)c_2}{\big(c_2 - r(1-\theta)(1-c_1)\big)^2} < 0 \tag{46}$$

$$\frac{\partial d^*}{\partial c_2} = -\frac{(1-\theta)\big(1+r(1-\theta)\big)(1-c_1)}{\big(c_2 - r(1-\theta)(1-c_1)\big)^2} < 0 \tag{45}$$

$$\frac{\partial d^*}{\partial \theta} = -\frac{(1-c_1+c_2)c_2}{\big(c_2 - r(1-\theta)(1-c_1)\big)^2} < 0 \tag{46}$$

$$\frac{\partial d^*}{\partial r} = \frac{(1-\theta)^2(1-c_1)(1-c_1+c_2)}{\big(c_2 - r(1-\theta)(1-c_1)\big)^2} > 0 \tag{47}$$

Given the combination of parameters that generates a stationary state, the public debt-to-GDP ratio decrease as the propensity to consume and the tax rate rises, it increases when the interest rate rises.

Although these results may seem paradoxical at first glance, they are actually in line with traditional conclusions of the SM literature. Given the exogenous level of primary government expenditure, the increase in the interest rate produces an endogenous increase in the growth rate of the autonomous component. Since the public debt-to-GDP ratio is negatively correlated with the growth rate of GDP and, hence, of the growth rate of the autonomous component, an increase in the interest rate can produce a decrease in such ratio.

Finally, the stability conditions do not change also considering an open economy and assuming that a share of public debt is held by foreign countries/investors. Firstly, it is worth noticing that this scenario implies that, necessarily, domestic households decide to hold a share of their saving in terms of deposits. The mechanism of stabilization is the following. The exchange of foreign currency for domestic currency required to sell public bonds to foreign investors raises CB foreign currency reserves. Once CB invests such an amount in foreign markets (for instance, buying the public debt of such foreign country), considering homogenous interest rates, the interests earned by CB will be equal to the amount of interest paid to foreign countries by the Government. As a consequence, keeping the hypothesis of CB profits distributed to Governments, the actual amount of interest paid by the Government on foreign debt is zero. In short, the acquisition of public debt from foreign countries corresponds to the case of public bonds held by CB.

4. Determinants and stability of the Public debt-to-GDP ratio: a dynamic SFC Supermultiplier model

In this section, we extend the SFC income-expenditure model presented in Section 3 including capital accumulation, bank loans and private money creation. To this extent, we develop a Supermultiplier SFC model and we solve it analytically to study the stability conditions of the public debt-to-GDP ratio and its determinants. The Supermultiplier model (Serrano, 1995, Freitas and Serrano 2015, Cesaratto et al. 2003, Pariboni and Girardi, 2016) is a demand-led growth model characterized by fully induced investments, exogenous normal (or desired) degree of capacity utilization and one (or more) autonomous component of demand. Firms invest and expand productive capacity to satisfy expected demand at the normal (or desired) degree of capacity utilization. The autonomous component represents the exogenous injection of purchasing power in the system which, triggering the interaction between multiplier and accelerator, determines the long-run pattern of GDP.

In the long-run, the growth rate of the economy converges to the growth rate of the autonomous component and the realized degree of capacity utilization converges to the normal one. In this counterpart of the monetary framework, savings adjust through a corresponding variation in productive capacity and stock of loans. As is commonly assumed in this literature, we are considering that the growth rate of labor force is higher than the growth rate of autonomous components of demand.

The following system of difference equations describes the out-of-equilibrium dynamic of the SFC-SM model[14]:

[14] Usually, traditional SM models consider an adaptive expectation function to define the expected growth rate of the economy. In this formulation, for tractability reasons, we consider that the adaptive parameter β is equal to one, thus the expected level of demand is equal to the one realized in the previous period ($Y_t^e = Y_{t-1}$). However, results do not change when adaptive expectation with $0 < \beta < 1$ are included in the model.

$$
\begin{cases}
Y_t = C_t + G_{t-1}(1+g_G) + I_t \\
C_t = (Y_{t-1} + B_{t-2}r(1-\alpha) - K_{t-1}\delta)(1-\theta)c_1 + S_{t-1}c_2 \\
I_t = Y_t^e v - K_t(1-\delta) \\
Y_t^e = Y_{t-1} \\
B_t = B_{t-1}(1+r) + G_t - \theta(Y_t + B_{t-1}r - K_t\delta) \\
S_t = S_{t-1}(1-c_2) + (Y_{t-1} + B_{t-1}r - K_t\delta)(1-\theta)(1-c_1) \\
K_t = K_{t-1}(1-\delta) + I_{t-1} \\
L_t = L_{t-1}(1-\delta) + I_t \\
B_t^h = \min(S_t, B_t) \\
M_t^h = S_t - B_t^h \\
B_t^{CB} = B_t - B_t^h \\
H_t = M_t^h - L_t
\end{cases}
\tag{48}
$$

Compared to the model in Section 3, this model includes I_t that is the investment demand, K_t that is the stock of capital, L_t is the stock of firms' debt, δ is the depreciation rate of capital, v that is the normal capital/output, M_t^h and B_t^h are, respectively, the stock of deposits and public bonds held by households. As in Section 3, we are assuming that households hold as much as possible of their wealth in terms of public bonds and, the remaining part, as deposits. Deposits and firm loans do not accrue interest rates. However, the results do not change relaxing these assumptions. The results of the general SFC-SM model can be sent upon request.

Since we are not including equities, stocks or the possibility for commercial bank to buy public bonds, the commercial bank holds the difference between deposits and loans as reserves at the CB. These, in turn, are equal to the volume of public bonds held by CB. Tables 2a and 2b report respectively the balance sheet and transaction matrix of the economy.

Assets	Household	Production	Bank	Government	BC	$\sum$
Check deposits	$+M1_w$	$+M1_c$	$-M1$			0
Time deposits	$+M1_w$		$-M2$			0
HPM			$+H_b$		$-H$	0
Advances			$-A$		$-A$	0
Loans		$-L_c$	$+L$			0
Fixed Capital		$+K_{f_c}$				$+K_f$
Public bonds				$-B$	$+B_{cb}$	0
Net wealth	$-V_{h,w}$	$-V_c$	0	$+GD$	0	$-K_f$
$\sum$	0	0	0	0	0	0

Table 2a: Balance sheet

	Households	Production		Government	Bank		Central Bank		Σ
		Current	Capital		Current	Capital	Current	Capital	
Consumption	$-C$	$+C$							0
Income	$+Y$	$-Y$							
Investments		$-I$	$+I$						
Public expenditure		$+G$		$-G$					0
Taxes	$-T$			$+T$					0
Depreciations		$-K_t\delta$	$+K_t\delta$						
Profits BC				$+F_{cb}$			$-F_{cb}$		0
Int. Deposit	$+rM_{t-1}$				$-rM_{t-1}$				0
Int. Loans					$+rL_{t-1}$				0
Int. Bond	$+rB_{t-1}^{h}$			$-rB_{t-1}$			$+rB_{t-1}^{bc}$		0
Int. Reserves					$+rH_{t-1}$		$-rH_{t-1}$		0
ΔDeposit time	$-\Delta M$					$+\Delta M$			0
Δ Deposit check	$-\Delta M^c$					$+\Delta M^c$			0
Δ Loans			$+\Delta L$			$-\Delta L$			0
Δ Bonds		$-\Delta B_h$		$+\Delta B$			$-\Delta B_{bc}$		0
Δ Reserves						$-\Delta H$	$+\Delta H$		0
Σ	0	0		0	0	0	0	0	0

Table 2b: Transaction matrix

Two implicit hypotheses characterize traditional SM models: investments are totally financed through credit creation (endogenous money) while no debt reimbursement is considered. The first hypothesis undergoes the ex-post determination of monetary savings to investments and is the necessary condition to have investment decisions independent from saving decisions (Daziel, 1996). Differently from the second hypothesis, in this formulation, we include debt repayment. As a consequence, the disposable income of households is lower than the value of GDP and corresponds to the NDP (Net Domestic Product). In detail, income before tax is equal to GDP net of $K_t\delta$. If the leverage is equal to one and the useful life of capital goods is equal to the debt repayment time, the reimbursement in each period corresponds to the value of amortization of the capital stock[15]. This reflects the fact that profits are net of capital amortization and the distributed income corresponds to NDP. The absence of $K_t\delta$ would imply that firms are not reimbursing the debt and all revenues are distributed to households. If the disposable income is not computed net of amortization (or debt service), the Government surplus would explode and the public debt-to-GDP ratio would be imploding to - ∞ values [16].

The system can be rewritten in a reduced form as a system of fourth-order difference equations with three dynamic variables and three equations (considering no hoarding on public bonds interests):

[15] If the leverage is equal to one, the amortization is equal to the debt service.

[16] Note that this misleading assumption is implicit in traditional SM model where the available income of household is equal to GDP.

$$\begin{cases} Y_t = (Y_{t-1} + B_{t-2}r(1-\alpha) - Y_{t-3}v\delta)(1-\theta)c_1 + S_{t-1}c_2 + G_t(1+g_t) + Y_{t-1}v - Y_{t-2}v(1-\delta) \\ B_t = B_{t-1}(1+r) + G_t - \theta(Y_t + B_{t-1}r - Y_{t-2}v\delta) \\ S_t = S_{t-1}(1-c_2) + (Y_{t-1} + B_{t-1}r - Y_{t-2}v\delta)(1-\theta)(1-c_1) \end{cases} \tag{49}$$

The redundant equation is:

$$B_t = S_t - I_t - K_t * (1-\delta) + (Y_t + B_{t-1}r - K_t\delta)(1-\theta) \tag{50}$$

That can also be rewritten as:

$$B_t = S_t - L_t - Y_t^F(1-\theta) = S_t - L_t - M_t^c \tag{51}$$

The redundant equation states that the stock of savings is equal to the sum of public debt, private debt (firms' stock of debt aimed at financing investments) and check deposits. If the leverage is equal to one, the firms debt is equal to the value of the capital stock.

As for the redundant equation in the income/expenditure model, the equalization between activities and liabilities at the end of the period is net of the net income distributed at the end of the period this figures as check deposits. It is worth noticing that, since also investments and associated firms' debt create a correspondent amount of savings, in the stationary state the stock of households' savings is higher than public debt. For this reason, a share of savings can be held in terms of deposits.

In this section, we first derive the stability conditions of the system to find whether the growth rates of public debt and GDP converge to the same rate. Secondly, we find the endogenously determined long-term growth rates of macro variables and the steady-state values of public debt and wealth-to-GDP ratios.

After some manipulations and substitutions with the redundant equation, we can rewrite the system as a fourth-order VAR model with three variables (public debt, stock of savings and income):

$$\begin{cases} Y_t = Y_{t-1}(c_1(1-\theta) + v) + S_{t-1}c_2 - Y_{t-2}v(1-\delta) + B_{t-2}rc_1(1-\theta) - Y_{t-3}v\delta c_1(1-\theta) + G_t \\ B_t = Y_{t-1}\theta\big((1-\theta)(1-c_1) - v\big) + B_{t-1}(1+r)(1-\theta) + \theta S_{t-1}(1-c_2) - \theta Y_{t-2}v\delta(1-\theta) + \theta B_{t-2}r(1-\theta)(1-c_1) + \\ \quad +\theta Y_{t-3}v\delta(1-\theta)c_1 + G_t(1-\theta) \\ S_t = Y_{t-1}(1-\theta)(1-c_1) + S_{t-1}(1-c_2) + B_{t-2}r(1-\theta)(1-c_1) - Y_{t-3}v\delta(1-\theta)(1-c_1) \end{cases} \tag{52}$$

The VAR system is stable and the growth rates converge to the same values if the eigenvalues of the companion-form matrix are lower than one in the absolute value. The companion-form matrix is:

$$\Gamma = \begin{pmatrix} c_1(1-\theta)+v & 0 & c_2 & -v(1-\delta) & rc_1(1-\theta) & 0 & -v\delta c_1(1-\theta) & 0 & 0 \\ \theta\big((1-\theta)(1-c_1)-v\big) & (1+r)(1-\theta) & \theta(1-c_2) & -\theta v\delta(1-\theta) & r\theta(1-\theta)(1-c_1) & 0 & v\theta\delta(1-\theta)c_1 & 0 & 0 \\ (1-\theta)(1-c_1) & 0 & (1-c_2) & 0 & r(1-\theta)(1-c_1) & 0 & -v\delta(1-\theta)(1-c_1) & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

(53)

The eigenvalues of the matrix Γ are represented by λ in the expression:

$$|\Gamma - \lambda I| = 0 \tag{54}$$

The roots of the eigenvalues can be derived from the following expression:

$$x^6 + x^5 A + x^4 B + x^3 C + x^2 D + E x + F = 0, \tag{55}$$

where:

$$A = (1-\theta)(c_1 - r) + c_2 + \theta - v - 2$$

$$B = 1 + (c_1 - c_2)(2+r) + r(1+c_1) + (c_1 + r)\theta^2 + v(3 - c_2 - \delta + r) + \theta(-1 - 3c_1 + 2c_2 + r(c_2 - 2 - c_1) - v(1+r)))$$

$$C = (r+1)(\theta-1)^2(-(c_1 - c_2)) - v(\delta\theta(c_1 + r + 1) - \delta(c_1 + r + 2) + c_2(\delta - r(1-\theta) + \theta - 2) + r(\theta-2)(\theta-1) - 2\theta + 3)$$

$$D = (\theta-1)v(\delta(c_1(-\theta - (1-\theta)\theta r + r + 2) + r(1-\theta) + 1) + c_2(-\delta(r+2) + r + 1) - (1-\theta)r - 1)$$

$$E = \delta(\theta-1)^2 v((1+r)(c_1 - c_2) + r\theta(c_2 - 2c_1))$$

$$F = \delta r(\theta-1)^2 \theta v(c_1 - c_2)$$

Although a sixth-degree polynomial equation does not have an analytical solution, it is possible to numerically verify that if all parameters are lower than 1, the absolute value of the eigenvalues is lower than one too (solving numerically the polynomial). We have computed local solutions by performing a multivariate analysis in the following space vector (see the algorithm in Appendix E):

$$c_1 \in (0{,}01{:}\,0.99;\ 0.05)$$
$$c_2 \in (0{:}\,0.99; 0.05)$$
$$\theta \ \in (0.01{:}\,0.99;\ 0.05)$$
$$\delta \ \in (0.01{:}\,0.99; 0.05)$$
$$v \ \in (0.01{:}\,0.99; 0.05)$$
$$r \ \in (0{:}\,0.99; 0.05)$$
$$g_G \ \in (0{,}0.01)$$

The fulfilment of this condition ensures that the growth rate of the stock of debt and savings converges to the growth rate of GDP or that both converge to a stationary level. As for the model in Section 3, the first regime is realized for high values of the propensity to save or $g_G > 0$, the second scenario is realized for low values of the propensity to save and $g_G = 0$.

There is one specific case in which one eigenvalue is higher than one, that is when the growth rate of public spending, the interest rate and the propensity to consume out-of-wealth are simultaneously equal to zero. Only in this case, growth rates not converge to the same value and the public debt-to-GDP ratio explodes. As for the income/expenditure model, income reaches a stationary state with zero growth while the growth rate of savings and public debt is positive.

Again, as for the income/expenditure model, given the absence of fiscal rules and no hoarding of the interest accrued on public bonds, the stability of the public debt-to-GDP ratio is granted if one of the following conditions is fulfilled:

$$g_G > 0 \ \vee r > 0 \ \vee c_2 > 0 \tag{56}$$

That is, the public debt-to-GDP ratio converges to its “natural” value if one across the growth rate of primary public spending, interest rate and the propensity to consume is higher than zero. However, the more general condition for stabilization is:

$$g_G > 0$$

Indeed, in this case, public debt-to-GDP stabilizes also in presence of hoarding on public debt interests.

Now, from the general system, we can derive the analytical expression of long-run growth rate of macroeconomic variables.

After several manipulations, the entire system can be re-expressed as a fourth-order difference equation describing the level of income as an autoregressive function (see Appendix for all mathematical derivations):

$$\begin{aligned} &Y_{t+1} - Y_t\big((1-\theta)(c_1+r)+v+1-c_2\big) + Y_{t-1}\big(vr(1-\theta)+v(2-\delta-c_2)+(1-\theta)(1+r)(c_1-c_2)\big) \\ &\quad +Y_{t-2}v\Big(\delta\big((1-\theta)(c_1+r)+1-c_2\big)-(1-c_2)\big(1+r(1-\theta)\big)\Big) - Y_{t-3}(1+r)(1-\theta)(c_1-c_2) \\ &\qquad = G_0(1+g_G)^{t-2}\Big(r(1-\theta)(c_2-c_1)+(1+g)\big((1-c_2)+r(1-\theta)(1-c_1)\big)\Big) \end{aligned} \quad (57)$$

The solution of the difference equation expressing the level of the income in each period is:

$$Y_t = C_1x_1^t + C_2x_2^t + C_3x_3^t + C_4x_4^t - \frac{G(1+g_G)^{1+t}[\mathrm{c}_2 + g_G(1+\mathrm{c}_2+g_G) - (1-\theta)r(1-\mathrm{c}_1-\mathrm{c}_2-g_G)]}{(1-g_G+X_1)(1+g_G-X_2)(1+g_G-X_3)(1+g_G-X_4)} \quad (5847)$$

$x_{1,2,3,4}$ stands for the solution of the following fourth-degree polynomial:

$$x^4 + bx^3 + cx^2 + dx + \mathrm{e} = 0 \quad (59)$$

Where:

$$b = -(1-\mathrm{c}_2+(1-\theta)(\mathrm{c}_1+r)+v)$$

$$c = (1-\theta)r(c_1-\mathrm{c}_2)(1+r)+v(2-\mathrm{c}_2-\delta+(1-\theta)r)$$

$$d = -v\Big(1-\mathrm{c}_2-\delta\big(1-\mathrm{c}_2+\mathrm{c}_1(1-\theta)(1+r)\big)+(1-\theta)r(1-\mathrm{c}_2)\Big)$$

$$e = \mathrm{c}_2\delta v(1-\theta)(1+r)(c_2-c_1)$$

The solutions of x are:

$$x_{1,2} = -\frac{b}{4} - Q \pm \sqrt{-4Q - \frac{8ac-3b^2}{4} + \frac{8d-4bc+b^3}{3Q}} \quad (6048)$$

$$x_{3,4} = -\frac{b}{4} + Q \pm \sqrt{-4Q - \frac{8ac-3b^2}{4} - \frac{8d-4bc+b^3}{3Q}} \quad (61)$$

where:

$$Q = \frac{1}{2}\sqrt{-\frac{8a-3b^2}{12}+\frac{1}{3}\frac{\sqrt[3]{\left(27d^2-72ce+27b^2e-9bcd+2c^3+\sqrt{\frac{s^2-4q^3}{2}}\right)^2}+12e-3bd+c^2}{\sqrt[3]{27d^2-72ce+27b^2e-9bcd+2c^3+\sqrt{\frac{(27d^2-72ce+27b^2e-9bcd+2c^3)^2-4(12e-3bd+c^2)^3}{2}}}}}$$

The maximum value of the root x determines the long-run growth rate of GDP. In particular, if the values of exogenous parameters (tax rate, propensity to consume, interest rate and capital/output) are such that the module of x is higher than $1+g_G$, the gross growth rate of the economy converges to the maximum value of x, that is the growth rate endogenously determined by the interests accrued on public debt. Conversely, the economy converges to the growth rate of the exogenous component (g_G). Indeed, the GDP growth rate converges to the highest value of the growth rate of the two autonomous components, one of the two is endogenous in the system and is determined by the growth rate of public debt service. The growth rates of macroeconomic variables along the steady-state are:

$$if\ -\frac{b}{4}-Q+\sqrt{-4Q-\frac{8ac-3b^2}{4}+\frac{8d-4bc+b^3}{3Q}}-1<g_G: \quad g^*=g_Y=g_B=g_S=g_G$$

$$else: \ : \quad g^*=g_Y=g_B=g_S=-\frac{b}{4}-Q+\sqrt{-4Q-\frac{8ac-3b^2}{4}+\frac{8d-4bc+b^3}{3Q}}-1$$

On the other hand, a stationary state of GDP is reached if the growth rate of public spending is zero and c_2 is sufficiently higher compared to r. In this case, the income generated by the stock of wealth do not have any growth effect and the economy reach a stationary state where the level of income result to be:

$$Y^* = \frac{\overline{\mathrm{G}}\big(\mathrm{c}_2-(1-\theta)r(1-\mathrm{c}_1+\mathrm{c}_2)\big)}{\mathrm{c}_2\theta(1-\delta v)+(1-\theta)r[(1-\mathrm{c}_1+\mathrm{c}_2)(\delta v-1)+c_2 v]} \qquad (6249)$$

Notice that this is the value of the SM model including propensity to consume out-of-wealth and endogenous public debt service. Instead, the value of the super multiplier when $c_2>0$ and $r=0$ is the following:

$$Y^* = \frac{\overline{\mathrm{G}}}{\theta(1-\delta v)} \qquad (63)$$

From a reduced form of the general model, it is also possible to derive the stationary income of the traditional SM considering that the disposable income of households is net of amortization (without interest rates, stocks and propensity to consume out-of-wealth):

$$Y^* = \frac{\overline{\mathrm{G}}}{1-c_1(1-\theta)(1-\delta \mathrm{v})-\delta \mathrm{v}} \qquad (64)$$

The value of the supermultiplier is slightly lower than traditional widespread formulations of the SM since we are considering that the income distributed to households is net of amortization (or debt reimbursement).

Finally, we can derive the steady-state value of the public debt-to-GDP ratio. Taking exogenously the interest rate fixed by CB, the propensities to consume, the tax rate and the capital/output, the public debt-to-GDP ratio and household wealth-to-income ratio are:

$$d^{*}=\frac{B^{*}}{Y^{*}}=\frac{(1-\theta)(1+c_{2}-c_{1}+g^{*})((1+g^{*})^{2}-\delta v)-v((1+g^{*})(c_{2}+g^{*}))}{(1+g^{*})\left((c_{2}+g^{*})(g^{*}-r(1-\theta)+1)-(1-c_{1})r(1-\theta)\right)} \tag{65}$$

$$s^{*}=\frac{S^{*}}{Y^{*}}=\frac{(1-c_{1})\left((1-\theta)(1+g^{*})^{2}-v\left(r(1-\theta)+\delta(1-\theta)\right)\right)}{(1+g^{*})\left(r(1-\theta)(-1+c_{1}-c_{2}-g^{*})+(1+g^{*})(c_{2}+g^{*})\right)} \tag{66}$$

In the stationary state, they are:

$$\frac{B^{*}}{Y^{*}}=\frac{(1-\theta)(1-c_{1}+c_{2})(1-\delta v)-c_{2}v}{c_{2}-r(1-\theta)(1-c_{1}+c_{2})} \tag{67}$$

$$\frac{S^{*}}{Y^{*}}=\frac{(1-c_{1})(1-\theta)\left((\delta+r)v-1\right)}{(1-c_{1})r(1-\theta)+c_{2}(r(1-\theta)-1)} \tag{68}$$

Latter expressions will be useful to clarify some common arguments about the possibility to persistently reduce the public debt-to-GDP ratio through a permanent expansion in the level of primary public spending.

As we can see from equation (65), in the model including investments and bank money, the public debt-to-GDP ratio depends also on v, that is the normal capital/output ratio. In detail, the public debt-to-GDP ratio is negatively affected by the capital/output:

$$\frac{\partial d^{*}}{v}=-\frac{(1+g)(c_{2}+g)+(1-c_{1}+c_{2}+g)\delta(1-\theta)}{(1+g)\left((c_{2}+g)\left(1+g-r(1-\theta)\right)-(1-c1)r(1-\theta)\right)}<0 \tag{69}$$

A raise in the capital intensity of the production process implies that both the amount of private investments per unit of public spending and elasticity of private indebtedness with respect to a one per cent increase in public spending raise. If capital/output increases, given the level of public spending, the amount of investment raises and fiscal revenues generated by the injection of purchasing power financed by private debt (and not by public spending) increases as well. Parallelly, through the interaction between the accelerator and multiplier, GDP expands. This leads to an increase in the private debt-to-GDP ratio and a corresponding reduction in the public debt-to-GDP ratio. That is, the higher the capital/output, the higher the share of household savings covered by private debt and the lower the one covered by public debt.

As for the baseline model without investment, the public debt-to-GDP ratio is positively correlated to the saving rate and it is negatively correlated to the tax rate and the growth rate of primary public spending, while the effect of the interest rate is non-linear. Figure 3 reports the relation between the propensities to consume (c_1 and c_2) and the public debt-to-GDP ratio as the interest rate increases.

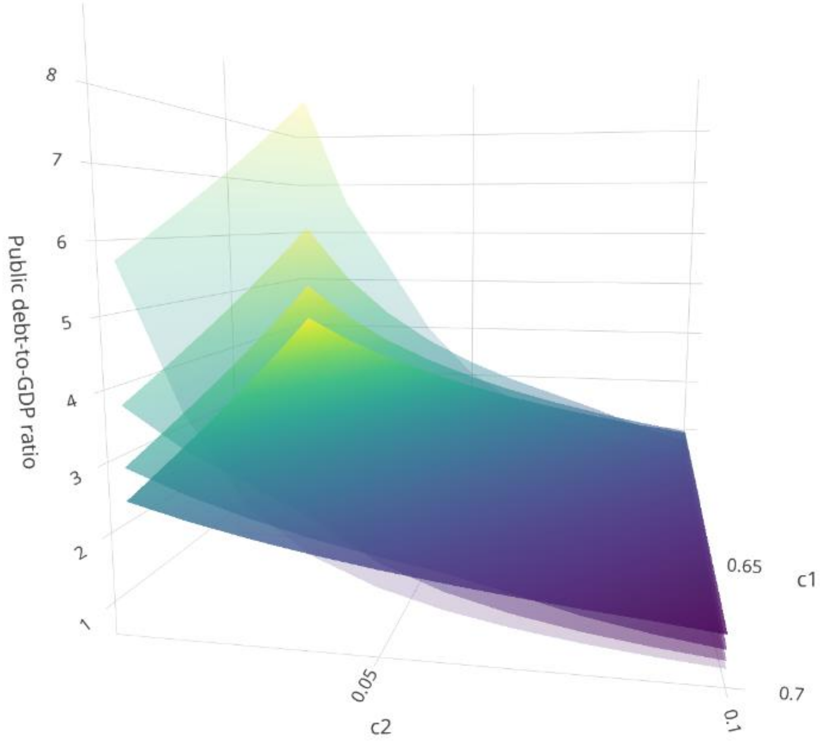

Figure 3: Steady-state public debt-to-GDP ratio for different combinations of the propensity to consume out of income (x-axis) and wealth (y-axis). Each surface represents increasing levels of the interest rate. The transparency of surfaces decreases as the interest rate increases.

The relationships depicted in Figure 3 confirm the results about the negative correlation between the propensities to consume and public debt-to-GDP ratio and the non-linear effect of the interest rate. Computing the partial derivatives of the compound function of the public debt-to-GDP ratio we can analytically identify the sign of each relationship. We consider the two regimes arising from the model: (a) $g^* > 0$; (b) $g = g^* = g_G = 0$, where g^* is the growth rate endogenously determined by the interest rate. In regime (a), the partial derivatives of public debt-to-GDP ratio with respect to each parameter are:

$$\frac{\partial d^*}{c_1} = -\frac{(c_2+g)\big(1+g(2+g)-v(r+\delta)\big)(1-\theta)}{\Big(g(1+g)+c_2\big(1+g-r(1-\theta)\big)-gr(1-\theta)-r(1-c_1(1-\theta)-\theta)\Big)^2} < 0 \tag{70}$$

$$\frac{\partial d^*}{c_2} = -\frac{(1-c_1)\big(1+g(2+g)-v(r+\delta)\big)(1-\theta)}{\Big(g(1+g)+c_2\big(1+g-r(1-\theta)\big)-gr(1-\theta)-r(1-c_1(1-\theta)-\theta)\Big)^2} < 0 \tag{71}$$

$$\frac{\partial d^*}{\theta} = -\frac{(c_2+g)(1-c_1+c_2+g)(1+2g(1+g)-v(r+\delta))}{\Big(g(1+g)+c_2\big(1+g-r(1-\theta)\big)-gr(1-\theta)-r(1-c1(1-\theta)-\theta)\Big)^2} < 0 \tag{72}$$

$$\frac{\partial d^*}{r} = \frac{(1-c_1+c_2+g)\big(-(1+g)(c_2+g)v+(1-c_1+c_2+g)((1+g)^2-v\delta)(1-\theta)\big)(1-\theta)}{(1+g)\Big((c_2+g)\big(1+g-r(1-\theta)\big)-(1-c_1)r(1-\theta)\Big)^2} > 0 \tag{73}$$

$$\frac{\partial d^*}{g} = -\frac{(1+g)^2(-2(1+g)(1-\theta)+v) - c_1(-(1+g)^2+v\delta)(1-\theta) - c_2((1+g)^2+v\delta)(1-\theta)}{(1+g)^2\left(g(1+g)+c_2\big(1+g-r(1-\theta)\big)-gr(1-\theta)-r(1-c1(1-\theta)-\theta)\right)} -$$

$$\frac{\big(1+c_2+2g-r(1-\theta)\big)\big(-(1+g)(c_2+g)v+(1-c_1+c_2+g)((1+g)^2-v\delta)(1-\theta)\big)}{(1+g)\left((c_2+g)\big(1+g-r(1-\theta)\big)-(1-c_1)r(1-\theta)\right)^2} < 0 \quad (74)$$

However, since each parameter affect also the growth rate we have to compare the derivative of the compound function ($\frac{\partial g^*}{\partial c_1}\frac{\partial d^*}{\partial g}$) with the partial derivative (eq.s 71,72,73,74):

$$\frac{\partial g^*}{\partial c_1}\frac{\partial d^*}{\partial g} < \frac{\partial\partial d^*}{\partial c_1} \rightarrow \frac{\partial d^*}{\partial c_1} < 0 \quad (75)$$

$$\frac{\partial g^*}{\partial c_2}\frac{\partial d^*}{\partial g} < \frac{\partial\partial d^*}{\partial c_2} \rightarrow \frac{\partial d^*}{\partial c_2} < 0 \quad (76)$$

$$\frac{\partial g^*}{\partial \theta}\frac{\partial\partial d^*}{\partial g} < \frac{\partial\partial d^*}{\partial \theta} \rightarrow \frac{\partial d^*}{\partial \theta} < 0 \quad (77)$$

$$\frac{\partial g^*}{\partial r}\frac{\partial\partial d^*}{\partial g} > \frac{\partial\partial d^*}{\partial r} \rightarrow \frac{\partial d^*}{\partial r} \geq< 0 \quad (78)$$

In the steady-growth, the public debt-to-GDP ratio is negatively affected by the tax rate, the propensities to consume and the growth rate of primary public spending, while the effect of the interest rate is non-linear.

Instead, in the stationary state, the interest rate positively affects the public debt-to-GDP ratio:

$$\frac{\partial d^*}{c_1} = -\frac{\big(1-v(r+\delta)\big)(1-\theta)c_2}{\big(r(1-\theta)(1-c_1)-\big(1-r(1-\theta)\big)c_2\big)^2} < 0 \quad (79)$$

$$\frac{\partial d^*}{c_2} = -\frac{\big(1-v(r+\delta)\big)(1-\theta)(1-c_1)}{\big(r(1-\theta)(1-c_1)-\big(1-r(1-\theta)\big)c_2\big)^2} < 0 \quad (80)$$

$$\frac{\partial d^*}{c_2} = -\frac{\big(1-v(r+\delta)\big)(1-c_1+c_2)c_2}{\big(r(1-\theta)(1-c_1)-\big(1-r(1-\theta)\big)c_2\big)^2} < 0 \quad (81)$$

$$\frac{\partial d^*}{r} = \frac{(1-c_1+c_2)\big(-c_2v+(1-c_1+c_2)(1-v\delta)(1-\theta)\big)(1-\theta)}{\big(c_2-(1-c_1+c_2)r(1-\theta)\big)^2} > 0 \quad (82)$$

In conclusion, the SM-SFC model confirms the results about the determinants and the relationships between the interest rates, the propensities to consume and the tax rate depicted in section 3. Including investments and bank money, it puts in light the negative relation between capital/output and the public debt-to-GDP ratio. This model extension allows for analysing the differential impact of demand components financed by public debt or private debt on the public debt-to-GPD ratio.

5. The impact of a permanent expansion in the level of public expenditure

In this section, we briefly assess the effect of permanent expansion in the level of public spending on the long-run public debt-to-GDP ratio. To this end, we adopt the range of parameters that generates a stationary-state of GDP[17]. In particular, we consider a growth rate of primary public spending equal to zero, while the propensity to consume out-of-wealth is positive and sufficiently higher than the interest rate. Figure 4 reports the results of the simulation, showing the impact on GDP, macroeconomic stocks and stock-flow ratios.

A permanent increase in public expenditure leads to a persistent rise in the level of output accompanied by a persistent expansion in the stock of savings, and public and private debt while leaving unmodified stock-flow ratios.

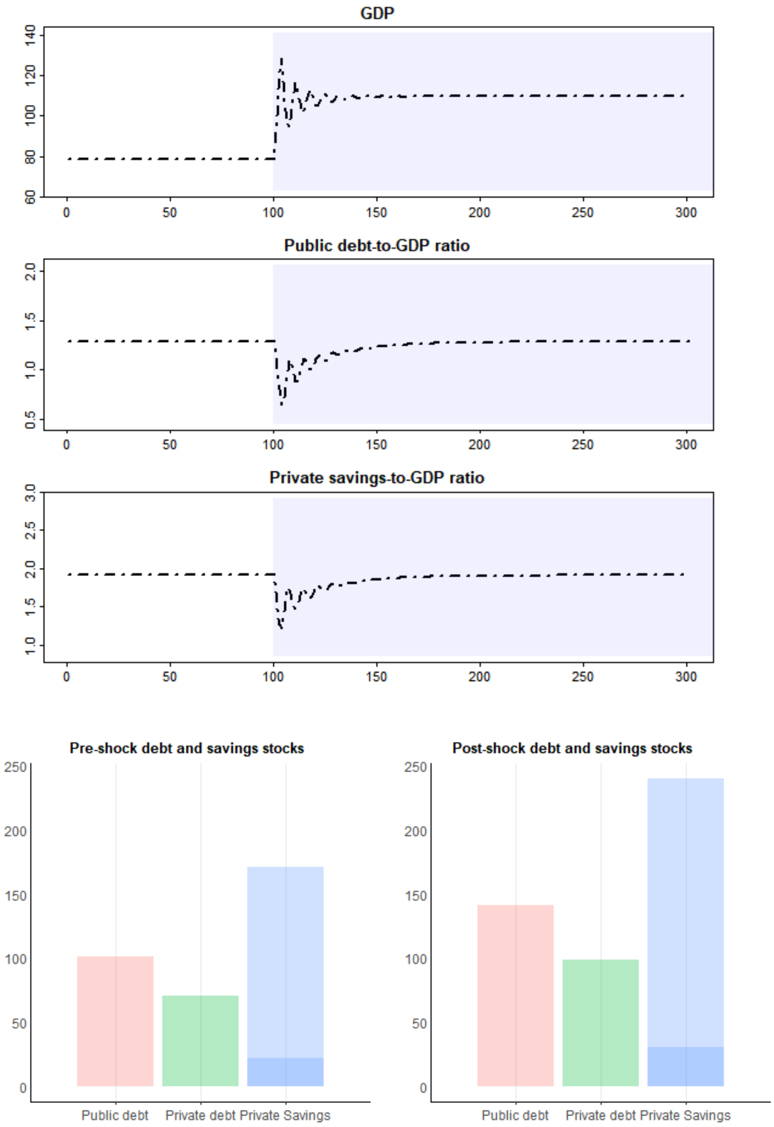

Figure 4: Impact on macroeconomic stocks produced by a permanent expansion in the level of public spending.

[17] The effect of an expansion in the level of public spending would be undetectable in a steady-growth configuration.

As highlighted in Sections 3 and 4, the analytical expression defining the stationary state of public debt-to-GDP does not include the value of the public spending, that is, stock and flow ratios are not affected by the level of public expenditure. The impact on the public debt-to-GDP ratio is only transient. Government can modify the steady-state value of the public debt-to-GDP ratio only through a persistent rise in the growth rate of primary public spending that triggers a series of level effects that never run out. To this extent, given the interest rate, the tax rate and the propensities to consume, the floor level of public debt-to-GDP ratio is fixed by the growth rate of labor force, labor productivity growth and the maximum tax rate.

Finally, it is worth noticing that, as it is clear from the SFC macroeconomic accountability, public expenditure and public deficit can only increase the amount of deposits and, more generally, the financial resources available to the private sector. In this regard, if one assumes - as mainstream authors do - a negative relationship between the amount of "loanable funds" and interest rates on loans, investments could only benefit from an increase in public debt. Consistently with the principle of endogenous money, the financial resources available in the economic system are not given but are endogenous to the expenditure of the system as a whole. Therefore, they do not respond to the logic of "scarcity" dictated by the laws of supply and demand.

6. The impact of fiscal rules

In this section, we introduce two different fiscal rules and we assess their impacts on the long-run public debt-to-GDP ratio. The first rule targets an exogenously-given public deficit (f^T) (1), the second one targets the attainment over time of an exogenously-given public debt-to-GDP ratio (d_*^T) (2). In particular, we impose a target threshold lower than the one autonomously achieved in the steady-state.

Table 1 reports the value of parameters in the baseline scenarios. In this setup, the growth rate of primary public spending is higher than the growth rate endogenously determined by the interest rate. Thus, the GDP growth rate converges to the growth rate of primary public spending.

	c_1	c_2	θ	v	g_G	r	g^*	g_{GDP}
S_1	0.7	0.02	0.4	0.6	1 %	2 %	0.0724 %	1 %

Table 3: Table of parameters

In both schemes, Government adjusts public spending in order to respect the planned public debt-to-GDP ratio of the given period.

In the first case, Government spending after the imposition of the fiscal rule is:

$$G_t = \max(0, f^T Y_{t-1}(1 + g_{t-1}) + Tax_{t-1}(1 + g_{t-1}) - rB_{t-1},$$

where the expected growth rates of GDP and fiscal revenues are equal to the realized growth rate in the previous period. In the second case, we adopt two different schemes of fiscal rules targeting a long-run public debt-to-GDP ratio. In the first one, Government attempts to reach d_*^T over a time span T with a constant (de)growth rate such as:

$$g_d^* = \left(\frac{d_*^T}{d_{s-1}}\right)^{\frac{1}{T}} - 1,$$

$$d_*^t = d_{t-1}(1 + g_d^*)$$

$$G_t = \max(0, d_*^t Y_{t-1}(1 + g_{t-1}) + Tax_{t-1}(1 + g_{t-1}) - B_{t-1} - rB_{h_{t-1}}),$$

where s is the period in which the fiscal rule is introduced, d_{s-1} is the value of public debt-to-GDP ratio in the last period before implementing the fiscal rule and d_*^t is the period-target public debt-to-GDP ratio. In the second case, Government adopts an active fiscal discipline rescheduling the planned (de)growth according to actual values. To this extent, Government adjusts the required (de)-growth of public debt-to-GDP ratio for each period in order to reach d^* within the target time T:

$$g_{d_t} = \left(\frac{d_*^T}{d_{t-1}}\right)^{\frac{1}{T-t+s}} - 1$$

However, the quantitative results between these two schemes are the same. Here, we are reporting the results of the first scheme.

Figure 4 shows that the imposition of the fiscal rule is effective in reaching a 3% deficit-to-GDP ratio, but it causes a persistent expansion in the public debt-to-GDP ratio. Indeed, it reduces the growth rate of primary public spending which, affecting the GDP growth rate, is one of the determinants of the public debt-to-GDP ratio.

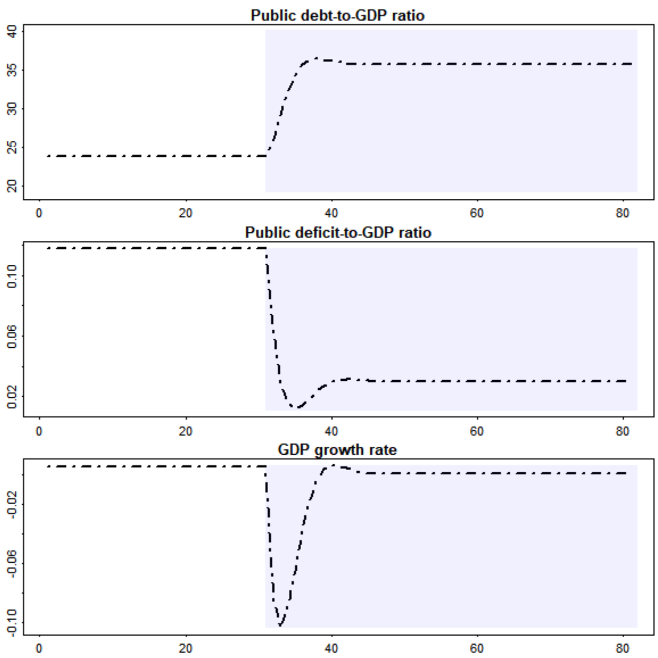

Figure 4: Impact of the introduction of fiscal rule (1) on selected macroeconomic variables

Figure 5 reports the impact of the introduction of fiscal rule (2). In this simulation, we consider two scenarios: in S_1 the interest rate is higher than 0, while in S_2 is equal to zero.

In both scenarios, the succession of cuts in public spending attempting to lower the public debt-to-GDP ratio has a self-defeating effect. If Government follows keeping this rule over time, the public debt-to-GDP ratio explodes. Indeed, since each cut implies a short-term increase in public debt-to-GDP ratio, the rule generates a succession of level effects that cause an ever-increasing public debt-to-GDP ratio, bringing also zero primary public spending. As shown in the figure, when $r = 0$, the fiscal rule causes a screwing between public spending and public debt-to-GDP, the former implodes and the latter explodes. The public debt-to-GDP ratio would stabilize at a higher level only if the interest rate is higher than zero. In this case, the continuous decrease in aggregate demand caused by a prolonged reduction in primary public spending is partially compensated by the public debt service, which the Government cannot refuse to pay.

In particular, the prolonged rise in public debt-to-GDP ratio will stop when primary public spending becomes zero. At this point, the economy will stabilize at a higher level of public debt-to-GDP ratio determined by the growth rate endogenously determined by the interest rate.

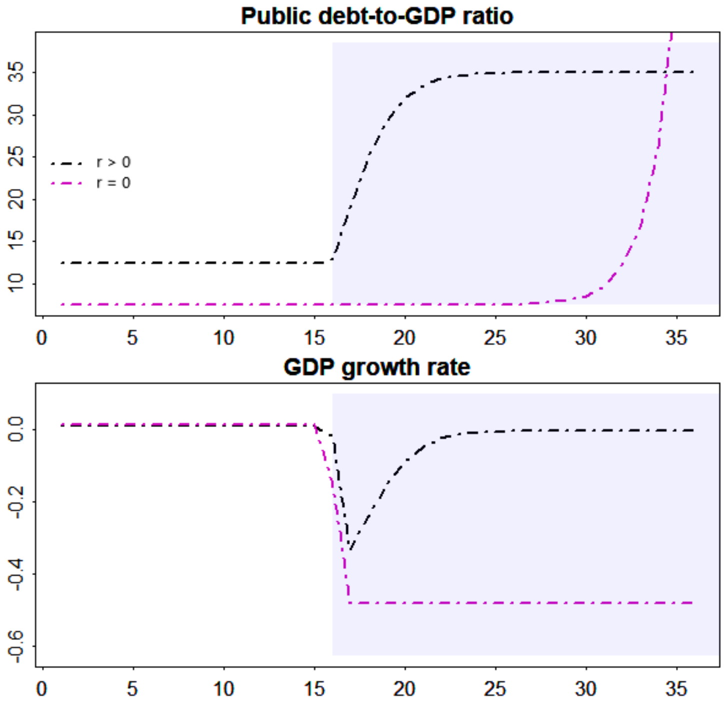

Figure 4: Impact of the introduction of fiscal rule (2) on selected macroeconomic variables

7. Conclusions

In this paper, we have outlined that, in a framework reproducing the main features of a monetary economy of production, the "natural" level of the public debt-to-GDP ratio is determined by the propensity to consume, the tax rate, the capital intensity of the economic system and the interest rate fixed by the Central Bank. While the first three parameters negatively impact the public debt-to-GDP ratio, the interest rate has a non-linear effect. Given an economic system defined by such parameters, there exists a steady-state value of the public debt-to-GDP ratio ingrained in the system, and towards which the economy converges in the long-run.

The necessary condition for the stabilization is the absence of full-hoarding on the interests accrued on public bonds and fiscal rules attempting to reach an exogenous target of public debt-to-GDP ratio. This condition becomes sufficient when one of the growth rate of (primary) public expenditure, the interest rate and the propensity to consume out of wealth is higher than zero.

It has been pointed out that the Government cannot modify the public debt-to-GDP ratio through permanent expansion in the primary public expenditure. Indeed such ratio, depending on the above-mentioned parameters, is always determined ex-post with respect to government spending. A permanent rise in public spending has only a transient effect on the public debt-to-GDP ratio, it goes back to its previous level in the long-run.

The only fiscal manoeuvres the Government has at its disposal to lower the ratio are: a persistent increase in the growth rate of primary public spending or an increase in the tax rate.

Finally, fiscal rules targeting an exogenously given public debt-to-GDP ratio that is lower than the "natural" one are self-defeating: they always result in an expansion or in an ever-increasing public debt-to-GDP ratio.

The future development of this work involves the inclusion of other autonomous components of demand and, in particular, the integration of the system aimed at determining quantity with the system voted to determine relative prices. This makes it possible to assess the simultaneous effect that changes in the interest rate have on the public debt-to-GDP ratio, both through the effects on aggregate demand and on the capital intensity of the economic system.

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Appendix A

This appendix shows why in a closed economic system with both fiat and bank money, and CB acting as lender of last resort, the interests accrued on deposits are indirectly paid by the Government. In particular, we show why the actual interest rate paid on public debt is a weighted average of the interest on deposits and public bonds. The weight depends on the share of wealth that households want to hold in the form of public bonds.

Let's first analyze the simplest case in which households hold all their wealth in terms of deposits, while the commercial bank does not buy public bonds. In this case, the public debt is entirely held by CB and, in turn, the stock of public bonds held by CB (B_{cb}) is equal to the volume of deposits held by households (M_h). To this extent, the commercial bank held the stock of collected deposits as reserves at the CB (H_b). This allows the bank to transform a liability into an activity and to generate an income flow to pay interests on deposits. Since we are assuming that the interests accrued on deposits are equal to the interests accrued on reserves, the commercial bank does not realize profits. The balance sheet and transaction matrix describing this economy are the following.

Assets	Households	C-Sector	Government	Bank	BC	$\sum$
Check deposits	$+M_c$			$-M_c$		0
Time deposits	$+M$			$-M$		
HPM				$+H$	$-H$	0
Public bonds			$-B$		$+B_{cb}$	0
Net wealth	$-V_h$		$+V_b$			0
$\sum$	0	0	0	0	0	0

Table 1a: Balance sheet

	Households	C-Sector	Government	Bank		Central Bank		∑
				Current	Capital	Current	Capital	
Consumption	$-C$	$+C$	0					0
Gov. spending	0	$+G$	$-G$					0
Income	$+Y$	$-Y$						0
Taxes	$-T$		$+T$					0
Profit CB			$+f_{CB}$			$-f_{CB}$		0
Int. Bond			$-r_b B_{t-1}$			$+r_b B_{t-1}^{cb}$		0
Int. deposits	$+r_m M_{t-1}$			$-r_m M_{t-1}$				
Δ Check deposits	$-\Delta M^c$				$+\Delta M^c$			0
Δ Time deposits	$-\Delta M$				$+\Delta M$			0
Δ Bonds	$-\Delta B_h$		$+\Delta B$				$-\Delta B_{bc}$	0
Δ HPM					$-\Delta H$		$+\Delta H$	0
∑	0	0	0	0	0	0	0	0

Table 1b: Transaction matrix

The stock of public debt is equal to the stick of deposits, that in turn is equal to the stock of reserves:

$$B = B_{cb} = M = H_b$$

The profits of CB are the difference between the interests accrued on public debt and those that it has to pay on reserves.

$$F_{CB} = \mathrm{r}_b B_{cb} - r_m H_b = \mathrm{r}_b B_{cb} - r_m M = \mathrm{r}_b B_{cb} - r_m M$$

The profits of the commercial bank are:

$$F_{Bank} = r_m H - r_m M = 0$$

The government debt service net of the profits distributed by CB is:

$$G_B = \mathrm{r}_b B - F_{CB} = \mathrm{r}_b B - \mathrm{r}_b B_{cb} + r_m M_h = r_m M = r_m B$$

The actual interest paid by Government on public debt is equal to the interest rate on deposits, while the total amount of interests paid on public debt corresponds to the interest accrued by households on deposits. Indeed, public debt is indirectly held by households in terms of deposits

Thus, as the equation shows, if households held all their wealth in terms of deposits, the government is indirectly paying the interest rate on deposits. Since the profits of CB are distributed to Government, the latter pays indirectly the interest on public debt through the CB payment of the interest on reserves to the commercial bank.

We can show that results are sim if we consider that households can hold a share of their wealth in terms of public bonds. Assuming that $B_h = \beta S$, since the public debt is the only liability in the economy, the amount of savings of households is always equal to the public debt, thus $B_h = \beta B$ the stock of public bonds held by CB remains equal to the stock of deposits, while the public debt is held by both CB and households:

$$B = B_{cb} + B_h = M + B_h = H_b + B_h$$

$$B_{CB} = M = H_b = B(1-\beta)$$

The public debt service is:

$$G_B = \mathrm{r}_b B - F_{CB} = \mathrm{r}_b B - (1-\beta)B(\mathrm{r}_b - r_m) = B[\beta r_b + r_m(1-\beta)]$$

The actual interest paid by the Government is a weighted average of the interest on bonds and deposits that depends on the share of public debt held by CB that is determined by the share of the wealth that households want to hold in terms of deposits $(1-\beta)$. As equation () shows, β is equal to one or $r_m = r_b$, the expression of public debt service is equal to the expression of the model presented in section 3.

Finally, the same results apply if we include bank money and loans. As before, let's firstly analyse the simplest case in which households detain all their wealth in terms of deposits. In this case, the stock of deposits is higher than public debt and corresponds to the sum of public debt plus loans:

$$M = L + B = L + B_{CB}$$

Since the commercial bank does not buy equities or public bonds, it holds the surplus between deposits and loans as reserves at the CB. As in the model with only fiat money, since the surplus of deposits corresponds to public debt which is completely held by CB, the amount of public bonds held by CB is equal to reserves.

$$H_b = M - L = B_{CB} = B$$

The profits of CB are:

$$F_B = r_b B_{CB} - r_m H = r_b B - r_m(M - L) = (r_b - r_m)B$$

The public debt service is:

$$G_B = \mathrm{r}_b B - F_{CB} = \mathrm{r}_b B - (r_b - r_m)B = r_m B$$

The actual interest rate paid on public bonds is equal to the interest rate on deposits, while the amount of interest accrued on the deposits symmetrically generated by public debt $(M - L)$, and that corresponds to the amount of public bonds held by CB, is indirecly paid by the Government.

Now, if we consider that households hold a share of their wealth in terms of public bonds such that:

$$B_h = \beta S = \beta\,(B + L)$$

$$B_h = \min(B, \beta\,(B + L))$$

The stock of public bonds held by CB remains equal to the stock of reserves of commercial bank:

$$B_{CB} = H_b$$

the stock of deposits, while the public debt is held by both CB and households:

$$B = B_{cb} + B_h = M + B_h = H_b + B_h$$

$$B_{CB} = M = H_b = B(1-\beta)$$

The public debt service is:

$$G_B = \mathrm{r}_b B - F_{CB} = \mathrm{r}_b B - (1-\beta)B(\mathrm{r}_b - r_m) = B[\beta r_b + r_m(1-\beta)]$$

Appendix B

In a sequential income-expenditure model where households hold all their wealth in form of public bonds, the amount of public debt held by CB is equal to the volume of check deposits. Table 3 reports the dynamic of macroeconomic variables in the stationary-state scenario: $c_1 = 0.8, c_2 = 0.1, r = 0.01, \theta = 0.05$.

	G	Y	C	T	B	B^{cb}	B^h	M^c	S
1	50	50	0	25	25	25	0	25	0
2	50	70	20	35	40	35	5	35	5
3	50	78	28	39	51	39	12	39	12
4	50	83	33	41	60	41	18	41	18
5	50	85	35	43	67	43	25	43	25
6	50	86	36	43	74	43	31	43	31
7	50	88	38	44	80	44	36	44	36
8	50	89	39	45	86	45	41	45	41
9	50	90	40	45	91	45	46	45	46
10	50	91	41	46	96	46	51	46	51
…	…	...	...	...	...	...	...	...	...
…	…	...	...	...	...	...	...	...	...
…	50	101	51	51	153	51	102	51	102
…	50	101	51	51	153	51	102	51	102

Table 4: Intertemporal dynamic of macroeconomic variables

Appendix C

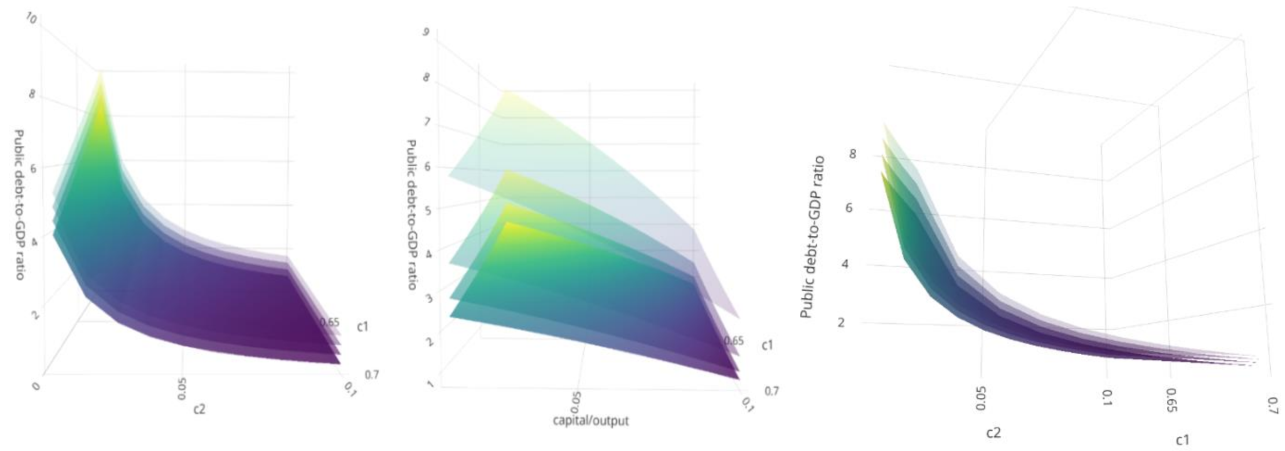

Figure: Steady-state public debt-to-GDP ratios (z-axis)

(a) c_1 and c_2 on x- and y-axis respectively, each surface represents a different level of the capital/output

(b) c_1 and capital/output on x- and y-axis respectively, each surface represents a different level of interest rate

(c) c_1 and c_2 on x- and y-axis respectively, each surface represents a different capital deterioration rate

Appendix D

R code of the SFC- Income/expenditure model

```
rm(list=ls())


# PARAMETERS
T=2000      # Time periods
tau=0.3     # Tax rate
c1=0.7      # Propensity to consume out-of-income
c2=0.01     # Propensity to consume out-of-wealth
Gin=50      # Initial or constant level of public spending
r=0.05      # Interest rate
alfa=1      # Percentage the interest accrued on public bonds which is hoarded
perc=1      # Percentage of savings held in terms of public bonds

# VARIABLES

# Income from production or GDP
Y=matrix(data=0, ncol=T)
# Net Income from production (NDP) plus financial income
YF=matrix(data=0, nrow=T)
# Households consumption
C=matrix(data=0, ncol=T)
# Households stock of savings
S=matrix(data=0, ncol=T)
# Pulic spending
G=matrix(data=0, ncol=T)
# Stock of public debt
B=matrix(data=0, ncol=T)
# Disposable income
YD=matrix(data=0, ncol=T)
# Fiscal revenues
Tax=matrix(data=0, nrow=T)
# Stock of public bonds held by households
Bh=matrix(data=0, nrow=T)
# Stock of deposits
M=matrix(data=0, nrow=T)
# Check deposits
Mc=matrix(data=0, nrow=T)
# Stock of public bonds held by Central Bank
B_CB=matrix(data=0, nrow=T)
# Public debt-to-GDP ratio
debpil=matrix(data=0, ncol=T)
# Stock of Savings-to-GDP ratio
spil=matrix(data=0, ncol=T)
# GDP growth rate
g=matrix(data=0, ncol=T)
# Stock of savings growth rate
gs=matrix(data=0, ncol=T)
# Variation in savings
```

```
deltasaving=matrix(data=0, ncol=T)
# Public deficit
deficit=matrix(data=0, ncol=T)
# Redundant equation
redundant=matrix(data=0, nrow=T)
# Stock of public debt over total income (including financial rents)
debpil2=matrix(data=0, nrow=T)
# Stock of private savings over total income (including financial rents)
spil2=matrix(data=0, nrow=T)

# Model: Define time loop

for (t in 1:T){
G[t]=Gin
if(t>1){
G[t]=G[t-1] }
if(t>1){
# Consumption function
if(t>2){
C[t]=((C[t-1]+G[t-1])+r*S[t-2]*alfa)*(1-tau)*c1+S[t-1]*c2
}else{
C[t]=((C[t-1]+G[t-1])*(1-tau))*c1+S[t-1]*c2}
# GDP
Y[t]=C[t]+G[t]
# Total income inclusing financial rent
YF[t]=C[t]+G[t]+r*S[t-1]
# Public debt
B[t]=B[t-1]+G[t]-YF[t]*tau+r*S[t-1]
# Private savings
S[t]=S[t-1]+YF[t-1]*(1-tau)-C[t]
# Growth rates
g[t]=(Y[t]-Y[t-1])/Y[t-1]
gs[t]=(S[t]-S[t-1])/S[t-1]
# Public deficit
deficit[t]=G[t]-YF[t]*tau+r*Bh[t-1]
deltasaving[t]=YF[t]*(1-tau)-YF[t-1]*(1-tau)*c1+S[t-1]*(c2+r)
# Public bonds demand of households
Bh[t]=min(S[t]*perc,B[t])
# Fiscal revenues
Tax[t]=YF[t]*tau
# Check deposits
Mc[t]=YF[t]*(1-tau)
}else{
Y[t]=G[t]
YF[t]=G[t]
B[t]=G[t]-YF[t]*tau
deficit[t]=G[t]-Y[t]*tau
Tax[t]=YF[t]*tau
Mc[t]=YF[t]*(1-tau)  }
# Bonds held by CB
B_CB[t]=B[t]-Bh[t]
# Disposable income
YD[t]=YF[t]*(1-tau)
debpil[t]=B[t]/Y[t]
```

```
debpil2[t]=B[t]/YF[t]
spil[t]=S[t]/Y[t]
spil2[t]=S[t]/YF[t]
redundant[t]=B[t]-S[t]-B_CB[t]}

m=rbind(c(1),c(2),c(3),c(4),c(5))
layout(m)
par(tcl=-0.1,mgp=c(2,0.4,0))
par(mar=c(1.4,1.8,1.5,2))
b=2000
#par(mar=c(4,1.5,4,1.5))
#par(mar=c(1.5,1.5,1,1.5))
mini=min(Y[1:b])
maxi=max(Y[1:b])
plot(Y[1:b], ylim=range(mini,maxi),main="Y",type='l',xlab="Tempo",ylab="")
mini=min(debpil[1:b])
maxi=max(debpil[1:b])
plot(debpil[1:b],col=3, ylim=range(mini,maxi),main="Deb/GDP_p",type='l',xlab="Tempo",ylab="")
mini=min(debpil2[1:b])
maxi=max(debpil2[1:b])
plot(debpil2[1:b],col=3, ylim=range(mini,maxi),main="Deb/GDP_tot",type='l',xlab="Tempo",ylab="")
mini=min(spil[1:b])
maxi=max(spil[1:b])
plot(spil[1:b], col=5,ylim=range(mini,maxi),main="S/PIL",type='l',xlab="Tempo",ylab="")
mini=min(redundant[1:b])+1
maxi=max(redundant[1:b])-1
plot(redundant[1:b], col=6,ylim=range(mini,maxi),main="Redundant equation",type='l',xlab="Tempo",ylab="")

if(g[T]==0){
# Stock-flow ratio in stationary state:#######
print ("Private savings over total income (production plus financial income) (analytical solution):")
print((1-c1)*(1-tau)/c2)
print ("Private savings over total income (production plus financial income) (simulated solution):")
print(S[T]/YF[T])
print("Private savings over income from production (analytical solution):")
print((1-tau)*(1-c1)/(c2-r*(1-tau)*(1-c1)))
print("Private savings over income from production (simulated solution):")
print(S[T]/Y[T])

print("Public debt over total income (production plus financial income) (analytical solution): ")
print(((1-tau)*(1+c2-c1))/c2)
print("Public debt over total income (production plus financial income) (simulated solution): ")
print(B[T]/YF[T])
print("Public debt over income from production (analytical solution): ")
print(((1-tau)*(1-c1+c2))/(c2-r*(1-tau)*(1-c1)))
print("Public debt over income from production (analytical solution): ")
print(B[T]/Y[T])
}else{
# The growth rate endogenously determined by public debt service
print("The endogenous growth rate is (analytical solution)")
print((((c2 - 1 - c1*(1 -tau))^2 +   4*(tau - 1)*(c1 - c2 - r*(1 - c1)))^0.5 + 1 +    c1*(1 - tau) -   c2)/2 - 1)
print("The endogenous growth rate is (simulated solution)")
print(g[T])
```

```
### Stock-flow ratios in steady growth #############
# Private savings over total income (production plus financial income)
print ("Private savings over total income (production plus financial income) (analytical solution):")
print((1-c1)*(1-tau)/(c2+g[T]))
print ("Private savings over total income (production plus financial income) (simulated solution):")
print(S[T]/YF[T])
# Private savings over income from production
print("Private savings over income from production (analytical solution):")
print((1+g[T])*(1-tau)*(1-c1)/((1+g[T])*(g[T]+c2)-r*(1-tau)*(1-c1)))
print("Private savings over income from production (simulated solution):")
print(S[T]/Y[T])
# Public debt over total income (production plus financial income)
print("Public debt over total income (production plus financial income)  (analytical solution): ")
print(((1-tau)*(1+c2-c1+g[T]))/(c2+g[T]))
print("Public debt over total income (production plus financial income) (simulated solution): ")
print(B[T]/YF[T])
# Public debt over income from production
print("Public debt over income from production (analytical solution): ")
print((-(1+g[T])*(-1+tau)*(1+g[T]-c1+c2))/((1+g[T])*(g[T]+c2)-r*(1-tau)*(1-c1)))
print("Public debt over income from production (simulated solution): ")
print(B[T]/Y[T])
}
```

R code of SFC-SM model

```
rm(list=ls())
# PARAMETERS
T=2000          # Time periods
tau=0.25        # Tax rate
c1=0.7          # Propensity to consume out-of-income
c2=0.02         # Propensity to consume out-of-wealth
Gin=50          # Initial or constant level of public spending
r=0.045         # Interest rate
alfa=1          # Percentage of income generated by which is hoarded
v_n=0.6         # Normal capital/output
kin=Gin*v_n     # Initial stock of capital
delta=0.1       # Capital depreciation rate
gp=0            # Growth rate of primary public spending


# VARIABLES
# Income from production or GDP
Y=matrix(data=0, ncol=T)
# Net Income from production (NDP) plus financial income
YF=matrix(data=0, nrow=T)
# Households consumption
C=matrix(data=0, ncol=T)
# Households stock of savings
S=matrix(data=0, ncol=T)
# Pulic spending
G=matrix(data=0, ncol=T)
```

```
# Stock of public debt
B=matrix(data=0, ncol=T)
# Disposable income
YD=matrix(data=0, ncol=T)
# Investments
I=matrix(data=0, nrow=T)
# Capital Stock
k=matrix(data=0, nrow=T)
# Stock of loans
L=matrix(data=0, nrow=T)
# Fiscal revenues
Tax=matrix(data=0, nrow=T)
# Stock of public bonds held by households
Bh=matrix(data=0, nrow=T)
# Stock of deposits
M=matrix(data=0, nrow=T)
# Check deposits
Mc=matrix(data=0, nrow=T)
# Stock of public bonds held by Central Bank
B_CB=matrix(data=0, nrow=T)
# Public debt-to-GDP ratio
debpil=matrix(data=0, ncol=T)
# Stock of Savings-to-GDP ratio
spil=matrix(data=0, ncol=T)
# GDP growth rate
g=matrix(data=0, ncol=T)
# Stock of savings growth rate
gs=matrix(data=0, ncol=T)
# Variation in savings
deltasaving=matrix(data=0, ncol=T)
# Public deficit
deficit=matrix(data=0, ncol=T)
# Redundant equation
redundant=matrix(data=0, nrow=T)
v=v_n


# Model: Define time loop
for (t in 1:T){
G[t]=Gin
# Public spending
if(t>1){
G[t]=G[t-1]*(1+gp)  }
if(t>1){
# Updating capital stock
k[t]=k[t-1]*(1-delta)+I[t-1]

# Investment function
I[t]=Y[t-1]*v_n-k[t]*(1-delta)

if(t>2){
# Consumption equation
C[t]=(Y[t-1]+B[t-2]*r*alfa-k[t-1]*delta)*(1-tau)*c1+S[t-1]*c2
```

```
}else{
C[t]=(Y[t-1]-k[t-1]*delta)*(1-tau)*c1+S[t-1]*c2  }

# Total production
Y[t]=C[t]+G[t]+I[t]

# Computing public debt
B[t]=B[t-1]+G[t]-(Y[t]+B[t-1]*r-k[t]*delta)*tau+r*B[t-1]

 # Computing Stock of Savings
 if(t>2){
     S[t]=S[t-1]*(1-c2)+(Y[t-1]+B[t-2]*r-k[t-1]*delta)*(1-tau)*(1-c1)}else{
     S[t]=S[t-1]*(1-c2)+(Y[t-1]-k[t-1]*delta)*(1-tau)*(1-c1)}

    #Computing total income (NDP + financial income)
    YF[t]=C[t]+G[t]+I[t]+B[t-1]*r-k[t]*delta

    # Computing the GDP and savings growth rate
    g[t]=(Y[t]-Y[t-1])/Y[t-1]
    gs[t]=(S[t]-S[t-1])/S[t-1]

    # Computing public deficit
    deficit[t]=G[t]-YF[t]*tau+r*B[t-1]

    # Stock of loans
    L[t]=I[t]+L[t-1]*(1-delta)
    #redundant[t]=S[t]-L[t]+Mc[t]-B[t]

    # SFC check
    redundant[t]=S[t]-I[t]-k[t]*(1-delta)+(Y[t]+B[t-1]*r-k[t]*delta)*(1-tau)-B[t]
  }else{
    G[t]=Gin
    I[t]=max(G[t]*v_n-k[t]*(1-delta),0)
    Y[t]=G[t]+I[t]
    YF[t]=G[t]+I[t]
    B[t]=G[t]-Y[t]*tau
    deficit[t]=G[t]-YF[t]*tau
    L[t]=I[t]
    redundant[t]=S[t]-I[t]-k[t]*(1-delta)+(Y[t]-k[t]*delta)*(1-tau)-B[t]  }

  # Public bonds held by Households
  Bh[t]=min(B[t],S[t])

  # Fiscal revenues
  Tax[t]=YF[t]*tau

  # Updating CB bonds
  B_CB[t]=B[t]-Bh[t]

  # Updating stock of deposits
  M[t]=S[t]-Bh[t]

  # Check deposits
```

```
  Mc[t]=YF[t]*(1-tau)

  # Disposable income
  YD[t]=Y[t]*(1-tau)
   debpil[t]=B[t]/Y[t]
  spil[t]=S[t]/Y[t] }

c_1=c1
c_2=c2
f=1-tau
m=rbind(c(1),c(2),c(3),c(4),c(5))
layout(m)
par(tcl=-0.2,mgp=c(2,0.4,0))
par(mar=c(1.4,1.8,3.6,2))
mini=min(Y)
maxi=max(Y)
plot(Y[1:T], ylim=range(mini,maxi),lwd=2,main="GDP",type='l',xlab="Tempo",ylab="")
mini=min(debpil)
maxi=max(debpil)*1.1
plot(debpil[1:T],col=3, lwd=2,ylim=range(mini,maxi),main="Public debt-to-GDP
ratio",type='l',xlab="Tempo",ylab="")
mini=min(spil)
maxi=max(spil)*1.1
plot(spil[1:T], col=3,lwd=2, ylim=range(mini,maxi),main="Private savings-to-GDP
ratio",type='l',xlab="Tempo",ylab="")
mini=min(g[1:T])
maxi=max(g[1:T])
plot(g[1:T], col=1,lwd=2,ylim=range(mini,maxi),main="GDP growth rate",type='l',xlab="Tempo",ylab="")
mini=min(redundant[1:T])+1
maxi=max(redundant[1:T])-1
plot(redundant[1:T],col=1,lwd=2,lty=4,ylim=range(mini,maxi),main="Redundant
equation",type='l',xlab="Tempo",ylab="")

  if(round(g[T],6)==0){
  #Stationary state income #
  print("Stationary GDP (analytical solution):")
  print((Gin*(c_2-(1-tau)*r*(1-c_1+c_2)))/(c2*tau*(1-delta*v)+(1-tau)*r*((1-c_1+c_2)*(delta*v-1)+c2*v)))
  print("Stationary GDP (simulated solution):")
  print(Y[T])
    # Stock-flow ratio in stationary state:#######
  print("Private savings over GDP (analytical solution):")
  print(((1-c_1)*(1-tau)*((delta+r)*v-1))/((1-c1)*r*(1-tau)+c2*(r*(1-tau)-1)))
  print("Private savings over GDP (simulated solution):")
  print(spil[T])
  print("Public debt over GDP (analytical solution): ")
  print(((1-tau)*(1-c1+c2)*(1-delta*v)-c2*v)/(c2-r*(1-tau)*(1-c1+c2)))
  print("Public debt over GDP (analytical solution): ")
  print(B[T]/Y[T])
  }else{
  # The endogenous growth rate:
  d=delta
  f=1-tau
  p=gp
```

```
v1=(-v + c2* v + d* v - c2* d* v + c1* d* f* v - f* r* v +c2* f* r* v + d* f* r* v) #1 +
v2=(c1* f - c2* f + c1* f* r - c2* f* r + 2* v - c2* v - d* v + f* r* v) #1^2
v3=(-1 + c2 - c1* f - f* r - v)
v4=-c1*d*f*v + c2*d*f*v-c1*d* f* r* v +  c2* d* f* r* v
a=1
b=v3
c=v2
d=v1
e=v4
q=12*a*e-3*b*d+c^2
s=27*a*d^2-72*a*c*e+27*b^2*e-9*b*c*d+2*c^3
z=(8*a*c-3*b^2)/(8*a^2)
d0=((s+(s^2-4*q^3)^0.5)/2)^(1/3)
S=(8*a^2*d-4*a*b*c+b^3)/(8*a^3)
Q=0.5*(-(2/3)*z+1/(3*a)*(d0+q/d0))^0.5
x1=-b/(4*a)-Q+0.5*(-4*Q^2-2*z+S/Q)^0.5
x2=-b/(4*a)-Q-0.5*(-4*Q^2-2*z+S/Q)^0.5
x3=-b/(4*a)+Q+0.5*(-4*Q^2-2*z-S/Q)^0.5
x4=-b/(4*a)+Q-0.5*(-4*Q^2-2*z-S/Q)^0.5
roots=c(x1,x2,x3,x4)
sol=max(roots[which(is.nan(roots)==FALSE)])
print("The endogenous growth rate is (analytical solution)")
print(sol-1)
print("The growth rate is (simulated solution)")
print(g[T])
if((sol-1)<gp){
print("The growth rate of the economy is higher than the endogenous growth rate since latter is lower than the growth rate of primary public spending. GDP growth converges to the highest value of the growth rate of autonomous or semi-autonomous components")
print(g[T])
}
### Stock-flow ratios in steady growth #############

# Private savings over income from production
print("Private savings over GDP (analytical solution):")
print(((1 - c1)*((1 - tau)*(1 + g[T])^2 - v*(r*(1 - tau) + delta*(1 - tau))))/((1 + g[T])*(r*(1 - tau)*(-1 + c1 - c2 - g[T]) + (1 + g[T])*(c2 + g[T]))))
print("Private savings over GDP (simulated solution):")
print(spil[T])

# Public debt over GDP
print("Public debt over GDP (analytical solution): ")
print(((1 -tau)*(1 + c2 - c1 + g[T])*((1 + g[T])^2 -delta*v) - v*((1 + g[T])*(c2 + g[T])))/ ((1 + g[T])*((c2 + g[T])*(g[T] - r*(1 - tau) + 1) - (1 - c1)*r*(1 - tau))))
print("Public debt over GDP (simulated solution): ")
print(B[T]/Y[T])
}
```