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Abstract

We analyze the *rationality* of a Decision Maker (DM) who chooses from *lists of sets of alternatives*. A new class of choice functions, representing the DM's choice-behavior, and a new rationality axiom are proposed and studied. We show that a property, that we call *No-Regret* suggests that alternatives disregarded as of no interest for the DM be ignored, is a rationality criterion that encompasses some compelling postulates of the classical choice model and extends them to the proposed general framework of choice from lists of sets of alternatives.

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Dedicated to the memory of Sydney N. Afriat (London, 1925 - Siena, 2021)

1 Introduction

We observe that there are a variety of real life decisions requiring selections from sets of alternatives presented in the form of an ordered list. Some possible examples are:

1. A DM, who manages a venture capital fund, chooses some bonds, stocks or equities from different stock exchanges (for instance, Singapore, New York, London) ordered in a list according to their opening time.

2. Consider the problem of finding an equilibrium in a market with excess demand. A DM with endowment ω faces prices p, choosing x_1 from the budget set $A_1 := \{x : (p^1, x) \leq (p^1, \omega)\}$ at current prices p^1. The DM's choice (and those of other agents), changes prices decreasing the excess demand. She subsequently chooses from $A_t = \{x : (p^t, x) \leq (p^t, x_{t-1})\}$, with $t = 2, \ldots, k$, by a kind of *iterative procedure* in which the budget set at time t depends on the choice (and equilibrium prices) at $t-1$. The DM chooses the set of alternatives $\{x_1, \ldots, x_k\}$ from list $\mathcal{A} = (A_1, \ldots, A_k)$.

3. The coach of a national team has to select players for the next World Cup from the eight teams that play a knockout-tournament in the national competition. There are four quarterfinals, where the losers are eliminated, followed by the semifinals, where the losers play an extra match to decide the bronze medal. The winner of the final gets the gold medal, the loser

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the silver. The coach selects players from teams according to two criteria: players with the freshest legs (hence, those whose team lost the match at the quarterfinals) and the strongest players (presumably those who won the final).

Many other instances from commonly observed situations could be added to these examples of choice from lists of sets. None of these cases interest us *per se*. Our main aim is, more generally, to study the *rationality* of a DM who has to solve her decision problem by choosing from lists of sets of alternatives. We consider the DM's behavior to be rational if she does not choose earlier ignored alternatives that do not interest her. This means that not-chosen alternatives can be removed from the set to choose without any impact on its value. This quite obvious consideration is the key principle on which the new rationality axiom, that we call *No-Regret* (NR), relies. The NR axiom is a *normative* prescription, saying what ought to happen. It should be used for evaluating the behavior of a DM who achieves desired outcomes in different choice contexts. We do not study here the NR principle as a solution to the many paradoxical choice-situations discussed in literature. We are aware that it could be used to prevent, for instance, the 'preference reversal phenomenon' the 'cyclicality of preferences' and other much discussed failures of the classical model of rationality in economics. However, this kind of exercise, that is partially carried out in the last section of the paper, is essentially best left for future research. Here, we just propose and study the NR axiom as a guide (a normative prescription, then) to how a DM ought to behave when she chooses from sets of alternatives presented in form of a list. We will confirm the validity of NR axiom as a rationality criterion to action by comparison with some other prominent rationality postulates well-established in theory of choice.

We in fact observe that the NR principle recalls the requirement of the *Outcast* rationality property (O) (see below), an axiom much studied in the classical context of choice from a set of alternatives (see e.g. Danilov (2012)). The Outcast postulate claims that if some alternatives are worthless for the DM, they can be removed from a set without affecting the DM's evaluation of it. In what follows, we show that the NR axiom encompasses two other prominent rationality notions of the classical model of choice from a set of alternatives, namely the *Heritage* axiom (H), (a fundamental rationality property, which states that if an alternative is chosen from a set it will be chosen from any subset of the latter containing it) and the *Path-independent* axiom (PI) (according to which the DM's choice does not depend on any particular order of the alternatives). We aim to test whether the new NR property is a (general) rationality axiom *by comparison* with these three (H, O, PI) classical notions of rationality.
After defining the class of choice functions on lists of sets of alternatives (hereafter CFL) and the NR axiom, we show that NR is a generalization of Outcast to the present setting. Since there exists no a unique procedure for selecting alternatives from lists of sets, we then propose considering four general choice mechanisms. In the first three, a DM chooses from a list of sets, taking one set of alternatives at a time. Credibly, she acts as in the classical choice model, namely her decision making behavior can be represented by a choice function that selects alternative(s) from one set at a time. Finally, the fourth general choice mechanism is induced by a binary relation defined over the sets (i.e. a *hyperrelation*) of a list. Thus:

(1) We first analyze an *iterative search with memory mechanism* (see among others Masatlioglu and Nakajima, (2013)), a choice procedure that typically arises, for instance, when a DM buys books grouped by categories (noir, short story, comedy, etc.) from an online library. As she moves

from one category to another, she sees a new set of books, that includes previously purchased books (as per the website algorithm's suggestion) with the invitation to buy them again (the memory), plus the books belonging to the new category explored. Following this example, we show that if a DM who complies with the NR-rationality postulate never buys boring and uninteresting books that she excluded in a previous screening, then her rationality is not sensitive to any particular order of presentation of the different sets of alternatives in a list (see Theorem 1 below). This type of rationality was called *pseudorationality* by Moulin (1985) and matches the PI rationality postulate.

(2) Then, we consider the issue of solving the smaller parts of a complex choice problem *sequentially* in order to gain a better understanding of the problem. This mental technique has been applied in mathematics, logic, and decision processes, since before Aristotle. In the cognitive sciences, Newell and Simon (1971) showed that one mental strategy for solving a complicated problem is to analyze parts of it sequentially so as to minimize dependence between parts and maximize the possibility of obtaining the best solution. In economics, people choose sequentially from a list of sets into which a set of alternatives has been divided. For instance, in many-to-many matching models (see e.g. Aygun and Sonmez (2012), Echenique and Oviedo (2006) and Roth and Sotomayor (1990)), colleges are faced with sets of candidates divided into groups according to their preferences for the colleges; candidates who have one college as first choice are grouped in the first set, others for whom that college is the second choice are in the second set of candidates and so on. Candidates thus partitioned form lists of sets of alternatives from which colleges are called upon to choose. A selection committee choosing sequentially from those sets decides to offer a place to some candidates, who therefore represent the committee's choice from a list of sets of alternatives. In order to provide a rationale for this issue, we construct a choice function on lists of sets by implementing the classical choice function f that selects sequentially from the element-sets of a list. We show that this choice function on lists satisfies NR if and only if the choice function f inducing the CFL satisfies the Path-independent axiom (see Theorem 2 below). This result is important because it can be interpreted as a new characterizations of the much studied class of path-independent choice functions. The NR property (see (1) below) can therefore be seen as a generalization (or a rationality requested extended to lists of sets) of the Path-independent axiom, characterizing the choice functions studied in Danilov and Koshevoy (2005) and Plott (1973), to the present setting.

(3) Next, we consider a sort of intertemporal choice model where commodities are collected not only by their physical attributes, but also by the date they are bought and consumed. So, the DM's choice problem consists in selecting a finite horizon consumption stream that is a selection of commodities from sets of alternatives available at each time t. With this interpretation in mind, we propose a third mechanism of choice from lists of sets of alternatives in which a DM chooses her best alternative(s) *separately* from each set. It is *rational* for her to focus on alternatives of great worth (*Matroidal axiom*, (M), see below and Danilov and Koshevoy, (2009)) and to ignore those that are valueless (Outcast axiom). We show that the behavior of a DM, who *separably* selects alternatives from each set in a list and is rational in the NR sense, is consistent with the M and O rationality principles and can also be represented by a *dichotomous choice function* (see Danilov and Koshevoy, (2009)), a contraction operator that divides alternatives in 'acceptable' and 'unacceptable' according to their value for the DM (see Theorem 3).

(4) Finally, we observe that if the alternatives in the sets of a list are *not mutually exclusive*, the DM needs to look for a heuristic tool to help her decide the best solution-set of the many available to solve her choice problem. So, we have to introduce *preferences over sets*, (called *hyperrelation* by Aizerman and Aleskerov (1995)), establishing whether a set of alternatives A is at least as good as another set B. This is the case, much studied in the literature on ranking of opportunity sets (see Barberà, Bossert and Pattanaik (2004)), of a DM who has preferences for wider sets of alternatives (*preference for flexibility*, see e.g. Kreps (1979)) and compares sets by considering opportunities *essential* for a future choice. We show that the 'choice function on lists of sets' induced by a hyperrelation fulfilling a *monotonicity* property, satisfies NR if and only if the hyperrelation over the sets of alternatives in a list is also transitive and satisfies a compelling rationality property (used among others by Kreps (1979) and Danilov, Koshevoy and Savaglio (2015)) requiring that the union of two sets, each of which is worse than a third set, is still worse than the latter (see Theorem 4 below). In particular, we show that when a hyperrelation induces a total well-order on the elements of a list, the CFL, that satisfies NR postulate, associates to any list of sets the maximal (i.e. undominated with respect to the hyperrelation) set.

We remark here the diversity between the classical model of choice from a set of alternatives and the model studied in the present work. We observe that it makes a significant and practical difference choosing from an (eventually large) set that is the union of sets of alternatives or from each of these sets presented (one after the other) in form of a list. The former choice could be regarded as a inherently difficult problem, whose solution requires a significant amount of computational resources (congestion effect). Moreover, the choice from a set in which a DM has to scrutinize many alternatives at a time completely neglects that the choice process typically takes place in time. It can be shown that the present model may be very helpful in both those connections.
The rest of the paper is organized as follows. Section 2 provides notation and the main definitions and reviews some elements of the classical choice model useful for comparison purposes. Section 3 and 4 show the main results and Section 5 contains some final comments, briefly discussing the literature and collecting some prominent examples.

2 Notation, definitions and preliminary results

2.1 Choice functions and rationality postulates

Let X be a finite set of alternatives and 2^X be the set of all possible subsets of X. A *choice function* $f : 2^X \to 2^X$ is a contraction operator, i.e. for any $A \in 2^X$, $f(A) \subseteq A$. The set of all choice functions is denoted with $CF(X)$. The empty choice is allowed, namely for some set $A \in 2^X$, it may be that $f(A) = \emptyset$.
A choice function $f \in CF(X)$ is usually regarded as representing the behavior of a *rational* DM if it somehow fulfills a *principle of consistency*, namely some suitable criterion which, if satisfied, prevents any logical contradiction. In the classical choice model, where a DM chooses an alternative(s) from a set, she is considered to be *rational* if the choice function, that describes her behavior, satisfies at least some of the following well-known rationality conditions, all thoroughly discussed in the theory of choice literature.

Heritage (H) For any A, $B \in 2^X$, if $A \subseteq B$ then $f(B) \cap A \subseteq f(A)$.

The Heritage condition (see e.g. postulate 4 in Chernoff (1954), axiom α in Sen (1976), Aizerman and Malishevski (1981), Aizerman and Aleskerov (1995)) means that if an alternative a is chosen from a set B, then it is also chosen from the smaller set $A \subseteq B$ including a. This is a very basic requirement in the classical approach to choice theory (see e.g. Moulin, (1985)).

Outcast (O) For any $A, B \in P(X)$, if $f(A) \subset B \subset A$, then $f(B) = f(A)$.

Outcast property (see e.g. postulate 5 in Chernoff (1954), Aizerman and Aleskerov (1995), Aizerman and Malishevski (1981), Independence of Irrelevant Alternatives axiom in Manzini and Mariotti (2007), Danilov (2012)) says that removing the alternatives that are not chosen from a set does not affect the worth of the set.
We recall here that for single-valued choice functions, the Outcast and Heritage axioms are equivalent and provide a rational choice with respect to a linear order defined on the set of alternatives. We also underline here that any choice function may be obtained as the intersection of Outcast choice functions (see Aizerman and Malishevski (1981)).
Plott (1973) proposed considering a choice to be *rational* if it does not depend on the way we divide the set of alternatives, namely the DM's choice does not depend on any particular order of presentation of the alternatives. This means that if a set A is divided into two subsets B and C, then making the choice from A must be the same as making a choice first from B and then from C, or the other way round, and finally making a choice from the union of these two choice sets. Analytically:

Path-independence (PI) For any $A \in 2^X$, with $A = B \cup C$, $f(A) = f(f(B) \cup f(C))$.[1]

A choice function that satisfies Path-independence is called a Plott function in Danilov and Koshevoy (2005) and usually *path-independent*, with the difference that the non-emptiness of the choice sets is not required for Plott functions. We underline that for a choice function $f \in CF(X)$ satisfying both Heritage and Outcast axioms is tantamount to satisfy Path-independence (see Aizerman and Malishevski (1981) and Lemma 6 in Moulin (1985)).
All the above axioms guarantee a (certain) *rational choice behavior.*

2.2 Choice functions on lists of sets of alternatives and the No-Regret axiom

We now analyze when a DM, who chooses from *lists of sets of alternatives*, could be considered *rational.*

We first define a *list* $\mathcal{A} = (A_1, \ldots, A_k)$ as a finite collection of sets of alternatives from $A \in 2^X$, where $k := l(\mathcal{A})$ is the *length* and $\cup A_i$ is the *support* of the list $\mathcal{A}$. The set of all lists with support in all subsets of X is denoted by $\mathcal{L}$. Then, we call a mapping $F : \mathcal{L} \to 2^X$, from a list $\mathcal{A} = (A_1, \ldots, A_k) \in \mathcal{L}$ into a subset of its support, i.e. $F(\mathcal{A}) \subseteq \cup_i A_i$, a **choice function on lists of sets** (CFL). A CFL represents the behavior of a DM who chooses from a list $\mathcal{A} = (A_1, \ldots, A_k)$ of

[1] We acknowledge here that Afriat (1967) was actually the first to propose 'path independence' as a suitable rationality property of a choice function.

different sets at times $t = 1, \ldots, k$. We claim that A DM can be considered *rational* in the present context if she will never feel *regret* for what she could have but did not choose: the alternatives not chosen may be removed from the list because they are uninteresting and will never be considered. Then, a *rational* DM will never rethink or revise her decision, but stands firm without any second thought.

The rational behavior of a DM, who chooses from lists of sets and follows the (normative) principle of disregarding worthless alternatives in her inter-temporal choice process, is analytically captured by the following:

Definition 1 *A choice function on lists of sets $F : \mathcal{L} \to 2^X$ satisfies No-Regret property (NR) if, for any list $\mathcal{A} = (A_1, \ldots, A_k) \in \mathcal{L}$ and each list $\mathcal{B} = (F(\mathcal{A}) \cup B_1, B_2, \ldots, B_k) \in \mathcal{L}$ such that $\cup_j B_j \subseteq \cup_i A_i \setminus F(\mathcal{A})$:*

$$F(\mathcal{B}) \cap (\cup_j B_j) = \emptyset. \tag{1}$$

We notice that $F(\mathcal{A})$ may be empty. In such a case, if the choice from some list $\mathcal{A}$ is empty, then the choice from any list, the support of which is a subset of the support of $\mathcal{A}$, is also empty.

No-Regret postulate requires that, in a choice procedure, the alternatives that a DM disregards because they are of no use for her will never again be chosen. Definition 1 entails a sort of *choice-consistency requirement* in two-steps. First, some alternatives are chosen from sets ordered in a list. If a new list composed of the set of alternatives chosen at the first step and (some of the) sets of alternatives previously rejected, is proposed to the DM she will then continue to only select alternatives from the former set, ignoring those from the latter sets. The following example could help to better understand how NR axiom properly works:

Example 1. Let $\mathcal{A} = (A_1 = \{a, b\}, A_2 = \{c, d, e\}, A_3 = \{f, g\})$ be a list of sets of alternatives. A DM chooses the best alternative (if any) from each set in the list according to a preference relation $\prec$, where $\prec$ is a complete and transitive binary relation (a linear order) defined on the set of alternatives. The DM's choice behavior is summarized by a the following CFL $F(\mathcal{A}) = \{a, c\}$. Suppose $\mathcal{B} = ((F(\mathcal{A}) \cup (B_1 = \{e\}), B_2 = \{f, b\})$, then, if the DM is consistent with the NR-rationality postulate, $F(\mathcal{B}) \subseteq F(\mathcal{A})$.

We observe that the sets of alternatives in a list are available for choice one after the other. A DM can modify her selection dynamically on the basis of the sets she faces each time. This dynamical procedure does not require an updating of beliefs and no new information comes from list $\mathcal{B}$ of Definition 1. Indeed, if a DM makes a choice $F(\mathcal{A})$ from a list $\mathcal{A} = (A_1, \ldots, A_k) \in \mathcal{L}$, then, for any list, with $F(\mathcal{A})$ as the first set and all other sets made up of alternatives that were not chosen before, her choice will again be from $F(\mathcal{A})$, i.e. $\mathcal{B}$ does not contain any new alternative worth choosing except those already chosen and in $F(\mathcal{A})$. Moreover, we note that the choice consistency implied by the NR axiom has nothing to do with the so-called Bayesian dynamical consistency since the list of sets $\mathcal{B}$ of Definition 1 brings no new information. The DM is therefore NR-rational, namely is consistent in her choice from lists of sets, if she simply never regrets alternatives that she previously judged as uninteresting.

We finally remark that the primacy of $F(\mathcal{A})$ in list $\mathcal{B}$ of Definition 1 emphasizes the dynamical aspect of the NR axiom. $F(\mathcal{A})$ is the choice *already made* when the DM faces list $\mathcal{B}$: she has already selected what interests her. Moreover, since a variety of cognitive and procedural effects suggest that people pay more attention to the first few alternatives they face, we also want to emphasize the role of $F(\mathcal{A})$ by putting such item first in list $\mathcal{B}$. The first element in a list somehow serves as a reference to which subsequent sets of alternatives are compared (see. e.g. Tversky and Kahnemann (1991)).

2.3 No-Regret axiom as a generalization of the Outcast property

The NR property is a variant of the principle of *independence of irrelevant alternatives* (see e.g. Manzini and Mariotti (2007)). It indeed is a generalization to the present more comprehensive model of the classic Outcast axiom that can be defined equivalently as:

$$for\ any \quad B \subset A \setminus f(A), \quad f(B \cup f(A)) \subset f(A), \tag{2}$$

meaning that alternatives not-considered will never be chosen and as such they can be removed without any loss of value for the set.[2]

**That NR axiom is a generalization of Outcast postulate is evident if we consider a list $\mathcal{A} = (A_1, A_2)$, a choice function on lists of sets $F : \mathcal{L} \to 2^X$, and suppose that $F(\mathcal{A}) \subset A_1$, $A_2 \subset A_1 \setminus F(\mathcal{A})$ and $\mathcal{B} = (F(\mathcal{A}), A_1 \setminus (F(\mathcal{A}) \cup A_2), A_2)$. Then, by NR, $F(\mathcal{B}) \cap ((A_1 \setminus (F(\mathcal{A}) \cup A_2)) \cup A_2) = \emptyset$, that is tantamount to $F(A_2 \cup F(\mathcal{A})) \subset F(\mathcal{A})$, i.e. (2) above.

It is known that the class of choice functions satisfying Outcast axiom is stable under union (but not under intersection), namely for any two choice functions $f, g \in CF(X)$ satisfying (2), $f \cup g$, defined as $(f \cup g)(A) = f(A) \cup g(A)$, also satisfies the Outcast axiom. Analogously, observe that, if F, G are two choice functions on lists of sets satisfying NR-postulate, then their union, defined as $(F \cup G)(\mathcal{A}) = F(\mathcal{A}) \cup G(\mathcal{A})$ is also a CFL that satisfies NR. Namely:

Proposition 1 *The set of "choice functions on lists of sets of alternatives" which satisfy No-Regret axiom is stable under union.*

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ satisfy the NR-property. We have to check that for any list $\mathcal{A} = (A_1, \ldots, A_k)$ and $\mathcal{B} = (B_1, \ldots, B_k)$ such that

$$\cup_t B_t \subset \cup_t A_t \setminus (\mathcal{F}(\mathcal{A}) \cup \mathcal{G}(\mathcal{A})),$$

$$(\mathcal{F} \cup \mathcal{G})(\mathcal{F}(\mathcal{A}) \cup G(\mathcal{A}) \cup \mathcal{B}) \cap \mathcal{B} = \emptyset \tag{3}$$

Since $\mathcal{F}$ and $\mathcal{G}$ satisfy N, we have:

$$\mathcal{F}(\mathcal{F}(\mathcal{A}) \cup G(\mathcal{A}) \cup \mathcal{B}) \cap \mathcal{B} = \emptyset, \tag{4}$$

and

$$\mathcal{G}(\mathcal{F}(\mathcal{A}) \cup G(\mathcal{A}) \cup \mathcal{B}) \cap \mathcal{B} = \emptyset \tag{5}$$

From (4) and (5) we obtain (3).

□

That NR is a generalization of Outcast is also evident by observing what follows.

We know that if $B \subset X$ is a set of "bliss" elements of X, then the choice function $f_B(A) = B \cap A$, selecting the bliss-alternatives that, if available, are in A, is an Outcast choice function. At

[2]We recall that the class of choice functions satisfying the Outcast property is studied in Aizerman and Malishevski (1981) and Brandt and Harrenstein (2011), and is characterized in Danilov (2012).

the same time, if $B \subset X$ is any set of *bliss* alternatives, the choice function on lists of sets $F_B(\mathcal{A}) = \cup(B \cap A_1, \ldots, B \cap A_k)$ satisfies the NR property.
Finally, we note that the CFL satisfying NR has a maximal element, defined as: **

$$\mathbf{1}(\mathcal{A}) = \cup_i A_i.$$

Therefore, $(CFL, \cup)$ is a *semi-lattice* with respect to the union operator, exactly like the class of choice functions satisfying the Outcast postulate.

We are now ready to discuss four *general choice mechanisms* from lists of sets of alternatives.

3 "Choice functions on lists of sets" relying on choice functions

**Economic decisions typically have an inter-temporal dimension and next-date available alternatives often are part of the DM's choice problem. The standard, largely accepted, approach to choice over time is the exponential discounting model (*edm*) that evaluates an alternative x available at any time t, (with $t = 0, 1, \ldots, T$), at time $t = 0$. We are aware that the assessment of a far in time alternative in terms of its present value is a non-trivial matter, however we wonder whether the *edm* is a reliable and effective dynamic model of choice over time. The evaluation in the present of an alternative that will be available in the future does not in fact correspond to a truly dynamic approach to the choice. A DM compares alternatives available in the time to come according to her current preferences, but the latter (and the choice) could evolve over time. The temporal dimension is therefore (partially) neglected by *edm* where the choice problem is just solved at $t = 0$. On the contrary, the present framework allows for a real choice over time: the DM faces one set after another in a list and the order of the sets exactly expresses the dynamics of a choice that occurs through time.
Now, if a DM chooses from a list of sets and takes *one set after another*, then her behavior can plausibly be described by a (classical) choice function $f \in CF(X)$ as applied to a (single) set of alternatives (at a time). For the case of choice from a list with a single set, we indeed have $F(A) = f(A)$.

We further observe that a unique prescribed selection mechanism to choose from lists of inter-temporally available sets does *not* exist. We then analyze here three general and compelling choice procedures of *inter-temporal* choice from lists of sets, all relying on choice functions defined on each set (of alternatives) in a list and all analytically corresponding to everyday examples.

3.1 Iterative search with memory

A college (DM) chooses from sets of candidates divided into groups (the sets of a list) according to their preferences for different colleges. Candidates who listed the college as tth in their preference order appear at time t. Hence, a college's choice from a list of candidates proceeds as follows: C_1 is the choice from the set of candidates who ranked this college as their first choice, C_2 is the choice from the union of C_1 (used for comparison purposes) and the set of candidates who were not chosen by the colleges they ranked first and who ranked this college as their second option, and so on. At each step, a college considers and chooses those who were not considered by other colleges and eventually adds them to the previously chosen candidates before making a new choice.

We observe that in this procedure, a college *can not discard candidates chosen in previous steps* and it also keeps them in mind when it undertakes a new selection.

Analytically, for any list $\mathcal{A} = (A_1, \ldots, A_k)$, we define the following *iterative* choice procedure for $f \in CF(X)$:

$$\begin{aligned} C_1 &= f(A_1), \\ C_2 &= f(C_1 \cup A_2), \\ &\ldots \\ C_k &= f(C_{k-1} \cup A_k). \end{aligned} \tag{6}$$

and set a CFL relying on f as follows:

$$F_f(\mathcal{A}) = \bigcup_{i=1}^{k} C_i.$$

This means that if a list is composed of only two sets (A_1, A_2), we have:

$$F_f(A_1, A_2) = f(A_1) \cup f(f(A_1) \cup A_2).$$

In words, a DM considers the set A_1 at time 1 and makes the choice $f(A_1)$. Then, she considers set A_2 at time 2, but she has already made choice $f(A_1)$, that is now *memorized* and acts as a reference for comparison purposes, so she has to choose from $A_2 \cup f(A_1)$. Finally, since choice $f(A_1)$ was made at time 1, the DM has to add $f(A_1)$ to the choice at time 2. It is worth noticing that the present choice procedure crucially *depends on the order of the sets in the list.* Indeed, for the list $\mathcal{A} = (A_2, A_1)$, we obtain $F_f(A_2, A_1) = f(A_2) \cup f(f(A_2) \cup A_1)$, which is a choice set different from $F_f(A_1, A_2) = f(A_1) \cup f(A_2 \cup f(A_1))$. It entails that the choice mechanism underlying F_f is non-commutative, namely it is not invariant under permutations of addenda. This makes the choice procedure in (6) an antisymmetric operation yielding a result that depends on the swapping of the arguments of the list.
The selection mechanism (6) describes how a *search* for desired alternatives depends *iteratively* on when the DM considered sets to choose from and on what she has already selected (*memory*): the inter-temporal choices are in fact closely connected.

For an iterative choice (with memory) from sets in a list, the NR axiom entails that, for any list $\mathcal{A} = (A_1, \ldots, A_k)$ and any set $B \subset \cup A_i \setminus \cup C_i$, with C_i defined in (6):

$$f(f(C_1 \cup \cdots \cup C_k) \cup B) \cap B = \emptyset.$$

Now, a normative issue rapidly arises: how should a DM, who chooses from lists of sets according to the choice mechanism in (6) and disregards those alternatives that are unsuitable, behave in order to be consistent and non-contradictory in her choice and therefore to be considered rational? The answer is in the following:

Theorem 1 *For $f \in CF(X)$, F_f is a choice function on lists of sets that satisfies No-Regret axiom if and only if f is Path-independent.*

Proof. ($\Leftarrow$) Let f be a Path-independent choice function and, for a list $\mathcal{A} = (A_1, \ldots, A_k)$, the sets C_i be defined as in (6). Then,

$$f(C_1 \cup \cdots \cup C_k) = C_k.$$

For a two-step list $(f(C_1 \cup \dots \cup C_k), D)$, we obtain:

$$C'_1 = C_k,\ \ C'_2 = f(C_k \cup D).$$

By Path-independence, we get:

$$f(A_1 \cup A_2 \cup \dots \cup A_k) = f(f(A_1) \cup A_2 \cup \dots \cup A_k) = f(f(f(A_1) \cup A_2) \cup \dots \cup A_k) = \dots = C_k.$$

So, $C_2 = f(f(A_1 \cup \dots \cup A_k) \cup D) = f(A_1 \cup \dots \cup A_k \cup D) = f(A_1 \cup \dots \cup A_k) = C_k$, i.e. the No-Regret axiom is verified.

($\Rightarrow$) Let F_f be a CFL that satisfies NR. Then f satisfies the Outcast axiom and is idempotent. We now have to check that the Heritage axiom is also satisfied. Suppose it is not, i.e. that H is violated for a pair $A \subset B$. This means that $A \cap f(B)$ is not a subset of $f(A)$. Let us denote $C := (A \cap f(B)) \setminus f(A)$, with C that is nonempty. Consider the following partition:

$$B = A \coprod B \setminus A.$$

We denote $D := f(f(A) \cup (B \setminus A))$. Then, since C is non-empty, $E := f(B) \setminus D$ is also non-empty.

So, $f(B) \subset D \cup E \subset B$, due to Outcast, we have:

$$f(D \cup E) = f(B) \ \Rightarrow\ E = f(B) \cap E.$$

That is not the case due to NR. The implication therefore holds true.

□

Theorem 1 concludes that a DM who chooses alternatives from lists of sets following the procedure (6) and ignores unsuitable alternatives is *rational* if her choice behavior on each set in a list can be represented by a Path-independent choice function.

3.2 Sequential search

The behavior of a firm (hospital) that chooses *sequentially* from sets of applicants (doctors), (divided according to specializations), those who best fit the different vacancies it offers (see e.g. Chambers and Yenmez, (2017)) can be represented by the following ‘choice function on lists of sets’:

$$G_f(\mathcal{A}) = C_k,$$

where C_k is defined as in (6), namely:

$$G_f(\mathcal{A}) := f(A_k \cup (f(A_{k-1} \cup (\dots \cup f(A_2 \cup f(A_1)))))), \tag{7}$$

with $\mathcal{A} = (A_1, \dots, A_k) \in \mathcal{L}$ and $f \in CF(X)$.

G_f could equally represent the behavior of a college in an admission model. It stands for the mechanism of selection from sets of candidates divided according to their preferences for the different colleges. So, a college first chooses from the set A_1 of candidates who ranked the college as their first choice, then from the set of candidates who indicated the college as their second choice and so on, where the college can *reject* candidates already chosen. We recall that the *choice procedure* (7) was widely used for establishing the existence of *stable matching* (see e.g. Aygun and Sonmez (2012), Echenique and Oviedo, (2006) and Roth and Sotomayor (1990)).

Now, in order to copy with the choice-tasks complexity of the sequential choice protocol (7), a DM must (necessarily) neglect those alternatives that are of no value for her, namely she must behave according to the following adaptation to the choice mechanism under (7) of the No-Regret Axiom:

for any $f \in CF(X)$, any list $\mathcal{A} = (A_1, \cdots, A_k)$, and any $B \subset \cup_i A_i \setminus C_k$:

$$G_f(G_f(\mathcal{A}) \cup B) \cap B = \emptyset. \tag{8}$$

We show that the class of CFL in (7) that satisfies NR in (8) coincides with the class of choice functions introduced by Plott (1973), namely:

Theorem 2 *For any $f \in CF(X)$ and any $\mathcal{A}= (A_1, \ldots, A_s) \in \mathcal{L}$, the following statements are equivalent:*

1. *G_f satisfies NR property in (8);*
2. *f is a Path-independent choice function.*[3]

The rationality of a DM who chooses *sequentially* from sets of alternatives in a list, disregarding alternatives that do not help solving her choice problem, therefore coincides with the (pseudo-)rationality (see Moulin, (1985)) of a DM whose choice is not influenced by the particular position of the alternatives, namely a DM who is 'immune' to *manipulation* of the alternatives (*strategy-proof property*). It is also worth noticing that Theorem 2 can be considered a *new characterization* of the rich class of choice functions proposed by Plott (1973) and the *No-Regret* property can therefore be read as an extension of the Path-independent rationality axiom for the present sub-class of CFL.

**We recall that Manzini and Mariotti (2007) have analyzed in depth one kind of sequential choice procedure that identifies a unique alternative from any feasible set after that one preference relation after another is applied to remove all the worthless alternatives. In other words, in Manzini and Mariotti (2007) a first choice function represents a screening process (focalisation phase), then a second one is a single-valued choice function, that is rationalizable as maximization of a linear order. Here, we do not revise the choice from a set, obtained with a choice function, by using another choice function, but the choice from a set has to be added or combined with the choice from another set and so on.

3.2.1 Iterative and sequential search: the difference

The following example illustrates the difference between a choice from lists of sets adopting the iterative search (with memory) procedure in (6) and one applying the sequential search approach in (7), when both are based on the same path-independent choice function f.

Suppose a set of the integer points $[N] := \{1, 2, ..., N\}$ of the real line $\mathbb{R}$ is the set of alternatives. For a subset $\{i_1 < i_2 < .. < i_k\} \subset [N]$, a path-independent choice function f selects the extremal points i_1 and i_k if $k \neq 1$ and i_1 for the singleton $\{i_1\}$, that is:

$$f(i_1 < i_2 < .. < i_k) = \{i_1, i_k\}.$$

For $k < N/2$, take a list $\mathcal{A} = (A_1, A_2, \ldots, A_k)$ consisting of the set $A_1 = \{k, k+1, ..., N-k, N-k+1\}$, and two-elements sets $A_2 = \{k-1, N-(k-1)+1\}$, $A_3 = \{k-2, N-(k-2)+1\}$, $\ldots$, $A_{k-1} = \{1, N\}$. Then, by the iterative search with memory procedure (Theorem 1), the choice

[3] The proof follows the same line of arguments as that of Theorem 1 and is therefore omitted.

from list $\mathcal{A}$ is $\{1, 2, ..., k, N-k+1, N-k+2, .., N\}$, and the rejected elements are $\{k+1, ..., N-k\}$, namely a subset of A_1. Then, NR axiom entails a choice that is always the two-elements set $\{1, N\}$.

Notice that a choice from a list of sets that does not satisfy NR postulate, for example, $\mathcal{B}' = (B_1, B_2)$ with $B_1 = \{k+1, .., N-k\}$ and $B_2 = \{1, 2, ..., k, N-k+1, N-k+2, .., N\}$, contains two elements previously disregarded, namely $\{k+1, N-k\}$, since $F_f(\mathcal{B}') = \{1, k+1, N-k, N\}$.

On the other hand, for the sequential search procedure (Theorem 2), the choice from list $\mathcal{A}$ is $\{1, N\}$, and due to NR we get the same choice from all lists combined according to the NR axiom. Notice that in this case, the choice is stable with respect to the permutation of elements of the list.

3.3 Separable choice

Another interesting (sub-)class of 'choice functions on lists of sets', built by applying a choice function $f \in CF(X)$ to each set in a list, is the following:

$$\begin{aligned} C_1 &= f(A_1), \\ C_2 &= C_1 \cup f(A_2), \\ &\dots \\ C_k &= C_{k-1} \cup f(A_k). \end{aligned} \tag{9}$$

with:

$$I_f(\mathcal{A}) = C_k = f(A_1) \cup f(A_2) \cup \cdots \cup f(A_k).$$

At least three possible suggested interpretations of the choice procedure (9) can help the reader understand the underlying selection mechanism. I_f may represent the behavior of a DM (*i*) who compiles her wish-list choosing from (a list of) different departments (the sets of alternatives) of an online market; (*ii*) who chooses a meal selecting from appetizers, main courses, desserts etc, i.e. the sets of dishes customary on a menu (the list); (*iii*) who buys different bonds from the sets A_t of securities available at stock exchange t, for $t = 1, \dots, k$. According to (9), the DM chooses the best elements from the sets of alternatives available in different periods, irrespective of the choices she made in other moments.

In the present case, the NR property reads as follows:

$$f(A \cup B) \subset f(A) \cup f(B), \tag{10}$$

$$f(A \setminus f(A)) = \emptyset, \tag{11}$$

and

$$for\ any \quad B \subset A \setminus f(A), \quad f(B \cup f(A)) \subset f(A) \tag{12}$$

where (10) says that the choice from the union of two sets is a smaller set than the set obtained as the union of the choice from the two sets; (11) is the so-called *Matroidal* property identified in Danilov and Koshevoy (2009), meaning that removing a 'good' alternative does not make 'bad' alternatives 'good'; and (12) is a restatement of the *Outcast* property mentioned above. In order to characterize I_f, we need to introduce the notion of *dichotomous* choice function, namely a $f \in CF(X)$ such that, for any $A \in 2^X$, $f(A) = A \cap f(X)$. In words, a dichotomous choice function divides all alternatives of a set into "*acceptable*" (those belonging to $f(X)$) and *non-acceptable*. Thus, for any $A \in 2^X$, it only selects the acceptable alternatives in A. We recall here that a dichotomous choice function satisfies both the *Heritage* and the *Matroidal* property (see Danilov

and Koshevoy, (2009)).
We can therefore state that:

Theorem 3 *For any choice function $f \in CF(X)$, I_f is a choice function on lists of sets which satisfies NR axiom if and only if, for some $B \subset X$, f_B is dichotomous.*

Proof. It is easy to check that, for a dichotomous f_B, I_{f_B} satisfies the NR-property. Vice versa, from NR sub-(10), (11) and (12), for $x \in f(X)$, we obtain that $f(x) \neq \emptyset$, and, for any $y \in X \setminus f(X)$, that $f(y) = \emptyset$. Thus, we have:

$$f(A) = A \cap f(X).$$

In fact, if $f(A) = \emptyset$, then this obviously holds. Let $f(A) \neq \emptyset$, then, due to (10), for any $a \in f(A)$, $f(a) \neq \emptyset$ and hence $a \in f(X)$.

□

Theorem 3 highlights the normative content of NR postulate.**

A DM chooses all 'satisficing' (in the sense of Simon (1955)) alternatives from a set: a rational behavior compatible (consistent) with the kind of rationality expressed by Heritage axiom (see Danilov and Koshevoy (2009)).

A DM that collects all 'satisficing' alternatives following the choice protocol (9) has some difficulty in processing too much information, namely the set of alternatives chosen from a larger set is smaller than the set of alternatives chosen from smaller sets (see condition (10) above). This "congestion effect" makes clear how the issue of solving a complex problem can more easily be addressed step-by-step, analyzing parts of it sequentially so as to minimize dependencies between parts and maximize the possibility of obtaining the best solutions.

Moreover, the DM is aware that there is nothing of value in any set of alternatives A of a list except her choice ($f(A)$) (condition (11) above). Stated differently, if $x \in f(A \cup x)$, i.e. there is no alternative better than x in A, then $f(A) \subset f(A \cup x)$, a condition that is equivalent to (11) and that can be rephrased as "deleting a 'suitable' option does not make 'worthless' alternatives 'suitable' ".

Finally, in solving her choice problem, the DM is allowed to discard alternatives in a set that are not chosen and she is aware that such a removing operation does not affect the intrinsic value of the set (condition (12)).

Conditions (10, 11, 12) entail the NR axiom for the separable choice mechanism in (9) and prescribe the rule of behavior that a DM must obey: 'choose only the satisficing alternatives'.

It is worth concluding by observing the following easy:

Corollary 1 *If $f \in CF(X)$ is dichotomous, then Theorem 1, 2 and 3 entail the same outputs.*

4 'Choice function on lists of sets' induced by a hyperrelation

In many remarkable economic situations, for instance, in matching theory (see e.g. Echenique and Oviedo (2006) and Roth and Sotomayor (1990)), certain voting procedures (see e.g. Brams and Fishburn, (2002)), coalition formation (see e.g. Ray and Vohra, (2014)) and ranking sets of opportunity (see e.g. Barberà, Bossert and Pattanaik (2004)), the choice is from a set with elements that are *not mutually exclusive*. In such cases, the DM needs to make *comparisons between sets* in order to find solutions to her choice problem. It may therefore be necessary to make *preferences on sets*, i.e. binary *hyperrelations* (see e.g. Aizerman and Aleskerov (1995), Danilov, Koshevoy and Savaglio, (2015), Kreps (1979)), a primitive.

So, let $\preceq$ be a hyperrelation defined on the *not mutually exclusive* elements of a list $\mathcal{A} = (A_1, \ldots, A_k)$, a DM chooses by the following mechanism:

$$P_1 = A_1$$

$$P_2 = \begin{cases} P_1 \text{ if } P_1 \succ A_2 \\ A_2 \text{ if } P_1 \prec A_2 \\ P_1 \cup A_2 \text{ otherwise} \end{cases}$$

$$\ldots\ldots$$

$$P_k = \begin{cases} P_{k-1} \text{ if } P_{k-1} \succ A_k \\ A_k \text{ if } P_{k-1} \prec A_k \\ P_{k-1} \cup A_k \text{ otherwise} \end{cases}$$

that defines the following CFL:

$$R_{\preceq}(\mathcal{A}) = P_k.$$

representing the behavior of a DM who adds set A_t at time t if it is not dominated by the union of previously chosen sets. Equivalently, we write:

$$R_{\preceq}(A_1, \ldots, A_k) = \bigcup_{i=1}^{J} A_j, \tag{13}$$

where $j \in J$ if and only if there is no $j' < j$ such that $A_j \preceq A_{j'}$.

Expression (13) means that a choice function on lists of sets of alternatives only selects the top elements of a list of sets with respect to a hyperrelation $\preceq$.

This is a typical situation that happens, for instance, when an international organization is recruiting a team of experts to work jointly on some project. They are collected according to the different deadlines for application fixed by the organization for that year. At each step of the selection, the organization decides either to reject the set of experts already selected if the new set has better skills or not to consider the new set of applicants or just to enlarge the team with (some members of) the latter group. Another example of the aforementioned choice procedure concerns a DM who has preferences for wider sets of alternatives (*preference for flexibility*). She prefers one set to another if the latter is a subset of the former or if the first contains more valuable opportunities for a future choice than the second. If two disjoint sets have distinct opportunities worth considering for a future choice, then the DM chooses both.

Thus, the proposed choice procedure (13) is definitely related to the literature on 'preferences for flexibility' (see, among others, Kreps (1979)), in which a hyperrelation $\preceq$ typically satisfies (see e.g. Kreps (1979), Danilov, Koshevoy and Savaglio, (2015)) the following two compelling properties:

- **Monotonicity with respect to set inclusion** (Mon). For all $A, B \in 2^X$, $A \subseteq B$ implies $A \preceq B$.

- **Union** (U). For any A, A', $B \in 2^X$ $A \preceq B$ and $A' \preceq B$ imply $A \cup A' \preceq B$.

Mon is a suitable axiom, discussed in the economic literature on ranking sets of opportunities in terms of freedom of choice (see Barberà, Bossert and Pattanaik, 2004). It entails that any set is usually (weakly) preferred to any of its subsets. The Union property simply states that the union of two sets both worse than another is still worse than the latter. Both properties have a straightforward normative content: a rational DM should rather prefer more alternatives to choose than less (Mon) and a good alternative than two bad.

Now, what is the relation between a DM that subscribes to the normative requests of Mon and U in comparing sets of alternatives and one that conforms to the rationality expressed by the NR property when she chooses from a list of sets (of alternatives)? The following result provides an answer:

Theorem 4 *If $\preceq$ is a monotone hyperrelation, then $R_{\preceq}$ satisfies No-Regret axiom if and only if $\preceq$ is transitive and satisfies Union axiom.*

Proof. ($\Rightarrow$) We have to show that if $R_{\preceq}$ satisfies NR, then the monotone hyperrelation $\preceq$ is transitive, namely $A_3 \preceq A_2 \preceq A_1$ implies $A_3 \preceq A_1$. Consider the lists (A_1, A_2, A_3) and (A_1, A_3, A_2). In fact, by (13), $R_{\preceq}(A_1, A_2, A_3) = A_1$. Then, by NR, $R_{\preceq}(A_1, A_3, A_2) \subseteq A_1$ and by definition of $R_{\preceq}$, we get the equality not embedding. Hence, $A_3 \preceq A_1$. In order to show that $\preceq$ satisfies **U**, consider the list $(B, A, A^{'})$. By definition (13), we get $R_{\preceq}(B, A, A^{'}) = B$. Hence, by NR, $R_{\preceq}(B, A \cup A^{'}) \subseteq B$ and again, by (13), we have the equality instead of embedding. Thus, $A \cup A^{'} \preceq B$.

($\Leftarrow$) This implication is obvious.

□

The idea of connecting hyperrelations and choice functions dates back to Puppe (1996) who introduced the notion of *essential* alternatives in a set. Namely, for an hyperrelation $\preceq$, an alternative a is *essential* in A if $A \backslash a \preceq A$. In particular, for a monotone and transitive hyperrelation $\preceq$ that satisfies Union, an alternative $a \in A$ is essential if $a \in f_{\preceq}(A) = \{a \in A : a \npreceq A \backslash a\}$, where $f_{\preceq}$ denotes the choice function induced by hyperrelation $\preceq$, which selects those alternatives such that if removed the 'freedom of choice' associated with set A decreases. On the other hand, alternatives that are not essential can be removed without any serious harm to the DM's freedom (*independence of non-essential alternatives*, see Puppe (1996)). In our setting, this property reads as follows:

$$\cup_i A_i \preceq R_{\preceq}(\mathcal{A}) \tag{14}$$

The relation (14) implies that, for a list $\mathcal{A} = (A_1, \ldots, A_k)$, with support $\cup_i A_i$, $\cup_i A_i \preceq \cup_i A_i \backslash \cup_j A_j$ for all $A_j \in \cup_i A_i$ such that $A_j \notin R_{\preceq}(\mathcal{A})$. This means that removing sets that are not chosen by a DM from a list of sets does not affect her freedom of choice because those sets will never be considered for future choice (NR property) by a DM with preference for flexibility. ** Now, let $\prec$ be a hyperrelation that is a *total order* on 2^X. In particular, let $\prec$ be a *well-order*, namely a total order satisfying the no-infinite ascending chain condition (see e.g. Aizerman and Aleskerov (1995)). Then, for any list $\mathcal{A} \in \mathcal{L}$, with support $\cup_i A_i$, the well-order $\prec$ defines the following choice function on lists of sets:

$$S_{\prec}(\mathcal{A}) := \underset{\prec}{max}\{A \subset 2^X | A \in \cup_i A_i\} \tag{15}$$

We observe that (15) corresponds to (13) for the special case in which, in a list, there exists only one not-dominated set.

The suggested interpretation here refers, for instance, to a DM who scrutinizes different departments of an online market and buy items from only one of them.

We show that a CFL, satisfying No-Regret property, associates the maximal set with respect to some well-order on 2^X to any list of sets. Formally,**

Theorem 5 *A 'choice function on lists of sets' $S : \mathcal{L} \longrightarrow 2^X$ satisfies No-Regret axiom if and only if $S = S_\prec$, where $\prec$ is some well-order on 2^X satisfying Mon.*

Proof. ($\Rightarrow$) By definition, for any $\mathcal{A} = (A_i, \ldots, A_k) \in \mathcal{L}$, $S_\prec(\mathcal{A}) \subseteq \cup_i A_i$. Let A_i^* the greatest element in $\mathcal{A}$ according to the well-order $\prec$, then it is also the greatest element of any list $\mathcal{B} = (A_i^* \cup B_1, \ldots, B_k)$ whose first entry is A_i^* and all the others are sets B_j with $j = 1, \ldots, k$ such that $\cup_j B_j \subseteq \cup_i A_i \setminus A_i^*$. Thus, $S_\prec(\mathcal{B}) \cap \cup_j B_j = \emptyset$, i.e. NR axiom holds true.

($\Leftarrow$) Suppose S is a choice function on lists of sets that satisfies NR property. We call a set of alternatives $A \in 2^X$ *totally-good* if $S(A) = A$. By NR, for any $\mathcal{A} \in \mathcal{L}$, $S(\mathcal{A})$ is totally-good. For any totally-good set $A^* \in 2^X$, we call $\bar{\mathcal{A}} = \{\mathcal{A} \in \mathcal{L} | S(\mathcal{A}) = A^*\}$ the *range* of A^*. It is obvious that A^* is the only totally-good set in $\bar{\mathcal{A}}$. For any range $\bar{\mathcal{C}}$, let C_{top} be the only totally-good set in $\bar{\mathcal{C}}$. Then, we order any range $\bar{\mathcal{C}}$ by a well-order $\prec_{\bar{\mathcal{C}}}$ such that $C_{top} = \max_{\prec_{\bar{\mathcal{C}}}}(\bar{C})$. We denote with $\mathcal{E}$ the set of all totally-good sets of the CFL S. By Monotonicity, $(\mathcal{E}, \subseteq)$ is a poset, namely, for any $A, B \in \mathcal{E}$ $A \subseteq B$ implies $A \leq B$. We extend such a partial order to a total well-order $\prec$ on $\mathcal{E}$ and on 2^X as follows:

- We have that $A^* \prec A'^*$ if $A^* = max(\bar{\mathcal{A}}) < max(\bar{\mathcal{A}}') = A'^*$;
- We have $A^* \succ A'^*$ if $max(\bar{\mathcal{A}}) > max(\bar{\mathcal{A}}')$;
- if $\bar{\mathcal{A}} = \bar{\mathcal{A}}'$, we put $A^* \prec A'^*$ if $A^* \leq_{\bar{\mathcal{A}}} A'^*$ and $A^* \succ A'^*$ if $A^* \geq_{\bar{\mathcal{A}}} A'^*$.

The binary relation $\prec$ is then a well-order. We note that, for any list $\mathcal{A} = (A_1, \ldots, A_k)$, $\cup_i A_i \prec S(\mathcal{A})$ by NR. We further observe that for any list $\mathcal{B}$ whose support is such that $\cup_j B_j = \cup_i A_i \setminus S(\mathcal{A})$, one has $\cup_j B_j \leq S(\mathcal{A})$. Indeed,

- if $S(\mathcal{B}) \prec S(\mathcal{A})$, $\cup_j B_j \prec S(\mathcal{B}) \prec S(\mathcal{A})$;
- if $S(\mathcal{B}) \not\prec S(\mathcal{A})$, then, by definition of $\prec$ on $\mathcal{E}$, $A_i \nsubseteq A_i^{'}$, that is a contradiction.

We conclude that $S = S_\prec$.

□

Finally, we recall that Danilov, Koshevoy and Savaglio (2015) studied the *closure operators* constructed from hyperrelations that are transitive and satisfy Union and other compelling properties (see Proposition 5 in Danilov, Koshevoy and Savaglio (2015)). It might be worth studying the connections between 'choice functions on lists of sets' satisfying NR and the *closure operators*, but this topic is best left for future research.

5 Summary and relation with the literature

We extend the classical choice set-up, in which a DM chooses from a set of alternatives, to the case in which she makes her selection from *lists of sets of alternatives*. We collect some general procedures of choice and study a corresponding *new rationality property*, showing that it is quite general to encompass and extend other rationality notions already discussed in the theory of choice literature.

Our paper is definitively linked to Rubinstein and Salant (*RS*, (2006)), that study a choice model in which the DM encounters alternatives in the form of *lists of singletons*. In that paper, the set of all possible lists of alternatives is a set of linear orders on X, namely the set of the permuted elements $(x_{\pi(1)}, \ldots, x_{\pi(n)})$ of $X = \{x_1, \ldots, x_n\}$, where $\pi : \mathtt{N} \to \mathtt{N}$ is a permutation function. For this special case of lists of singleton sets, our rationality property of No-Regret requires that if x_i is the choice from a list, then it will be the choice from any list that starts with x_i. *RS* (2006) shows that a choice function f on a list of alternatives $\mathcal{A} = (a_1, \ldots, a_n)$ satisfies an *adapted* version of the Path-independent property if and only if f satisfies a conveniently *modified* version of the Outcast axiom. Namely, a DM chooses one alternative from a list or equivalently that same alternative contained in any of the possible disjointed sub-lists in which the list could be split (*pseudorationality*) and this is consistent with choosing that alternative neglecting all others as useless (*Outcast* or 'Independence of Irrelevant alternatives' rationality). In our setting, *RS* (2006) can be seen as a model of choice from sub-lists. **Indeed, for a list $\mathcal{A} = (x_1, \ldots, x_k)$, consider as a choice $T(x_1, \ldots, x_k) = (f(x_1), \cdots, f(x_k)) = \{x_i\}$, where f is a dichotomous function and x_i is the first elements in the list such that $f(x_i) \neq \emptyset$. The CFL T satisfies the No-Regret property, that, in such a specific case, is equivalent to the Outcast axiom, i.e. a DM chooses x_i if and only if it is the earliest occurrence of the elements in the sub-list that is "acceptable". Since NR axiom, for such a case, is equivalent to Outcast property and, by Proposition 1 in RS (2006), to the adapted version of Path-independent axiom, the example makes explicit the relation between choice from lists of sets with choice from sub-lists.
RS (2006, pgg 5-6) identify several examples on which to apply their model of choice from lists of singletons. We check below the implications of some of them for our (more general) model of choice from lists of sets of alternatives. In particular, we check whether the *rationality* of a DM, represented by a choice function on lists of sets that satisfies the NR axiom is 'consistent' (or not) with that one analyzed in some remarkable proposals from the literature of theory of choice.

Example 1. (*Rational choice*) Given a complete, asymmetric and transitive binary relation $\prec$ over a finite set X, the DM chooses the maximal element of each A_i of $\mathcal{A} = \{A_1, ..., A_k\}$, then selects one element from the k-elements previously chosen. It is easy to check that the corresponding CFL satisfies the No-regret property.

Example 2. (*Satisficing* (see e.g. Simon (1955))) Given a complete, asymmetric and transitive binary relation $\prec$ over a finite set X, the DM chooses the first element of each set A_i in a list $\mathcal{A} = \{A_1, ..., A_k\}$ that lies above a given bliss point $x^* \in X$, and if there is not any one she chooses the last element of A_i. We observe that the present choice procedure does not induce a a choice function on lists of sets satisfying NR postulate.

Example 3. (*Place-dependent rationality*) The goods are divided on the shelves of a supermarket on the basis of the seller's own interest in selling the product. Supermarket-marked goods are to the center and the other goods are to the bottom or top of the shelves, more hidden. The DM has a binary preference relation $\prec$ over the set $X \times \{1, 2, ..., k\}$ where the pair (x, j) denotes the good x in the jth place on the shelf. Typically, the consideration of the goods depends on their relative positions on the shelves. So, a DM chooses the goods x_j for which $(x_i, i) \prec (x_j, j)$

for all $1 \leq i \leq k$, where each good-position comparison occurs in every one of the *n-partition* sets into which the goods of the supermarket are divided. Therefore, a generic x will be chosen if and only if there exists a permutation σ such that $(\sigma(x), |\sigma(x)|) \succ (\sigma(b), |\sigma(b)|)$ for any b such that $\sigma(b) \prec \sigma(x)$. We observe that, in the present context, for $A \in 2^X$, if $A = B \cup C$, with $f \in CF(X)$, $f(A) \neq f(f(B) \cup C)$. Hence, the foregoing (sequential) choice procedure is not compatible with a path-independent choice function and as such (by Theorem 1) does not induce a CFL satisfying NR.

Example 4. (*The first element establishes the ordering*) For every element $a \in A$, that works as a reference point, there exists a corresponding ordering $\prec_a$ with respect to which the DM chooses as in e.g. Tverski and Kahneman (1991). In particular, for $\mathcal{A} = (A_1 \cup A_2 \cup, ..., \cup A_k)$ and a choice function f, if $x_1 = f(A_1)$, then consider x_1 as a given target and rank all the alternatives in X according to their desirability (as expressed by a utility function or a suitably defined distance) with respect to x_1, in order to have the corresponding ordering $\prec_{x_1}$. Then, suppose that $\mathcal{A} = (A_1 = \{x_1\}, A_2 = \{x_2, x_4\}, A_3 = \{x_3, x_5, x_6, x_7\})$, that the corresponding ordering induced by $\prec_{x_1}$ goes as follows: $x_1 \prec_{x_1} x_2 \prec_{x_1} x_4 \prec_{x_1} x_7 \prec_{x_1} x_5 \prec_{x_1} x_6 \prec_{x_1} x_3$ and the DM selects the closest alternative to x_1 from each set of the list $\mathcal{A}$, namely x_1, x_2, x_7. It is possible to check that the CFL so obtained does not satisfy the NR axiom.

Example 5. (*Limited Attention*) Choice with limited attention (see Masatlioglu, Nakajima and Ozbay, (2012)) is regarded as a novel approach to choice theory that accommodates several frequently observed behaviors not captured by the standard choice model as e.g. Attraction Effect, Cyclical Choice or Choosing Pairwisely Unchosen. Such a literature considers, for instance, a DM looking for something on the web using a search tool. She has a limited amount of time to check all the possible alternatives that the search engine shows, so she only considers a fixed number of links proposed by the search engine out of the whole list provided by the web. She then chooses from this limited set of alternatives.
We extend the problem of choice with limited attention to a problem of choice on a *limited* list of sets, namely:

$$F_{f,k}(A_1, \ldots, A_s) = f(f(A_1) \cup, \ldots, \cup f(A_{\min(k,s)})), \tag{16}$$

where $f \in CF(X)$ and $k < s$ is the number of sets in the list, whose length is $s = l(\mathcal{A})$, on which the DM focuses her attention. Then, it is possible to show that the CFL $F_{f,k}$ satisfies NR either if $k = 1$ or $k = |X|$, and f is *path-independent*.

Since the relation between a choice with limited attention and the new axiom of rational behavior presented here relies on the path-independence property of a choice function, we are confident that our model could also be behaviorally testable. We remark that (16) is *non-commutative*, namely it is not invariant under permutations of addenda. This makes the *limited attention choice procedure*, just extended to the present more general setting, an *antisymmetric operation* yielding a result that too much depends on the swapped arguments that attract the attention of the DM.

We conclude mentioning the following prominent papers analyzing the choice from lists of alternatives: Guney (2014), Masatlioglu and Nakajima (2013), Dimitrov, Mukhererjee and Mutu (2016), Yildiz (2016), Ishii, Kovach and Ulku (2021). All these works propose quite complete guidelines for choice protocols from lists of alternatives, which invariably differ from our own proposal in some significant respects. They study the case of choice from *lists of alternatives* as opposed to our more general case of *lists of sets of alternatives.* They provide a rational to some (real-life met) selection procedure of a single best alternatives out of a list, while we study the *NR-rationality* of a DM who makes selections from lists of sets of alternatives by using general choice mechanisms, a definitively different exercise.

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