UNIVERSITA' DEGLI STUDI DI SIENA

FACOLTA' DI SCIENZE ECONOMICHE E BANCARIE

ISTITUTO DI ECONOMIA

QUADERNI DELL'ISTITUTO DI ECONOMIA

N. 9

GIULIO CIFARELLI

THE NATURAL RATE OF UNEMPLOYMENT WITH RATIONAL EXPECTATIONS HYPOTHESIS. SOME PROBLEMS OF ESTIMATION.

SIENA, NOVEMBRE 1980

ELENCO DEI QUADERNI PUBBLICATI

1. MASSIMO DI MATTEO: Alcune considerazioni sui concetti di lavoro improduttivo
2. MARIA L. RUIZ: Mercati oligopolistici e scambi internazionali di manufatti. Alcune ipotesi e un'applicazione all'Italia.
3. DOMENICO MARIO NUTI: Le contraddizioni delle economie socialiste: una interpretazione marxista.
4. ALESSANDRO VERCELLI: Equilibrio e dinamica del sistema economico-semantica dei linguaggi formalizzati e modello kyenesiano.
5. A. RONCAGLIA-M. TONVERONACHI: Monetaristi e neokeynesiani: due scuole o una?
6. NERI SALVADORI: Mutamento dei metodi di produzione e produzione congiunta.
7. GIUSEPPE DELLA TORRE: La struttura del sistema finanziario italiano: considerazioni in margine ad un'indagine sull'evoluzione quantitativa nel dopoguerra (1948-1978)
8. AGOSTINO D'ERCOLE: Ruolo della moneta ed impostazione antiquantitativa in Marx: una nota.

UNIVERSITA' DEGLI STUDI DI SIENA

FACOLTA' DI SCIENZE ECONOMICHE E BANCARIE

ISTITUTO DI ECONOMIA

QUADERNI DELL'ISTITUTO DI ECONOMIA

N. 9

GIULIO CIFARELLI

THE NATURAL RATE OF UNEMPLOYMENT WITH RATIONAL EXPECTATIONS HYPOTHESIS. SOME PROBLEMS OF ESTIMATION.

SIENA, NOVEMBRE 1980

I. Introduction.

The Natural Rate of Unemployment with Rational Expectations model - in its various versions - implies that
- anticipated money supply changes(and anticipated changes in other policy instruments) are reflected in price changes only,having no effect on output and unemployment;
- unanticipated changes in economic policy instruments are reflected more on output changes than on price changes.

This approach cannot be properly tested by means of standard least squares techniques because the price expectations formation function is too similar in structure to the original price or output equations which have to be estimated.The authors who attempted such an estimation were forced - in order to obtain significant results - to make use of auxiliary approaches.(1)

More significant is the fact that - as we shall see - standard least squares approaches cannot be used to estimate what Rational Expectations econometricians label "observable reduced forms" of a model including Rational Expectations without a preliminary analysis of the ARIMA structure of the series which enter in the formantion of expectations. The focus has been set on the estimation of reduced forms more than on the estimation of the structural equations.It is evident however that, if the reduced

(1) There have been several attempts to estimate Rational Expectations relationships of tis kind in papers by R. Barro (1977,1978),A.Meltzer (1975), P.Korteweg (1978), P. Korteweg & A.Meltzer (1978),A.Fourcans (1978), M.Fratianni (1978), T.Mc Callum (1976) and E.J.Bohmoff (1980).

forms do not allow to compute some coefficients, the corresponding structural equations will be underidentified. If the reduced forms provide multiple estimates of some parameters, the corresponding structural equations will be overidentified.

ooo

II. An Outline of the Main Difficulties.

In a standard Rational Expectations model in which the Natural Rate of Unemployment hypothesis is tested, the system of equations which has to be estimated is as follows:

(II.1) $P_t = a' + b'P^a_t + c_1' X_{1t} + c_2' X_{2t} + u_{1t}$ $\quad u_{1t} \sim N(0, \sigma^2_{u_{1t}})$

(II.2) $y_t = y_{nt} + d_1 X_{1t} + d_2 X_{2t} + u_{2t}$ $\quad u_{2t} \sim N(0, \sigma^2_{u_{2t}})$ (1)

(II.2') $y_t = y_{nt} + \delta(P_t - P^a_t)$.

(Equation (II.2) has indeed little to do with the Rational Expectations hypothesis, being a plain reduced form relationship when estimated as

$(y_t - y_{t-1}) = d_1 X_{1t} + d_2 X_{2t} + u_{2t}$, where $y_{nt} = y_{t-1}$ by assumption. It will not be analysed in this form in this paper).

$P^a_t = E\left[P_t \text{ /Amount of Information Available at Time } t-1\right]$.

The assumption is that Rational Expectations about prices can be quantified as:

(1) P_t is the price level, y_t is the level of output, P^a_t is the expected price level and X_{it} ($i = 1,2$) are the exogenous variables (economic policy stimuli). Levels, rates of growth or logarithms of the series can alternatively be used.

(II.3) $P^a_t = \alpha + \beta \bar{X}_{1t} + \gamma \bar{X}_{2t} + u_{3t}$ $\quad u_{3t} \sim N(0, \sigma^2_{u_{3t}})$, (1)

(1) Consider Wallis' general model (K.Wallis: "Econometric Implications of the Rational Expectations Hypothesis", Econometrica, 1980).

Let $By_t + Ay^a_{1t} + Cx_t = u_t$, where B, A and C are matrices of dimension gxg, gxh, gxf respectively. Vectors y_t, y^a_{1t}, x_t and u_t have $g, h < g, k$ and g elements respectively. y^a_{1t} represents (unobservable) anticipations, formed in period $t - 1$, about values of h of the endogenous variables y_{1t}.

$y_t = \left[y_{1t}, y_{2t}\right]$. y_{2t} = vector of endogenous variables about which no anticipations are made. y^a_{1t} is the expectation of y_{1t} implied by the model conditional on information θ_{t-1} available at time t-1. The reduced form of the model is:

$y_t = -B^{-1} A_1 y^a_{1t} - B^{-1} C x_t + B^{-1} u_t$.

It can be partitioned and written as:

1) $y_{1t} = \Pi_{11} y^a_{1t} + \Pi_{12} x_t + v_{1t}$

2) $y_{2t} = \Pi_{12} y^a_{1t} + \Pi_{22} x_t + v_{2t}$.

Taking conditional expectations in the first matrix equation gives:

$E(y_{1t}/\theta_{t-1}) = \Pi_{11} E(y^a_{1t}/\theta_{t-1}) + \Pi_{12} E(x_t/\theta_{t-1}) + E(v_{1t}/\theta_{t-1})$

So we have:

3) $y^a_{1t} = (I - \Pi_{11})^{-1} \Pi_{12} E(x_t/\theta_{t-1}) = (I - \Pi_{11})^{-1} \Pi_{12} \hat{x}_t$.

Rational Expectations are given as linear combinations of the predictions of the exogenous variables, and the relevant informations on which to base them is the set of values $x_{t-1}, x_{t-2} \ldots$ Substituting in the system above, we obtain:

4) $y_{1t} = \Pi_{11} (I - \Pi_{11})^{-1} \Pi_{12} \hat{x}_t + \Pi_{12} x_t + v_{1t}$

5) $y_{2t} = \Pi_{21} (I - \Pi_{11})^{-1} \Pi_{12} \hat{x}_t + \Pi_{22} x_t + v_{2t}$.

(Suite of note (1) of page 3)
The estimation of the parameters of relationships analogous to 1), 2) and 3) constitutes the subject of this paper. Multicollinearity problems render the computation of the coefficients of equations with patterns similar to 1) and 2) difficult to perform. Relationships analogous to 1) - where anticipations of the dependent variables are included among the predetermined (right hand) variables - could be rewritten as (assuming that $x_t = \hat{x}_t + a_t$):

4') $y_{1t} = (1-\Pi_{11})^{-1}\Pi_{12}\hat{x}_t + \Pi_{12}a_t + v_{1t}$,

where $a_t = x_t - \hat{x}_t$ is a random shock. In that case, if a_t and $\hat{x}_t$ are orthogonal (i.e. independent), multicollinearity problems would be avoided. In order to be efficient, also this approach requires a proper evaluation of the ARIMA structure of the x_t time series. The approach set forth below is more general and applies to relationships of both patterns 1) and 2).
Multicollinearity problems would disappear if we were to assume a priori that the coefficients of y^a_{1t} are equal to one in both equations 1) and 2). At least at a theoretical level, some Natural Rate of Unemployment with R.E. studies proceed further and subtract 3) from 1) to obtain the endogenous variables vector - y_{1t} - as a function of the vector of the Rational Expectations of these very variables and of "unanticipated" shocks.

6) $y_{1t} = \Pi_{12}(x-\hat{x}_t) + y^a_{1t} + v_{1t} = (I-\Pi_{11})^{-1}\Pi_{12}\hat{x}_t +$

$+ \Pi_{12}(x_t-\hat{x}_t) + v_{1t} = a\hat{x}_t + b(x_t-\hat{x}_t) + v_{1t}.$

The output equation would be of the type:

7) $y_{2t} = a'\hat{x}_t + b'(x_t - \hat{x}_t) + v_{2t}.$

In practice however, most empirical approaches, being unable to compute satisfactorily $(x_t - \hat{x}_t)$, have proxied it with x_t. Moreover, the assumption of a one to one effect of y^a_{1t} on y_{1t} having been deemed not to be acceptable a priori, relationships analogous to 4) and 5) have been estimated in most of the cases.

where $\bar{X}_{1t}$ and $\bar{X}_{2t}$ are moving averages over two or three time periods of X_{1t} and X_{2t} lagged one period (sometimes unlagged). They are meant to represent anticipated values of X_{1t} and X_{2t}.
If the Natural Rate of Unemployment with Rational Expectations hypothesis is correct, we must have $c_1' \simeq c_2' \simeq 0$ and $b_1' \simeq 1$ (and $a' \simeq 0$) in equation (II.1). (We assume unanticipated shocks have no effect on prices and anticipated prices have a one to one effect on current prices).

If this is the case, in the following output equation

(II.4) $y_t = y_{nt} + bP^a_t + c_1X_{1t} + c_2X_{2t} + u_{4t} \quad u_{4t} \sim N(0, \sigma^2_{u_{4t}})$ (1)

we must find $b \simeq 0$ and c_1, c_2 different from zero since the Natural Rate of Unemployment with Rational Expectations hypotheses are assumed to be valid.

If $\bar{X}_{1t}$ and $\bar{X}_{2t}$ are significantly different from X_{1t} and X_{2t}, then the structural equations (II.1) and (II.2) can be properly estimated. This will be the case if both X_{1t} and X_{2t} are white noise processes, i.e. time series with no serial correlation; a rather strong restriction.

If $\bar{X}_{1t}$ and $\bar{X}_{2t}$ are closely correlated with X_{1t} and X_{2t} respectively, there will be collinearity between P^a_t (which depends upon $\bar{X}_{1t}$ and $\bar{X}_{2t}$) and X_{1t} and X_{2t}. This will impair

(1) Indeed it is equivalent to a test of an equation of the form: $y_t = y_{nt} + \delta(P_t - P^a_t) + \gamma P^a_t + u_t$. In order to simplify the notation, y_{nt} has been dropped from this equation in the following sections. This is of no consequence for our analysis.

the estimation of equations (II.1)and (II.4) above.In that case two stage least squares methods (2SLS) can no longer be used.In order for the two stages least squares method to be used,at least one of the predetermined variables in the model must be excluded from the structural equation we want to estimate.(1)

The problem then becomes one of economic common sensewe must assume that expectations about prices are influenced by more factors than output or prices in the structural equations.But the Rational Expectations hypothesis stresses that price expectations should depend upon the price equation in structure and upon the whole model. This in turn implies that their estimation should not include variables which are not explicitly included in the model.

This renders the problem outlined above difficult to solve.

A possible solution would be to include more and more lagged values of the exogenous variables in the information set (i.e. enlarging the 'memory' of the public) in an attempt to reduce somehow collinearity between $\bar{X}_{it}$ and X_{it}

(1) In order for an equation to be identified - in a simultaneous equations setting - a necessary condition is that the number of predetermined variables excluded from the equation must be greater than or equal to, the number of included endogenous variables minus one.Equations (II.1) and (II.3),(II.4) and (II.3) constitute two systems of equations.Equation (II.1) is identified.We have an endogenous variable included (P_t^a) and two exogenous variables excluded - $\bar{X}_{1t}$and $\bar{X}_{2t}$- if the latter differ from X_{1t} and X_{2t} respectively.The same holds true for equation (II.4). It can be shown that systems (II.1)-(II.3)and(II.4)-(II.3) satisfy the rank conditions for identification as well.

(i = 1,2). This approach is not very efficient. It will be investigated in the next section.

A more appealing method is that of isolating the correlation elements between $\bar{X}_{it}$ and X_{it}.This approach will be investigated in detail in section IV of this paper.

Some additional difficulties arise with respect to the economic interpretation of the estimation of equation (II.1).

The price level is regressed on an approximation of itself.

The anticipated price equation is estimated (by ordinary least squares) as:

$$P_t^a = \alpha + \beta\bar{X}_{1t} + \gamma\bar{X}_{2t} = P_t - \text{error},$$

and the structural equation (II.1) becomes:

$$P_t = a' + b'(P_t - \text{error}) + c_1'X_{1t} + c_2'X_{2t} + u_{1t}.$$

Correlation between $\bar{X}_{it}$ and X_{it} (i = 1,2) being usually strong,we should not be surprised that most of the predetermined variables in the structural price equation (II.1) have coefficients not significantly different from zero ($c_1 = c_2 = 0$): they have lost their explanatory power.

Thus the usual Rational Expectations finding that unanticipated changes in policy instruments are reflected more on output than on prices changes is disposed of.

The fact that a higher R^2 statistic was found in the empirical investigation of the price structural equation (II.1) than in the investigation of the anticipated price equation (II.3) is due to strong collinearity between P_t and P_t^a and does not have much significance.

As a consequence the standard Natural Rate of Unemployment with Rational Expectations finding of a one to one ef fect of anticipated prices on current prices can be easily explained.

These results hold even if a correct proxy for unanticipated stimuli were to be introduced.

They reduce the significance of any Rational Expectations application in which the endogenous variable includes among its determinants a Rational Expectation of itself.

ooo

III. Information Set and Multicollinearity.

Consider an equation of the form

(III.1) $\gamma_t = \alpha P^a_t + \sum_{k=1}^{n} \beta_k X_{t,k} + u_t \qquad u_t \sim N(0, \sigma^2_{u_t}) \qquad t=1,2,\ldots,T$

In matrix terms:

(III.2) $\gamma_t = P^a \alpha + X\beta + u_t$,

where γ_t, P^a_t and u_t are Tx1 vectors, X is a Txk matrix, β is kx1 and α is a scalar. The Rational Expectations hypothesis implies that:

(III.3) $P^a_t = E[P/\theta_{t-1}] = P_t - \eta$,

where η is a random vector that is uncorrelated with (orthogonal to) information available to and utilized by market participants in forming expectations about P_t.

To estimate P^a_t, the relevant available information - according to Rational Expectations hypotheses - is:

$\theta_{t-1} = (X_{t-1}, X_{t-2}, \ldots X_{t-k} \ldots)$, i.e. present (at time t-1) and past values of the variables actually involved in the process in question.

If the conditional expectation above is linear, we have:(1)

(III.4) $E[P_t/\theta_{t-1}] = P^a_t = X_{t-1}f_1 + X_{t-2}f_2 + X_{t-3}f_3 + \ldots + X_{t-n}f_n$.

The outcome is a two equations system:

(III.5) $\gamma_t = P^a\alpha + X\beta + u_t \qquad u_t \sim N(0, \sigma^2_{u_t})$

(III.6) $P_t = X_{t-1}f_1 + X_{t-2}f_2 + \ldots + X_{t-n}f_n + v$,

where γ_t can be either y_t or P_t.

In the above mentioned approaches use has been made of the two stages least squares technique. After a preliminary estimation of P^a_t by means of an ordinary least squares regression of P_t on a matrix Z of variables selected from θ_{t-1}, γ_t is regressed on P^a_t and other predetermined variables by means of ordinary least squares in equation (III.5).

$P^a_t = \tilde{P} = Z(Z'Z)^{-1}Z'P$.

The estimators of α and β are:

---

(1) Sometimes θ_{t-1} is seen as including also past values of P_t: $\theta_{t-1} = (X_{t-1}, \ldots X_{t-k}, P_{t-1}, \ldots P_{t-k})$.

So $E(P_t/\theta_{t-1}) = X_{t-1}f_1 + \ldots + X_{t-n}f_n + P_{t-1}h_1 + \ldots + P_{t-n}h_n$.

Using the relationship $P^a_t + \eta = E(P/\theta_{t-1}) + \eta = P_t$, we can substitute for the lagged values of P_t and obtain

$E(P_t/\theta_{t-1}) = X_{t-1}f_1 + X_{t-2}f_2 + \ldots + X_{t-n}f_n + e_t$, (where e_t is a term involving realized values of $\eta_{t-2}, \eta_{t-3}, \eta_{t-4} \ldots$ and also incorporates some parameters of the extended expectations function). In that way we are back to equation (III.6) above.

$$(III.7) \quad \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \left([\tilde{P}X]' \, [\tilde{P}X] \right)^{-1} [\tilde{P}X]' \gamma \quad \text{or}$$

$$(III.8) \quad \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} \tilde{P}'\tilde{P} & \tilde{P}'X \\ X'\tilde{P} & X'X \end{bmatrix}^{-1} \begin{bmatrix} \tilde{P}'\gamma \\ X'\gamma \end{bmatrix}$$

What constitutes an appropriate choice for the Z matrix? In the two stages least squares case $\hat{\alpha}$ and $\hat{\beta}$ will be true instrumental variables estimators (i.e. consistent) only if the Z matrix used in computing $P^a_t = \tilde{P}_t$ includes enough additional variables for the reverse matrix to exist.[1] (Two stages least squares estimators can be interpreted as instrumental variables estimators, where $\tilde{P}_t = P^a_t$ is used as instrument). Whereas McCallum (1976)[2] - in his example - estimates anticipated price changes within the context of a model which uses a relatively small number of variables of X (the matrix of all the exogenous variables vectors of the model), it being thus easy for him to use a large number of variables in X (and in Z) which are not in the equation to be examined, this is not the case in the models examined above.

In the Natural Rate of Unemployment with Rational Ex

---

(1) If Z were to include only those variables X_1 of X which are included in (III.5), the reverse matrix would not exist. If Z were to include only lagged values of X_1(we assume that these variables are serially correlated), the reverse matrix would be very small, providing unstable and unreliable estimates of the regression coefficients.

(2) B.T.McCallum: "Rational Expectations and the Natural Rate Hypothesis : Some Consistent Estimates", Econometrica, 1976.

pectations case the set of exogenous variables of the model (i.e. of X) not entering the structural equations to be estimated is empty. Only lagged values of the very variables appearing in the equations to be estimated can be used to solve this multicollinearity problem .

This is due to the fact that the structural equations are similar in pattern to the "observable reduced forms" of Rational Expectations.

But - as will be shown in the next section - the addition of lagged values of the exogenous variables is not a very efficient way of solving this problem. Lagged values of the exogenous variables appearing in a structural equation are not equivalent to different exogenous variables -in the context of this problem- unless they are white noise time series.

The idea is that the larger the lag, the smaller the correlation between the exogenous variables of the structural equation (III.5) to be estimated and the additional determinants of P^a_t in (III.6). If it is acceptable to assume that after n lags a variable x_{t-n} of the matrix X_{t-n} is independent of its counterpart - x_t- of X_t, so that - from our point of view - including x_{t-n} in Z would be equivalent to including a new exogenous time series, the effect of x_{t-n} upon P_t is not known.

If x_{t-n} has to be chosen with the lag - n - sufficiently large to ensure its serial independence from x_t, it is not likely that it will have a statistically significant and reliable impact upon P_t.

As a consequence its inclusion in the information set will do little to reduce multicollinearity.[1] The problem is then that of making a trade off between serial independence (i.e. independence of x_{t-n} from x_t) and the causal effect of x_{t-n} on P_t.

These difficulties show why it is inefficient (and unreliable) to use lagged values of the exogenous variables of the structural equation (III.5) to solve the multicollinearity problem outlined above.[2]

ooo

(1) Let the original relationship be: $\gamma_t = P^a_t \alpha + X_t \beta + u_t$; P^a_t is obtained from $P_t = X_{t-1} f_1 + \epsilon_t$. X_t and X_{t-1} have serially correlated elements. Assume that to solve the multicollinearity problem we add x_{t-n} to the determinants of P^a_t, where x_{t-n} is serially independent from x_t. γ_t, in reduced form, can be written as: $\gamma_t = [X_{t-1} f_1]\alpha + X_t \beta + x_{t-n} h + u_t$. If x_{t-n} has little or no effect on P_t, $h = 0$; $\gamma_t = [X_{t-1} f_1]\alpha + X_t \beta + u_t$ and multicollinearity is not eliminated.

(2) Multicollinearity problems of this kind would evidently be reduced if we were to assume that individuals make several periods ahead forecasts. In that case correlation between the information set -on the basis of which expectations are made- and the exogenous variables of the structural equation would be reduced. (In other words we must assume that there is a 'black out' period in the information availability of the public). We could assume that individuals make 2-3 years ahead forecasts whenever the information is based on annual data, or 2-3 quarters, one year ahead forecasts whenever their information is obtained on a quarterly basis.

IV. An Estimation Proposal.

In this section we shall first show why estimation by traditional methods of Rational Expectations models does not give satisfactory results. We shall then suggest an approach for dealing with these difficulties and obtaining consistent estimates of the parameters of these models.

1

Let the structural equation be:

$$\text{(IV.1)} \quad \gamma_t = a + bP^a_t + cx_t + u_t \qquad u_t \sim N(0, \sigma^2_{u_t}),^{(1)}$$

where P^a_t is a linear least squares forecast (or extraction) based upon observations of an exogenous set of variables $x_{t-1}, x_{t-2}, \ldots, x_{t-n}$ influencing P_t. We assume that P^a_t is - following the Rational Expectations hypothesis - the anticipated value of P_t held at period t-1, which must be equal to the expected value (in a statistical sense) of P_t conditional on all observations of the system available up to time t-1. More specifically, P^a_t is the projection of P_t on past x_t.

$$\text{(IV.2)} \quad P^a_t = E_{t-1}[P_t].$$

x_t is an exogenous variable. There may be several of them, but this would not - at this stage - alter the nature of this analysis.

$\{u_t\}$ is a normally distributed random variable, with mean zero and variance σ^2_u, orthogonal to the P^a_t and x_t processes.

P^a_t is the one period ahead mean square error forecast based upon the following generating function:

(1) γ_t can be replaced either by y_t or by P_t.

(IV.3) $P_t = \alpha + \beta f(x_t) + \epsilon_t \qquad \epsilon_t \sim N(0, \sigma^2_{\epsilon t})$,

where ϵ_t is orthogonal to x_t.

$$P^a_t = {}_{t-1}E\left[P_t\right] = {}_{t-1}E\left[\alpha + \beta f(x_t)\right] = \alpha + \beta\, {}_{t-1}E\left[f(x_t)\right] = \alpha + \beta \bar{x}_t$$

$\bar{x}_t$ is some kind of expectation about the exogenous variable x_t at time t-1.

There are various ways of expressing $\bar{x} = {}_{t-1}E\left[f(x_t)\right]$.

A)Naive Expectations ${}_{t-1}E\left[f(x_t)\right] = \bar{x}_t = x_{t-1}$.

B)Extrapolation (unconstrained) of past values of x_t.

$${}_{t-1}E\left[f(x_t)\right] = \bar{x}_t = \sum_{i=1}^{n} \beta_i B^i x_t = \beta_1 x_{t-1} + \beta_2 x_{t-2} + \dots + \beta_n x_{t-n}.$$

A proper distributed lag specification might be determined empirically.

C)Moving Average ${}_{t-1}E\left[f(x_t)\right] = (x_{t-1} + x_{t-2} + \dots + x_{t-n})/n.$

D)One period ahead minimum mean square forecast based upon the (predetermined) knowledge of the structure of the time series (of its covariance generating function). This hypothesis implies that past information on the exogenous variable is used optimally.

Most analyses have used either approach C or approach D, which shall be investigated in detail. Approaches A and B are but special cases of C.

Under assumption C) P^a_t is evaluated as:

(IV.4) $P^a_t = {}_{t-1}E\left[P_t\right] = \alpha + \beta/2(x_{t-1} + x_{t-2})$

on the basis of:

(IV.4') $P_t = \alpha + \beta/2(x_{t-1} + x_{t-2}) + \epsilon_t = \alpha + \beta_1(x_{t-1} + x_{t-2}) + \epsilon_t$,

where we write $\beta_1 = \beta/2$ and assume anticipations are based

on a two periods moving average. Equations (IV.1) and (IV.4') constitute e system which has been estimated by means of least squares techniques.

Full Information Maximum Likelihood techniques should be preferred from a theoretical point of view, because they allow to dispose of cross equation dependence of the error terms. A simplified version, which corresponds to the estimation procedures used in practice, is discussed here.(1)

(1) The Maximum Likelihood estimates can be obtained by minimizing $|\Sigma|$, where

$$\Sigma = \begin{bmatrix} 1/T \sum_{t=1}^{T} u_t^2 & 1/T \sum_{t=1}^{T} u_t \epsilon_t \\ 1/T \sum_{t=1}^{T} \epsilon_t u_t & 1/T \sum_{t=1}^{T} \epsilon_t^2 \end{bmatrix} = \begin{bmatrix} \sigma_u^2 & \sigma_{u\epsilon} \\ \sigma_{\epsilon u} & \sigma_\epsilon^2 \end{bmatrix}$$

The minimization is carried over the parameters α, β, a and b. Eliminating the intercepts by measuring all variables in deviations from their means, the log likelihood function to be maximized is:

$$1)\ L = (1/(2\Pi)^{T/2}) \cdot (1/|\Sigma|^{T/2}) \cdot \exp\Big\{-(1/2)\Big[(\sigma_u^2)^{-1} \sum_t (Y_t - bP^a_t - cx_t)^2 + 2(\sigma_{u\epsilon})^{-1} \sum_t \big((Y_t - bP^a_t - cx_t)(P_t - \beta_1(x_{t-1} + x_{t-2}))\big) + (\sigma_\epsilon^2)^{-1} \sum_t (P_t - \beta_1(x_{t-1} + x_{t-2}))^2\Big]\Big\}$$

If we assume that the error terms are not correlated with each other, $\sigma_{\epsilon u} = \sigma_{u\epsilon} = 0$, the relationship above becomes:

$$2)\ L = (1/(2\Pi)^{T/2}) \cdot (1/|\Sigma|^{T/2}) \cdot \exp\Big\{-1/(2\sigma_u^2) \sum_t (Y_t - bP^a_t - cx_t)^2 - 1/(2\sigma_\epsilon^2) \sum_t (P_t - \beta_1(x_{t-1} + x_{t-2}))^2\Big\}.$$

Taking logarithms, we have: $\ln|\Sigma| = \ln \sigma_u^2 + \ln \sigma_\epsilon^2$. Equation 2) can be written as (IV.5).

In the examples of this paper, insofar as we assume that the error terms are normal, that the correlation between the error terms of the equations is nil and that the sample is large, Maximum Likelihood approaches are equivalent, with some minor discrepancies, to least squares approaches.

Assume that correlation between $\{\epsilon_t\}$ and $\{u_t\}$ is nil.The log likelihood function becomes:

(IV.5) $$\ln L = \left[-(T/2)\ln(2\Pi)\right] + \left[-(T/2)\ln\sigma_u^2 - 1/(2\sigma_u^2)\sum_t(\gamma_t - bP_t^a - cx_t)^2\right] + \left[-(T/2)\ln\sigma_\epsilon^2 - 1/(2\sigma_\epsilon^2)\sum_t(P_t - \beta_1(x_{t-1}+x_{t-2}))^2\right] = \text{constant}+\ln L_1+\ln L_2$$

Intercepts are suppressed (by taking variables as deviations from means) in order to simplify computation.

Eliminating the unobservable variable, P_t^a, by substituting from (IV.4') into (IV.1), we obtain the following recursive set:

(IV.6)
a) $\gamma_t = b\beta_1(x_{t-1} + x_{t-2}) + cx_t + u_t$ $\quad u_t \sim N(0,\sigma_u^2)$
b) $P_t = \beta_1(x_{t-1} + x_{t-2}) + \epsilon_t$ $\quad \epsilon_t \sim N(0,\sigma_\epsilon^2)$.

The corresponding log likelihood function is:

(IV.7) $$\left[-(T/2)\ln(2\Pi)\right] + \left[-(T/2)\ln\sigma_u^2 - 1/(2\sigma_u^2)\sum_t(\gamma_t - b\beta_1(x_{t-1}+x_{t-2}) - cx_t)^2\right] + \left[-(T/2)\ln\sigma_\epsilon^2 - 1/(2\sigma_\epsilon^2)\sum_t(P_t - \beta_1(x_{t-1}+x_{t-2}))^2\right] = \ln L.$$

Provided x_t is a white noise time series,(1) treating the two terms of the likelihood function separately allows for consistent estimates of β_1, $b\beta_1$, c,

(1) Indeed,insofar as x_t is not correlated with $x_{t-1} + x_{t-2}$, both equations are identified.

σ_u^2 and σ_ϵ^2.(1)

The parameter b is unidentified in this approach but from the estimates of β_1 from ln L_2 and the a priori knowledge about the structure of Rational Expectations formation, b can also be estimated as $\widehat{b\beta_1}/\hat{\beta}_1$. We obtain in that way a

(1) The estimation procedure is analogous to a least squares one.It involves the maximization of ln L_1, of (IV.7) with respect to $b\beta_1$, c, σ_u^2 and of ln L_2 with respect to β_1 and σ_ϵ^2. (The model is recursive.The Jacobian determinant of the likelihood function is unity and hence nonlinearities in the estimation of the parameters are avoided). Indeed, this approach is analogous to a plain OLS estimate of (IV.6. a)) and of (IV.6. b)).

$$\delta\ln L_1/\delta b\beta_1 = 1/\sigma_u^2 \sum_t(\gamma_t - b\beta_1(x_{t-1}+x_{t-2}) - cx_t)(x_{t-1}+x_{t-2}) = 0$$

$$\delta\ln L_1/\delta c = 1/\sigma_u^2 \sum_t(\gamma_t - b\beta_1(x_{t-1}+x_{t-2}) - cx_t)x_t = 0$$

$$\delta\ln L_1/\delta\sigma_u^2 = -(T/2)(1/\sigma_u^2) - 1/(2\sigma_u^4)\sum_t(\gamma_t - b\beta_1(x_{t-1}+x_{t-2}) - cx_t)^2 = 0$$

$$\delta\ln L_2/\delta\sigma_\epsilon^2 = -(T/2)(1/\sigma_\epsilon^2) - 1/(2\sigma_\epsilon^4)\sum_t(P_t - \beta_1(x_{t-1}+x_{t-2}))^2 = 0$$

$$\delta\ln L_2/\delta\beta_1 = 1/\sigma_\epsilon^2 \sum_t(P_t - \beta_1(x_{t-1}+x_{t-2}))(x_{t-1}+x_{t-2}) = 0$$

Maximizing lnL_1 with respect to $b\beta_1$, and c amounts to minimizing $\sum_t(\gamma_t - b\beta_1(x_{t-1}+x_{t-2}) - cx_t)$ with respect to these very variables.Maximizing lnL_2 with respect to β_1, amounts to minimizing $\sum_t(P_t - \beta_1(x_{t-1}+x_{t-2}))$ with respect to it.Hence M. L. estimators are -in this case- the same as the OLS estimators.This is not the case for M.L.estimators of σ_u^2 and σ_ϵ^2.

consistent estimate of b.[1] These parameters estimates are analogous to those obtained when linear least squares estimation of the system constituted by equations (IV.1) and (IV.4') is performed. It should be noticed that b and β_1 could be estimated directly from (IV.6 a)) by means of nonlinear estimation approaches. In that case the estimation of (IV.6 b)) is no longer needed. These approaches are, however, computationally expensive. If instead of hypothesis C) on Rational Expectations estimation, hypotheses A) or B) were to be adopted, the result above would not be altered.[2]

---

(1) ln L_1 can be written as:

$$-(T/2)\ln \sigma_u^2 - 1/(2\sigma_u^2) \sum_t (\gamma_t - bP_t^a(\beta_1) - cx_t)^2.$$

We can use the estimate of β_1 obtained by maximizing ln L_2 in the estimation of ln L_1, obtaining in that way Maximum Likelihood estimates of the remaining parameters. ln L_1 can be seen here as the 'concentrated likelihood' function version of (IV.7).

(2) If there is more than one exogenous variable, in cases A, C and B, i.e. if the anticipated price equation includes more than one exogenous variable on the right hand side, as is usually the case in hypothesis B, the approach mentioned above would bring about some problems since the γ_t structural equation is overidentified. If -as an example- we were to evaluate P_t^a from $P_t = \alpha + \beta_1 x_{1t-1} + \beta_2 x_{2t-1} + \ldots + \beta_n x_{nt-1} + e_t$, in finite samples $\widehat{b\beta_1}/\hat\beta_1, \widehat{b\beta_2}/\hat\beta_2, \ldots, \widehat{b\beta_n}/\hat\beta_n$ would not be equal, so that this approach would lead to multiple consistent estimates of b. Plugging estimates of $\beta_1, \beta_2, \ldots, \beta_n$ obtained by maxizing ln L_2 in the maximization process of ln L_1 would allow to solve this problem. However, as has been stressed by M.N. Nerlove, D.M. Grether and J.L. Carvalho ("Analysis of Economic Time Series", 1979), some loss of efficiency is connected with this estimation procedure, since none of the information in ln L_1 relevant to the determination of $\beta_1, \beta_2, \ldots, \beta_n$ has been used. Nerlove, Grether and Carvalho suggest an iterative approach to improve the efficiency of the estimators.

If we were to assume that the information set upon which P_t^a is based included unlagged values of x_t (and of any other exogenous variable) the approach above would still be appropriate.[1]

If x_t is a white noise process Rational Expectations estimate approach D), based on one period ahead minimum mean square error forecasts, is not possible within the context of this model since: ${}_{t-1}E\left[x_t\right] = 0$.
(Expected prices would then be independent of the model).

Hypothesis A), where P_t^a is obtained from $P_t = \alpha + \beta_1 x_{t-1} + \epsilon_t$, does not bring about estimation problems.

2

Some difficulties in cases A), B), and C) arise if x_t is not a white noise process.

---

(1) The log-likelihood function becomes (hypothesis C));

$$\ln L_1 = \left[-(T/2)\ln \sigma_u^2 - 1/(2\sigma_u^2)\sum_t(\gamma_t - bP_t^a - cx_t)^2\right] + \left[-(T/2)\ln \sigma_\epsilon^2 - 1/(2\sigma_\epsilon^2)\sum_t(P_t - \beta_1 x_t - \beta_1 x_{t-1})^2\right] + \text{constant}.$$

By direct estimation -after having made the usual substitution for the unobserved variable P_t^a - we get estimates of $b\beta_1$, β_1, σ_ϵ^2, σ_u^2, $(b\beta_1 + c)$. b and c would be indirectly approximated by noting that the coefficient of x_{t-1}, $b\beta_1$, divided by β_1 (obtained from ln L_2) gives b. The coefficient of x_t, $(b\beta_1 + c)$, minus the coefficient of x_{t-1}, $b\beta_1$, provides an estimator for c. Once more, if hypothesis B) about expectations of x_t is adopted, coefficient values obtained from ln L_2 have to be used for the estimation of ln L_1.

Since most economic data exhibit substantial autocorrelation the efficacy of these methods is seriously curtailed. The parameters estimated above either with linear or with nonlinear approaches are not the best consistent estimators of b, β_1 and c.

This can be shown in the following way: assume x_t follows - to keep the algebra as simple as possible - a first order autoregressive Markov process (an AR(1) process).

(IV.8) $x_t = \rho x_{t-1} + \mu_t \qquad |\rho| < 1, \mu_t \sim N(0, \sigma_\mu^2).$

The Rational Expectations system above becomes (intercepts are suppressed) :

$$\text{(IV.9)} \begin{cases} \gamma_t = b\beta_1 ((\rho x_{t-2} + \mu_{t-1}) + x_{t-2}) + c(\rho^2 x_{t-2} + \rho\mu_{t-1} + \mu_t) + u_t \\ P_t = \beta_1 ((\rho x_{t-2} + \mu_{t-1}) + x_{t-2}) + \epsilon_t \end{cases}$$

Coefficient estimators are flawed by a strong multicollinearity problem. Indeed, rearranging terms, we get:

$$\text{(IV.9')} \begin{cases} \gamma_t - (b\beta_1(\rho + 1) + \rho^2 c)\, x_{t-2} = (b\beta_1 + \rho c)\mu_{t-1} + c\mu_t + u_t \\ P_t - (\beta_1(\rho + 1))\, x_{t-2} = \epsilon_t + \beta_1\mu_{t-1}, \end{cases}$$

where we have: $x_{t-1} = \rho x_{t-2} + \mu_{t-1}, x_t = \rho^2 x_{t-2} + \mu_t + \rho\mu_{t-1}$

The system to be estimated becomes:

$$\text{(IV.9")} \begin{cases} \gamma_t - (b\beta_1(\rho + 1) + \rho^2 c)\, x_{t-2} = S_t \\ P_t - (\beta_1(\rho + 1))\, x_{t-2} = G_t, \end{cases}$$

where $S_t = u_t + (b\beta_1 + \rho c)\mu_{t-1} + c\mu_t \qquad G_t = \epsilon_t + \beta_1\mu_{t-1}.$

Insofar as plim $T^{-1}x_{t-2}\mu_{t-1} = 0 = $ plim $T^{-1}x_{t-2}\mu_t$, the esti-

mators are consistent, but collinearity does not allow to compute b and c separately. We cannot assert that $\{S_t\}$ and $\{G_t\}$ are independent serially uncorrelated time series and that $cov(G_t, S_t) = cov(S_t, G_t) = 0$, if $\{\epsilon_t\}$ and $\{u_t\}$ are as above assumed to be normal serially uncorrelated series, not correlated with each other.(1) In other words, (IV.1) is underidentified.

(1) This identification problem can be analyzed in terms of an ARIMA process such that $\varphi_x(B)x_t = \theta_x(B)a_t$. Assume x_t is a stationary time series which can be interpreted by means of an A.R.M.A. (p,q) model. x_t can be written as an infinite weighted sum of previous values of x_t plus a random shock.

$$x_t = \sum_{j=1} A_{j-1} x_{t-j} + a_t = A(B)x_{t-1} + a_t =$$
$$= \{\theta_x(B)^{-1}(\phi_x(B) - \theta_x(B))\} B^{-1}x_{t-1} + a_t.$$

(In case of an A.R.I.M.A. (p,d,q) model an analogous expression can be obtained, with

$\varphi_x(B) = \phi_x(B)\nabla^d$ replacing $\phi_x(B)$).

$x_{t-1} = A(B)x_{t-2} + a_{t-1}; \quad x_{t-2} = A(B)x_{t-3} + a_{t-2}.$

Proceeding as in the model (IV.8) above we obtain:

$$x_t = [A(B)]^3 x_{t-3} + [A(B)]^2 a_{t-2} + [A(B)]\, a_{t-1} + a_t$$

$x_{t-1} = [A(B)]^2 x_{t-2} + [A(B)]\, a_{t-2} + a_{t-1}$. Substituting into the system (IV.1)-(IV.4'), after some manipulation, we obtain:

$$\begin{cases} \gamma_t = (b\beta_1 \{[A(B)]^2 + A(B)\} + c[A(B)]^3)x_{t-3} + \Lambda_\gamma \\ P_t = \beta_1\{[A(B)]^2 + A(B)\}x_{t-3} + \Lambda_P, \end{cases}$$ where

$$\Lambda_\gamma = \beta_1 b((A(B) + 1)a_{t-2} + a_{t-1}) + c([A(B)]^2 a_{t-2} + A(B)a_{t-1} + a_t) + u_t$$

$$\Lambda_P = \beta_1((A(B) + 1)\, a_{t-2} + a_{t-1}) + \epsilon_t.$$

As seen above, it would be possible to compute β_1 from the anticipated price equation, provided the A.R.M.A. structure of x_t were known. This is not sufficient, however, to provide estimators for b and c from the γ_t equation.

Only if it were possible to estimate this system with the techniques set forth above, could we say that the standard linear least squares estimation approach of the Rational Expectations literature provides an appropriate estimation of the Natural Rate of Unemployment with Rational Expectations reduced forms.

This is not the case.

Even if we were to make the strong assumption that $\{S_t\}$ and $\{G_t\}$ are normal independent serially uncorrelated processes and that cov (S_t, G_t) = cov $(G_t, S_t) = 0$, the Maximum Likehood methods mentioned above would not provide a proper estimation of the parameters of interest. Direct estimation of ln L would provide estimators for $(b\beta_1(\rho+1)+c\rho^2)$ and $\beta_1(\rho+1)$. They cannot be disentangled, however, to obtain estimators of b, ρ_1 and c. This is the case even if we were to assume that the value of the correlation coefficient is known.[1]

In such a case we would have a proper estimation of β_1 (from ln L_2), but would still be unable to obtain a distinct estimation of b and c. This is a standard problem of multicollinearity.[2] The closer $|\rho|$ is to 1, the more serious such a

---

(1) This invalidates any iterative or "grid" method of approximating the parameters' true value by performing several estimations of the system, corresponding to different values of ρ, and choosing the value of ρ -and the corresponding parameter estimates- which minimize the covariance matrix of the system.

(2) Nonlinear methods of estimation can be used to appproximate the values of the parameters, provided the structure (i. e. the order) of the x_t time series is known. This gives rise to a set of new problems, convergence, uniqueness,....., which lie outside the scope of this paper.

problem is.[1]

Approach D) to expectations formation is connected with an attempt to solve this problem.

The time series x_t is divided into two components by **ARIMA** techniques. Assuming the hypothetical A.R.M.A.(p,q) structure of x_t to be given by $\phi_x(B)x_t = \theta_x(B)a_t$, we obtain as A.R.M.A. representation of the series $x_t = \sum_{j=1} \Pi_j x_{t-j} + a_t$, where a_t is an independent white noise process.

Once ascertained, the structure of the time series is used to evaluate the one period ahead minimum mean square error forecast of the series:[2]

---

(1) Problems would arise too with respect to the identification of equation (IV.1).

(2) Suppose x_t is a linearly regular stochastic process with covariance generating function $R_x(z) = \sigma^2 b(z)b(z^{-1})$, where $b(z) = \sum_{j=0}^{\infty} b_j z^j$, $\sigma^2 > 0$. Any linearly regular (linearly nondeterministic) stochastic process has a covariance generating function that can be represented in that way. By the Wiener-Kolmogorov formula, the z transform of the projection of x_{t+v} against $x_t, x_{t-1}, \ldots$, is given by $[1/b(z)]\, [z^{-v}b(z)]_+$ i.e. $E_t[x_{t+v}] = \delta(B)x_t$, where $\delta(z) = \sum_{j=0}^{\infty} \delta_j z^j = 1/b(z)\,[b(z)/z^v]_+$. $1/b(z)$ is assumed to be analytic in a circle with radius greater than unity, so that the representation is invertible ($b(z)$ is invertible). The minimum mean square error forecast for lead time 1 is the conditional expectation $E[x_t]$ of x_t at origin t-1. The residuals a_t which generate the process (being independent random variables or shocks) turn out to be the one step ahead forecast errors: $x_t - {}_tE_1[x_t] = a_t$. It follows that for a minimum mean square error forecast, the one step ahead forecast errors must be uncorrelated.

$$\underset{t-1}{E}\left[x_t\right] = \sum_{j=1}^{\infty} \Pi_j \underset{t-1}{E}\left[x_{t-j}\right] + \underset{t-1}{E}\left[a_t\right] = \sum_{j=1}^{\infty} \Pi_j \underset{t-1}{E}\left[x_{t-j}\right] =$$

$$= \sum_{j=1}^{\infty} \Pi_j \, x_{t-j}.^{(1)}$$

(D.a) If a model of the kind

$$\text{(IV.10)} \begin{cases} \gamma_t = bP_t^a + cx_t + u_t & u_t \sim N(0,\sigma_u^2) \\ P_t = \beta \bar{x}_t + \epsilon_t & \epsilon_t \sim N(0,\sigma_\epsilon^2) \end{cases}$$

is evaluated, $\bar{x}_t$ - the minimum mean square error forecast - and x_t will be correlated since x_t is not a

---

(1) The minimum mean square error forecast is defined in terms of the conditional expectation

$$\underset{t-1}{E}\left[x_t\right] = E\left[x_t / x_{t-1}, x_{t-2}, \ldots\right].$$

which theoretically requires knowledge of the xs' stretching to the infinite past. However, the requirement of invertibility, which we have imposed on the general A.R.I.M.A. model, ensures that the Π weights form a convergent series.
(In practice the Π weights usually decay rather quickly, so that whatever form of the model is employed, only a moderate lenght of series,

$x_{t-1}, x_{t-2} \ldots x_{t-n}$ is needed to calculate the forecasts with sufficient accuracy).

white noise process.(1)

$\underset{t-1}{E}\left[x_t\right] = \bar{x}_t = \sum_{j=1}^{\infty} \Pi_j \underset{t-1}{E}\left[x_{t-j}\right]$ and some of the problems examined above will have to be dealt with.

This difficulty is relevant because (IV.10) is the standard Rational Expectations model, which includes the 'optimal'

---

(1) Assume x_t has an A.R.M.A. (p,q) structure.

$x_t + \phi_{x1}x_{t-1} + \phi_{x2}x_{t-2} + \ldots + \phi_{xp}x_{t-p} = \theta_{xo} + \theta_{x1}a_{t-1} + \theta_{x2}a_{t-2} + \ldots + \theta_{xq}a_{t-q}$. $\phi_x(B)x_t = \theta_x(B)a_t$, where $\phi_x(B) = 1 + \phi_{x1}B + \phi_{x2}B^2 + \ldots + \phi_{xp}B^p$; $\theta_x(B) = \theta_{xo} + \theta_{x1}B + \theta_{x2}B^2 + \ldots + \theta_{xq}B^q$.

Assume $\theta_x(B)$ is invertible. The optimal one step ahead forecast is: $\underset{t-1}{E}\left[x_t\right] = \bar{x}_t$. (It is assumed that $\theta_o = 1$).

$\bar{x}_t = -\phi_{x1}x_{t-1} - \ldots - \phi_{xp}x_{t-p} + \theta_{x1}a_{t-1} + \ldots + \theta_{xq}a_{t-q} =$

$= \sum_{j=1}^{\infty} A_{j-1}x_{t-j} = A(B)x_{t-1}$. We can rewrite these expressions as:

$$x_t = \left\{\theta_x(B)^{-1}(\phi_x(B) - \theta_x(B))\right\} B^{-1}x_{t-1} + a_t$$

$$\bar{x}_t = \left\{\theta_x(B)^{-1}(\phi_x(B) - \theta_x(B))\right\} B^{-1}x_{t-1}.$$

Substituting in (IV.10) above we get:

$$\begin{cases} \gamma_t = b\beta(\bar{x}_t) + cx_t + u_t = (b\beta + c)\left\{\theta_x(B)^{-1}(\phi_x(B) - \theta_x(B))\right\} B^{-1}x_{t-1} + ca_t + \\ \quad + u_t = (b\beta + c)A(B)x_{t-1} + ca_t + u_t \\ P_t = \beta(\bar{x}_t) + \epsilon_t = \beta\left\{\theta_x(B)^{-1}(\phi_x(B) - \theta_x(B))\right\} B^{-1}x_{t-1} + \epsilon_t = \beta A(B)x_{t-1} + \epsilon_t, \end{cases}$$

Assume the A.R.M.A. structure of x_t is estimated. Direct evaluation of (IV.10) above provides the estimators of

$\beta A(B) = \beta(\left\{\theta_x(B)^{-1}(\phi_x(B) - \theta_x(B))\right\} B^{-1})$, and of

$(b\beta + c)(\left\{\theta_x(B)^{-1}(\phi_x(B) - \theta_x(B))\right\} B^{-1}) = (b\beta + c)A(B)$.

From the price equation we could then compute β. But this would not be sufficient to solve the identification problems of the structural γ_t equation.

forecast behaviour of the individuals.

Model (IV.10) is analogous in structure to the 'observable reduced forms' of any model in which expectations are assumed to be rational.

(D.b) If a model of the kind:

$$\text{(IV.11)}\quad \begin{cases} \gamma_t = bP^a_t + ca_t + u_t & u_t \sim N(0,\sigma^2_u) \\ P_t = \beta \bar{x}_t + \epsilon_t & \epsilon_t \sim N(0,\sigma^2_\epsilon) \end{cases}$$

is evaluated, where a_t is a white noise random shock, residual of the above mentioned A.R.M.A. one period ahead x_t forecast ($x_t - \underset{t-1}{E}[x_t] = a_t$),

a consistent estimation of the parameters can be performed using the approaches mentioned above.

If - however - $(\theta_x(B) / \phi_x(B))\, a_t$

does not capture all systematic components of the x_t time series, the a_t s might keep some of the systematic structure of the x_t time series and the multicollinearity problems examined above would remain

unsolved.(1)

3

A proper estimation of the system set forth above requires a preliminary manipulation of the time series.

---

(1) Provided x_t is the one period ahead minimum mean square error forecast of x_t, the estimation procedure above is correct.

Let $a_t = x_t - \underset{t-1}{E}[x_t]$. The system becomes:

$$\ln L = \text{const.} + \left[-(T/2)\ln\sigma^2_u - 1/(2\sigma^2_u)\sum_t(\gamma_t - b\beta\bar{x}_t - c(x_t-\bar{x}_t))^2\right] + \left[-(T/2)\ln\sigma^2_\epsilon - 1/(2\sigma^2_\epsilon)\sum_t(P_t-\beta\bar{x}_t)^2\right].$$

Let x_t be an ARIMA (1,0,0) process:

$x_t = \phi x_{t-1} + \mu_t$, $|\phi| < 1$, $\bar{x}_t = \underset{t-1}{E}[x_t] = \phi x_{t-1}$. Substituting in ln L we get:

$$\ln L = \text{const.} + \left[-(T/2)\ln\sigma^2_u - 1/(2\sigma^2_u)\sum_t(\gamma_t - b\beta\phi x_{t-1} - c\mu_t)^2\right] + \left[-(T/2)\ln\sigma^2_\epsilon - 1/(2\sigma^2_\epsilon)\sum_t(P_t - \beta\phi x_{t-1})^2\right],$$

which allows for a consistent estimation of the parameters of interest. Assume x_t is an A.R.M.A. (p,q) process, where $\phi_x(B)$, $\theta_x(B)$ are the autoregressive and moving average operators respectively, as above. Assume further that $\theta_{xo} = 1$. Written in inverted form the A.R.M.A. representation of x_t is

$$x_t = \{\theta_x(B)^{-1}(\phi_x(B) - \theta_x(B))\}\, B^{-1}x_{t-1} + a_t = A(B)x_{t-1} + a_t.$$

The one period ahead mean square error forecast can be written as: $\underset{t-1}{E}[x_t] = \{\theta_x(B)^{-1}(\phi_x(B)-\theta_x(B))\}\cdot B^{-1}x_{t-1} = A(B)x_{t-1}$.

Substituting in (IV.10) we get:

$$\gamma_t = bP^a_t + c(x_t - \bar{x}_t) + u_t = b\beta A(B)x_{t-1} + c(A(B)x_{t-1} + a_t - A(B)x_{t-1}) + u_t =$$
$$= b\beta A(B)x_{t-1} + ca_t + u_t = b\beta\{\theta_x(B)^{-1}(\phi_x(B)-\theta_x(B))\}B^{-1}x_{t-1} + ca_t + u_t.$$

$$P_t = \beta A(B)x_{t-1} + \epsilon_t = \beta\{\theta_x(B)^{-1}(\phi_x(B)-\theta_x(B))\}B^{-1}x_{t-1} + \epsilon_t.$$

A consistent estimation of the parameters of interest is then possible.

Assume that x_t is a time series which can be given an A.R.M.A. representation, $\phi_x(B)x_t = \theta_x(B)a_t$,[1] where $\phi_x(B)$ and $\theta_x(B)$ are defined as above. The moving average operator is assumed to be invertible. Multiplying both sides by $\theta_x(B)^{-1}$ we obtain $(\phi_x(B)/\theta_x(B))x_t = a_t$. In that way a higly autocorrelated time series has been transformed into a sequence of uncorrelated random variables, a_t. A possible solution to the multicollinearity problem which hinders the computation of coefficients b and c is the following: consider model (IV.6) under hypothesis C) about expectations formation

$$\text{(IV.6)}\begin{cases} \gamma_t = bP^a_t + cx_t + u_t & u_t \sim N(0,\sigma^2_u) \\ P_t = \beta_1(x_{t-1}+x_{t-2}) + \epsilon_t & \epsilon_t \sim N(0,\sigma^2_\epsilon) \end{cases}$$

Substituting for x_{t-1} and x_{t-2} in terms of their A.R.M.A. representation we obtain

$$\text{(IV.12)}\begin{cases} \text{(a)}\ \gamma_t = b\beta_1\left[\theta_x(B)/\phi_x(B)\right](a_{t-1}+a_{t-2}) + c\left[\theta_x(B)/\phi_x(B)\right]a_t + u_t \\ \text{(b)}\ P_t = \beta_1(x_{t-1}+x_{t-2}) + \epsilon_t, \end{cases}$$

(IV.12.a) can be written as

$$\text{(IV.12.a)}\quad \gamma_t = b\beta_1\left[\Psi_x(B)\right](a_{t-1}+a_{t-2}) + c\left[\Psi_x(b)\right]a_t + u_t,$$

where $\phi_x(B)\Psi_x(B) = \theta_x(B)$. (IV.12.a) -reduced - yields:

$$\text{(IV.12.a')}\left[\phi_x(B)/\theta_x(B)\right]\gamma_t = \left[\Psi_x(B)^{-1}\right]\gamma_t = b\beta_1(a_{t-1}+a_{t-2}) + ca_t + V_t.$$

$$\left(V_t = \left[\phi_x(B)/\theta_x(B)\right]u_t = \left[\Psi_x(B)^{-1}\right]u_t\right).$$

Since $\phi_x(B)/\theta_x(B)$ is a rational lag operator, i.e. contains an autoregressive component, the error term u_t, which is serially uncorrelated, will generate a serially correlated error

(1) The analysis assumes that any trend and/or any mean component -i.e. any non indeterministic component- of the time series has been eliminated. Otherwise the ARIMA representation of the series would be given by

$$\varphi_x(B)x_t = \nabla^d\phi_x(B)x_t = \theta_x(B)a_t,$$

where $\varphi_x(B)$ is the generalized autoregressive operator used for the representation of nonstationary time series, such that d of the roots of $\varphi_x(B) = 0$ are unity and the remainder lie out side the unit circle.

term $V_t = \left[\phi_x(B)/\theta_x(B)\right]u_t$ and equation (IV.12a') will be dynamic.

A three step approach could be used to estimate the model:

1) estimate the structure of the x_t time series (often this is performed by means of nonlinear methods) and estimate the weights of $\Psi_x(B)$ from $\phi_x(B)\Psi_x(B) = \theta_x(B)$.

$$\phi_x(B)x_t = \theta_x(B)a_t \qquad a_t \sim N(0,\sigma^2_a);$$

2) estimate the anticipated price equation coefficients from

$$P_t - \beta_1(x_{t-1}+x_{t-2}) = \epsilon_t \qquad \epsilon_t \sim N(0,\sigma^2_\epsilon);$$ [1]

3) substitute from 1) and 2) and estimate (IV.12.a').

$$\text{(IV.12.a')}\left[\Psi_x(B)^{-1}\right]\gamma_t - b\beta_1(a_{t-1}+a_{t-2}) - ca_t = \left[\Psi_x(B)^{-1}\right]u_t.$$ [2]

It can be written as: $\gamma^a_t - b\beta_1(a_{t-1}+a_{t-2}) - ca_t = V_t$, where $V_t = \left[\phi_x(B)/\theta_x(B)\right]u_t$, $u_t = \sum_{\tau=0}^{\infty}\lambda_\tau u_{t-\tau}$, $\gamma^a_t = \left[\Psi_x(B)^{-1}\right]\gamma_t$, and can be estimated with linear least squares, using the standard approaches for dealing with serial correlation of residuals.

$\gamma^a_t - b\beta_1(a_{t-1}+a_{t-2}) - ca_t = V_t$ could alternatively be estimated directly by means of nonlinear least squares.

(1) The price anticipations equation could be estimated as:
$P_t - \beta_1(\Psi_x(B)a_{t-1} + \Psi_x(B)a_{t-2}) = \epsilon_t, \epsilon_t \sim N(0,\sigma^2_\epsilon)$.
But this would imply the introduction of some unnecessary computations and reduce the accuracy of the findings insofar as $\Psi_x(B) = \left[\theta_x(B)/\phi_x(B)\right]$ is only an approximation.

(2) Consider the case in which x_t has an A.R.M.A.(1) = A.R.(1) structure. The procedure becomes :

1) Evaluate $\phi = \sum_t x_t x_{t-1} / \sum_t x^2_{t-1}$;
2) Evaluate β_1 with OLS procedures from $P_t - \beta_1(x_{t-1}+x_{t-2}) = \epsilon_t$;
3) Use these estimators to evaluate
$\gamma_t - \phi\gamma_{t-1} - b\beta_1(\mu_{t-1}+\mu_{t-2}) - c\mu_t = u_t + \phi u_{t-1}$.

The filtering approach expounded here is valid whether we assume x_t is a single time series or instead a vector of time series. Indeed in the Natural Rate of Unemployment with Rational Expectations model more than one exogenous policy variable is usually included.

Time series analysis allows for the development and estimation of the A.R.I.M.A. structure of a vector of time series.

Within the context of this approach however, this method would reduce the accuracy of the computations since different exogenous variables will have differing A.R.I.M.A. structures.

The common structure of anyone of the time series will probably be less accurate than that obtained in isolation, as it has to be averaged with those of the other time series of the vector.

As a consequence the larger the number of exogenous variables entering the model, the greater the loss in accuracy.[1]

This procedure however is still valid whatever hypothesis

---

(1) This can be seen intuitively by noting that if in the example above we were to have more than one exogenous time series $-x_{1t}, x_{2t}, \ldots x_{jt} \ldots,$ each with its own ARIMA structure - the whitening of the first exogenous variable would introduce autocorrelation and moving average elements in the others. On the other hand the whitening of the second would introduce nonwhite noise in the first exogenous variable.

is made about $\underset{t-1}{E}(x_t)$, whether hypotheses A,B,C or D.[1]

Alternatively, a second approach can be developed.

Instead of filtering γ_t - at stage three - multiplying it by $\Psi_x(B)^{-1}$, the following equation could be estimated directly.

$$(IV.12)\quad \gamma_t = b\beta_1\left\{\left[\Psi_x(B)\right]a_{t-1} + \left[\Psi_x(B)\right]a_{t-2}\right\} + c\left[\Psi_x(B)\right]a_t + u_t$$

$$u_t \sim N(0, \sigma_u^2).$$

Once more step two could be avoided if nonlinear methods of estimation were to be intro

---

(1) When hypothesis D) is made, the estimation procedure requires first that minimum mean square forecasts of x_t be expressed in inverted autoregressive form:

$$\bar{x}_t = A(B)x_{t-1} = \left\{\theta_x(B)^{-1}\left(\Phi_x(B) - \theta_x(B)\right)\right\} B^{-1}x_{t-1}.$$

The approach set forth above is then introduced:

$$\gamma_t = b\beta\bar{x}_t + cx_t + u_t;$$

$$\gamma_t = b\beta A(B)\left[\theta_x(B)/\Phi_x(B)\right]a_{t-1} + c\left[\theta_x(B)/\Phi_x(B)\right]a_t + u_t;$$

multiplying both sides by $\Phi_x(B)/\theta_x(B)$, we obtain:

$$\Phi_x(B)/\theta_x(B)\gamma_t = b\beta A(B)a_{t-1} + ca_t + \left[\Phi_x(B)/\theta_x(B)\right]u_t.$$

We can then proceed to the estimation as in the case of (IV.11.a) above. An alternative is as follows:

$$A(B)\left[\theta_x(B)/\Phi_x(B)\right] = \left\{\Phi_x(B)^{-1}\left(\Phi_x(B) - \theta_x(B)\right)\right\} B^{-1}$$

$$\gamma_t = b\beta\left\{\Phi_x(B)^{-1}\left(\Phi_x(B) - \theta_x(B)\right)\right\} B^{-1}a_{t-1} + c\left[\theta_x(B)/\Phi_x(B)\right]a_t + u_t,$$

which can be reduced as:

$$\Phi_x(B)\gamma_t = b\beta\left\{\left(\Phi_x(B) - \theta_x(B)\right)\right\}B^{-1}a_{t-1} + c\theta_x(B)a_t + \Phi_x(B)u_t$$

and computed as above.

duced.[(1)] Here too, a certain loss of accuracy occurs if

(1) Analogous problems, involving the iterative estimation of a general rational lag structure of the exogenous variables, have been investigated by P.J.Dhrymes ("Distributed Lags: Problems of Estimation and Formulation", 1971, chapters 9 and 10), and by K.Steiglitz & L.E. McBride ("Iterative Methods for Systems Identification", 1966). These approaches have been set forth in order to estimate the parameters of General Rational Lag models of the type:

$y_t = (A(B)/C(B)) x_t + u_t$, where x_t is a set of nonstochastic variables, u_t is identically independently distributed, such that

$E(u_t) = 0; E(u_t^2) = \sigma_u^2; A(B) = \sum_{i=0}^{m} a_i B^i; C(B) = \sum_{j=0}^{h} c_j B^j; c_o = 1.$

They can be used to estimate the model:

$$1)\ \gamma_t = \left[F(B)/\Phi_x(B)\right] d_{t-1} + \left[Z(B)/\Phi_x(B)\right] a_t + u_t \qquad u_t \sim N(0,\sigma_u^2)$$

$$2)\ P_t = \left[\xi(B)/\Phi_x(B)\right] d_{t-1} + \epsilon_t \qquad \epsilon_t \sim N(0,\sigma_\epsilon^2),$$

where $d_{t-1} = a_{t-1} + a_{t-2}$,

$$\xi = (\xi_o, \xi_1, \dots, \xi_m)' = (\beta_1\theta_{xo}, \beta_1\theta_{x1}, \dots, \beta_1\theta_{xm})',$$

$$F = (F_o, F_1, \dots, F_m)' = (b\beta_1\theta_{xo}, b\beta_1\theta_{x1}, \dots, b\beta_1\theta_{xm})',$$

$$Z = (c\theta_{xo}, c\theta_{x1}, \dots, c\theta_{xm})' \text{ and}$$

$\Phi_x = (\Phi_{x1}, \Phi_{x2}, \dots, \Phi_{xm})'$ are the coefficients of the rational lags derived from the A.R.M.A. structure of x_t. An iterative maximum likelihood approach can be developed, which, if convergent, provides consistent estimates of Z, ξ, F, Φ_x.

($\gamma_t^{\bullet} = (1/\Phi_x(B))\gamma_t$, $d_t^{\bullet} = (1/\Phi_x(B))d_t$, $a_t^{\bullet} = (1/\Phi_x(B))a_t$,

$d_t^{\bullet\bullet} = (F(B)/\Phi_x(B))d_t^{\bullet}$, $a_t^{\bullet\bullet} = (Z(B)/\Phi_x(B))a_t^{\bullet}$ are computed and the M.L. function, expressed in terms of these derived variables, is maximized with respect to the parameters of interest. New values of the variables are computed, with respect to the new parameters, the M.L. function is once more maximized

additional exogenous variables enter the picture, due to the differences between the A.R.I.M.A. structures of different series and the corresponding difficulty in deriving a common A.R.I.M.A. struc

(Suite of note (1) of page 32)
with respect to the parameters of interest and so on until convergence is achieved).
As a consequence, we have consistent estimators of

$\xi(B) = \beta_1\theta_x(B)$ from equation 2)

and consistent estimators of

$F(B) = b\beta_1\theta_x(B)$ and of $Z(B) = c\theta_x(B)$ from equation 1).

$\hat{b} = \widehat{F(B)} / \widehat{\xi(B)}$ can be obtained.

However, unless we estimate $\theta_x(B)$, we cannot obtain an estimator of c. This reduces somewhat the validity of this approach. A possible solution would be that of evaluating the A.R.M.A. structure of the x_t series, obtaining in that way $\theta_x(B)$ and $\Phi_x(B)$. (Alternatively a nonlinear approximation could be introduced at each stage, which would allow us to avoid the computation of equation 2). Such an approach would be computionally very expensive).

This iterative approach requires initial values of

F, θ_x, and Φ_x.

They can be obtained - as suggested by Dhrymes, pag.250 - by means of principal components techniques.
If the error terms in the first equation above are not serially independent, more sophisticated approaches have to be used, connected with spectral analysis.

(For more details, see Dhrymes, p. 239-250).

ture.[1] These difficulties are due to the fact that if x_t is a vector of time series,i.e. represents more than one time series,the structural γ_t equation is overidentified.The method seen above can be assimilated to the estimation of the following three equations (recursive) system:

$$
(IV.13)\begin{cases} 1)\ \gamma_t - bP^a_t - cx_t = u_t & u_t \sim N(0,\sigma^2_u) \\ 2)\ P_t - \beta_1(x_{t-1}+x_{t-2}) = \epsilon_t & \epsilon_t \sim N(0,\sigma^2_\epsilon) \\ 3)\ [\phi_x(B)/\theta_x(B)]\ x_t = a_t & a_t \sim N(0,\sigma^2_a). \end{cases}
$$

(1) Assume the model includes two time series, x_{1t} and x_{2t}, with A.R.I.M.A. structures

$[\theta_1(B)/\phi_1(B)]a_{1t} = \Psi_1(B)a_{1t}$ and $[\theta_2(B)/\phi_2(B)]\ a_{2t} = \Psi_2(B)a_{2t}$

respectively.

(IV.12) can be written as:

$$1)\gamma_t = b\beta_1\Psi_1(B)(a_{1t-1}+a_{1t-2})+b\beta_2\Psi_2(B)(a_{2t-1}+a_{2t-2}) +$$

$$+c_1\Psi_1(B)a_{1t}+c_2\Psi_2(B)a_{2t}+u_t,$$ which is likely to lead to more than one estimator of b(the structural γ_t equation is overidentified).If $\theta_2(B)/\phi_2(B)=\Psi_2(B)=\theta_x(B)/\phi_x(B)=\Psi_x(B)=\theta_1(B)/\phi_1(B)$ $=\Psi_1(B)$,we would have:

$$2)\gamma_t=\Psi_x(B)\{b(\beta_1(a_{1t-1}+a_{1t-2})+\beta_2(a_{2t-1}+a_{2t-2}))+c_1a_{1t}+c_2a_{2t}\} +u_t$$

which provides a single consistent estimator of b.(Since -in practice- it is as if we had a single exogenous variable, so that the equation is exactly identified.Or we can assume that an additional restriction ,that the two series have equal structure,has been imposed,with the same effect on the identification of the γ_t structural equation).

If the A.R.I.M.A. structures of x_{1t} and of x_{2t} are different, (IV.12) can be written as:

$$3)\gamma_t=\Psi_1(B)(b\beta_1(a_{1t-1}+a_{1t-2})+c_1a_{1t}) +$$

$$+\Psi_2(B)(b\beta_2(a_{2t-1}+a_{2t-2})+c_2a_{2t})+u_t,$$ which is likely to lead to two different consistent values of b.

With respect to models of type(D.a),in which expectations of x_t are based on the A.R.I.M.A. structure of this very series,in which no equation is therefore missing,the problem is one of improper use of this information.As a consequence models which are potentially identified remain unidentified because of the multicollinearity problems mentioned above.

ooo

V. Some Final Remarks.

This paper has allowed to clarify two main points.

Case(D.b) only,(p.26),where anticipations about the exogenous variables,upon which price anticipations are based, are mean square error forecasts based on the A.R.I.M.A. structure of the series and where unanticipated shocks are provided by one period ahead forecast errors of these very series leads to a correct empirical evaluation.

Once the ARIMA structure of the series of interest has been evaluated,the model (IV.11) can be estimated by means of a standard two stages least squares approach.

The weakness of this method is that the ARIMA structure of a series can at best be approximated.This reduces both the accuracy and the validity of such an estimation.

The remaining versions of the Rational Expectations model cannot be properly estimated by means of linear least squares techniques unless some very restrictive assumptions are made about the ARIMA structure of the exogenous variables. (This holds true also for nonlinear methods of estimation).

- The 'observable reduced form' of any model involving Rational Expectations cannot be evaluated by means of least squares techniques (linear or nonlinear).

The filtering methods expounded above have to be introduced in which account is taken of the ARIMA structure of the exogenous variables.

If nonlinear techniques are used however, it could theoretically be sufficient, to estimate the standard model of this paper, to set forth the order of the ARIMA structure of the exogenous variables (provided appropriate normalization restrictions are made about it, to allow for identification of the parameters) without having to estimate it in advance.

This approach reduces the accuracy of the analysis.

The difficulties we have tried to solve in this paper are due to the fact that standard evaluation approaches of the Rational Expectations models examined above have attempted to estimate a three equations system (the third equation providing the ARIMA structure of the exogenous time series) by estimating two equations only. We have a model misspecification.

Case (D.b) above leads to a correct estimation because it includes the investigation of this very equation: the ARIMA structure of the exogenous time series.

The computational cost of adding the ARIMA structures' analysis of the exogenous time series may be relevant if several exogenous variables are included in the 'observable reduced forms', and the increase in accuracy may be relatively small.

A possible suggestion is that whenever a 'reduced form' model has to be estimated involving this kind of multicollinearity problems, "Quasi Rational Expectations" be substituted to Rational Expectations of the unobserved variable. (The

former are obtained as one period ahead mean square error forecast based upon the ARIMA structure of the series to be anticipated). In that case the analysis would involve two steps:

identification, estimation and verification of the structure of the P_t time series;

estimation of the structural equation:

$$Y_t = a + bP^a_t + cx_t + u_t \, , \quad \text{where}$$

P^a_t is a more or less sophisticated function of its own past values. If u_t is serially correlated, consistency problems would arise in the estimation of the equation above if we set $Y_t = P_t$.

ooo

APPENDIX

The case of Some More Complex Reduced Forms.

The Natural Rate of Unemployment with Rational Expectations model does not include, usually, lagged exogenous variables in the relevant equations.

The 'observable reduced forms' of a Rational Expectations model may include lagged exogenous variables, which in turn appear in the relationship on the basis of which 'Rational Expectations' of the unobservable variable are computed.

If the following model has to be estimated,

$$
\text{(A.I)}\begin{cases} \text{a)}\ P_t = c_o x_t + c_1 x_{t-1} + \dots + c_n x_{t-n} + bP^a_t + u_t; \quad u_t \sim N(0,\sigma^2_u) \\ \text{b)}\ y_t = d_o x_t + d_1 x_{t-1} + \dots + d_n x_{t-n} + dP^a_t + v_t; \quad v_t \sim N(0,\sigma^2_v) \end{cases}
$$

where $P^a_t = \frac{c_o}{1-b}\bar{x}_t + \frac{c_1}{1-b} x_{t-1} + \dots + \frac{c_n}{1-n} x_{t-n}$.

It can be shown that the difficulties expounded above (multicollinearity between P^a_t - i.e. its determinants - and the exogenous variables) introduce identification problems also in this context.

An intuitive proof, to keep the algebra as simple as possible, is the following:

$$
\text{(A.II)}\begin{cases} \text{a)}\ P_t = c_o x_t + c_1 x_{t-1} + bP^a_t + u_t & u_t \sim N(0,\sigma^2_u) \\ \text{b)}\ y_t = d_o x_t + d_1 x_{t-1} + dP^a_t + v_t & v_t \sim N(0,\sigma^2_v) \end{cases}
$$

Assume x_t has an ARIMA (p,0,0) = AR(p) structure.

$$
x_t = \Phi_{x1} x_{t-1} + \Phi_{x2} x_{t-2} + \dots + \Phi_{xp} x_{t-p} + a_t \qquad a_t \sim N(0,\sigma^2_a).
$$

$$
P_t = \frac{c_o}{1-b}\bar{x}_t + \frac{c_1}{1-b} x_{t-1} = \beta_o \bar{x}_t + \beta_1 x_{t-1}
$$

We have to evaluate:

$$
\text{(A.III)}\begin{cases} \text{a)}\ P_t = c_o x_t + c_1 x_{t-1} + b\left(\frac{c_o \bar{x}_t}{1-b} + \frac{c_1 x_{t-1}}{1-b}\right) + u_t; \quad u_t \sim N(0,\sigma^2_u) \\ \text{b)}\ P_t = \frac{c_o}{1-b}\bar{x}_t + \frac{c_1}{1-b} x_{t-1} + \epsilon_t & \epsilon_t \sim N(0,\sigma^2_\epsilon). \end{cases}
$$

Assume expectations are made 'optimally', on the basis of the ARIMA structure of the x_t time series. The system satisfies the rank conditions for identification.(1)

$$
\text{(A.III')}\begin{cases} \text{a)}\ P_t = c_o x_t + \left(\frac{bc_o}{1-b}\Phi_1 + \frac{bc_1}{1-b} + c_1\right) x_{t-1} + \\ \qquad + \frac{bc_o}{1-b}\Phi_2 x_{t-2} + \dots + \frac{bc_o}{1-b}\Phi_p x_{t-p} + u_t \\ \text{b)}\ P_t = \left(\frac{c_o}{1-b}\Phi_1 + \frac{c_1}{1-b}\right) x_{t-1} + \frac{c_o}{1-b}\Phi_2 x_{t-2} + \\ \qquad + \dots + \frac{c_o}{1-b}\Phi_p x_{t-p} + \epsilon_t \end{cases}
$$

---

(1) The three equations system is:

a) $P_t = bP^a_t + c_o x_t + c_1 x_{t-1} + u_t$

b) $P_t = \beta_o \bar{x}_t + \beta_1 x_{t-1} + \epsilon_t$

c) $x_t = \Phi_1 x_{t-1} + \Phi_2 x_{t-2} + \dots + \Phi_p x_{t-p} + a_t$.

It can be shown that equation a) is exactly identified, equation b) is overidentified and equation c) is exactly identified.

The maximization of the joint likelihood function, following the approach set forth above, allows to compute:

$$(A.III'')\left[-(T/2)\ln(2\Pi)\right] + \left[-(T/2)\ln\sigma_u^2 - 1/(2\sigma_u^2)\sum_t \left(P_t - c_o x_t - \left(c_1 + \frac{bc_o}{1-b}\Phi_1 + \frac{bc_1}{1-b}\right)x_{t-1} - \frac{bc_o}{1-b}\Phi_2 x_{t-2} - \dots - \frac{bc_o}{1-b}\Phi_p x_{t-p}\right)^2\right]$$

$$+ \left[-(T/2)\ln\sigma_\epsilon^2 - 1/(2\sigma_\epsilon^2)\sum_t \left(P_t - \left(\frac{c_o}{1-b}\Phi_1 + \frac{c_1}{1-b}\right)x_{t-1} - \frac{c_o}{1-b}\Phi_2 x_{t-2} - \dots - \frac{c_o}{1-b}\Phi_p x_{t-p}\right)^2\right] = \ln L.^{(1)}$$

From $\ln L_2$ - i.e. (A.III''b)) - we obtain:

$\left(\widehat{\frac{c_o}{1-b}\Phi_1 + \frac{c_1}{1-b}}\right)$, $\widehat{\frac{c_o}{1-b}\Phi_2}, \dots, \widehat{\frac{c_o}{1-b}\Phi_p}$ and thus $\widehat{\frac{c_o}{1-b}}$ and $\widehat{\frac{c_1}{1-b}}$,

provided that $\widehat{\Phi}_1, \widehat{\Phi}_2, \dots, \widehat{\Phi}_p$ are known. (2)

From $\ln L_1$ - i.e. (A.III'a)) - we obtain:

c_o, $\widehat{\frac{bc_o}{1-b}\Phi_1}$ and $\left(\widehat{\frac{bc_o}{1-b}\Phi_1 + \frac{bc_1}{1-b} + c_1}\right)$, and thus $\hat{b}, \hat{c}_o$ and $\hat{c}_1$.

The same result holds true if we want to evaluate (A.II b)).

---

(1) We could as well add a third element,
$\ln L_3 = \left[-(T/2)\ln\sigma_a^2 - 1/(2\sigma_a^2)\sum_t (x_t - \Phi_1 x_{t-1} - \dots - \Phi_p x_{t-p})^2\right]$,
for the estimation of the x_t autoregressive structure.

(2) The price anticipations equation is overidentified so that p-1 consistent estimators of $\widehat{c_o/(1-b)}$ and of $\widehat{c_1/(1-b)}$ will be obtained as $\widehat{\frac{c_o}{1-b}\Phi_1}/\widehat{\Phi}_1, \dots, \widehat{\frac{c_o}{1-b}\Phi_p}/\widehat{\Phi}_p$ and as $\left(\widehat{\frac{c_o}{1-b}\Phi_1 + \frac{c_1}{1-b}}\right) - \widehat{\frac{c_o}{1-b}\Phi_1}$.

This will be the case if the order of the autoregressive component of the structure of the x_t time series is higher than the lag of the exogenous variables appearing in the equations. (1)

---

(1) Consider the following model:

$$(A.IV)\begin{cases} a)\ P_t = c(B)x_t + bP_t^a + u_t = c_o x_t + c_1 x_{t-1} + c_2 x_{t-2} + bP_t^a + u_t & u_t \sim N(0,\sigma_u^2) \\ b)\ P_t = \frac{c_o}{1-b}\bar{x}_t + \frac{c_1}{1-b}x_{t-1} + \frac{c_2}{1-b}x_{t-2} + \epsilon_t & \epsilon_t \sim N(0,\sigma_\epsilon^2), \end{cases}$$

where $x_t = \Phi_1 x_{t-1} + a_t$ and where $c(B)$ is a second order polynomial in the lag operator (B), whereas x_t has a first (or even a second) other autoregressive structure.

We have:

$$(A.IV')\begin{cases} P_t = c_o x_t + \left[c_1 + \frac{bc_o\Phi_1}{1-b} + \frac{c_1}{1-b}\right]x_{t-1} + \frac{c_2}{1-b}x_{t-2} + u_t & u_t \sim N(0,\sigma_u^2) \\ P_t = \left[\frac{c_o\Phi_1}{1-b} + \frac{c_1}{1-b}\right]x_{t-1} + \frac{c_2}{1-b}x_{t-2} + \epsilon_t & \epsilon_t \sim N(0,\sigma_\epsilon^2) \end{cases}$$

The maximization of the corresponding joint likelihood function does not allow to obtain distinct estimates of the parameters of interest.

Otherwise the rank conditions for identification ofthe equations of the system will not be satisfied.Neither(A.IV.a)) nor (A.IV.b)) will satisfy the rank conditions for identification.

It should be noticed however that precisely because x_t has an ARIMA structure which includes an autoregressive component (i.e. x_t is serially correlated),identification becomes problematic for the system,(more specifically for equation (A.III'),even if the order of the autoregressive component of the structure of x_t is higher than that of the lags of the exogenous variables in the reduced forms.

Substituting - in order to take account of collinearity problems - for

$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \dots + \phi_p x_{t-p} + a_t$ in (A.III.a)),we obtain:

$$
\text{(A.V)}\begin{cases}
\text{a) } P_t = \left[\frac{bc_0}{1-b}\phi_1 + \frac{bc_1}{1-b} + c_1 + c_0\phi_1\right]x_{t-1} + \left[\frac{bc_0}{1-b}\phi_2 + c_0\phi_2\right] \\
\qquad x_{t-2} + \dots + \left[\frac{bc_0}{1-b}\phi_p + c_0\phi_p\right]x_{t-p} + W_t \\
\text{b) } P_t = \left[\frac{c_0}{1-b}\phi_1 + \frac{c_1}{1-b}\right]x_{t-1} + \frac{c_0}{1-b}\phi_2 x_{t-2} + \dots + \\
\qquad + \frac{c_0}{1-b}\phi_p x_{t-p} + \epsilon_t, \quad \text{where } W_t = c_0 a_t + u_t.
\end{cases}
$$

This means that a restriction is imposed on the coefficients of the first equation.It will not satisfy the rank conditions for identification.

The maximization of the joint likelihood function with respect to $\sigma_u^2, \sigma_\epsilon^2$ and the coefficients of $x_{t-1}, x_{t-2}, \dots x_{t-p}$ does not allow to obtain estimates of c_0, c_1 and b. (A.V. a))

is underidentified.(Indeed,if it is possible to obtain estimates of $\frac{c_0}{1-b}$ and $\frac{c_1}{1-b}$ -(p - 1) couples of estimates which are not necessarily identical in finite samples - this is not sufficient to disentangle the parameters of interest.

The estimation of (A.II.b)) requires the evaluation of the following system:

$$
\text{(A.VI)}\begin{cases}
\text{a) } y_t = \left[d_0\phi_1 + d_1 + \frac{dc_1}{1-b} + \frac{dc_0}{1-b}\phi_1\right]x_{t-1} + \left(d_0 + \frac{dc_0}{1-b}\right)\phi_2 \\
\qquad x_{t-2} + \dots + \left(d_0 + \frac{dc_0}{1-b}\right)\phi_p x_{t-p} + V_t \\
\text{b) } P_t = \left[\frac{c_0}{1-b}\phi_1 + \frac{c_1}{1-b}\right]x_{t-1} + \frac{c_0}{1-b}\phi_2 x_{t-2} + \dots + \\
\qquad + \frac{c_0}{1-b}\phi_p x_{t-p} + \epsilon_t, \text{ where } V_t = d_0 a_t + v_t.
\end{cases}
$$

The maximization of the joint likelihood function - using the approach mentioned above - shows that if from equation (A.VI.b)) it is possible to estimate $\widehat{\frac{c_0}{1-b}}$ and $\widehat{\frac{c_1}{1-b}}$, this is not sufficient to obtain estimates of d,d_1 and d_2 from(A.VI.a)).

Only estimators for $\widehat{(d_0 + \frac{dc_0}{1-b})}$ and for $\widehat{(d_1 + \frac{dc_1}{1-b})}$ can be computed.

Equation (A.VI.a)) is not identified because of the multicollinearity problem mentioned above.

It can be shown that in the context of this model the filtering methods set forth in section IV no longer hold.

As a consequence Rational Expectations reduced forms which include lagged exogenous values of the very variables on the basis of which Rational Expectations are made cannot be properly estimated and are affected by multicollinearity and thus by identification problems.

ooo

REFERENCES

Barro,R.J.(1977):"Unanticipated Money Growth and Unemloyment in the United States",American Economic Review 67, no.2,(March),p.101-115.

Barro,R.J. (1978):"Unanticipated Money,Output,and the Price Level in the United States",Journal of Political Economy 86,no.4,(August),p.549-580.

Bomhoff,E.J. (1980):"Inflation,the Quantity Theory,and Rational Expectations",North Holland Publishing Company, Amsterdam.

Box,G.E.P. & Jenkins,G.M. (1970):"Time Series Analysis,Forecasting and Control",Holden Day,San Francisco.

Dhrymes,P.J. (1971):"Distributed Lags.Problems of Estimation and Formulation",Holden Day,San Francisco.

Fourcans,A. (1978):"Inflation and Output Growth:the French Experience,1960-1975",in K.Brunner & A.H.Meltzer (eds) 'The Problem of Inflation',Journal of Monetary Economics,Supplement no.8,p.81-140.

Fratianni,M. (1978):"Inflation and Unanticipated Changes in Output in Italy",in Brunner K. & Meltzer A.H. eds.:'The Problem of Inflation',Journal of Monetary Economics,Supplement no.8.

Korteweg,P. (1978):"The Economics of Inflation and Output Fluctuations in the Netherlands,1954-1975", in K. Brunner & A.H.Meltzer (eds) 'The Problem of Inflation', Journal of Monetary Economics,Supplement no.8,p.17-79.

Korteweg,P. & Meltzer,A.H. (1978): "Inflation and Price Changes:Some Preliminary Estimates and Tastes of Alternative Theories",in K.Brunner & A.H.Meltzer(eds)'The Problem of Inflation',Journal of Monetary Economics, Supplement no.8,p.325-353.

McCallum,B.T. (1978):"Rational Expectations and the Natural Rate Hypothesis:Some Consistent Estimates",Econometrica 44,no.1,(January),p.43-52.

Meltzer,A.H. (1977):"Anticipated Inflation and Unanticipated Price Change.A Test of the Price-Specie Flow Theory and the Phillips Curve",Journal of Money Credit and Banking 9,no.1,(February),p.182-205.

Muth,J.F. (1961):"Rational Expectations and the Theory of Price Movements",Econometrica 29,no.3,(July),p.315-319.

Nelson,C.R. (1976):"Rational Expectations and the Estimation of Econometric Models",International Economic Review 16,no.3,(October),p.555-561.

Nerlove,M.,Grether,D.M. & Carvalho,J.L. (1979):"Analysis of Economic Time Series.A Synthesis",Academic Press, New York.

Pashigian,P. (1970):"Rational Expectations and the Cobweb Theory",Journal of Political Economy 78,p.338-352.

Sargent, T.J. (1978):"Rational Expectations,Econometric Exogeneity and Consumption",Journal of Political Economy 86,no.4,p.673-699.

Steiglitz K. & McBride,L.E. (1966):"Iterative Methods for Systems Identification",Technical Report no.15,Department of Electrical Engeneering,Princeton University,Princeton.

Wallis,K.F. (1980):"Econometric Implications of the Rational Expectations Hypothesis",Econometrica 48,no.1,(January).

ooo

INDEX

oooooooo