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# Modelling green attitudes and informality along the North-South divide*

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**Abstract**

Public perceptions of the urgency of fighting climate change differ between countries and have fluctuated over time. Heterogeneity in ecological thinking poses a problem because limiting global warming requires cohesion and coordination among the socioeconomic system's leading players in developed and developing countries. Most studies in the field have wrongly treated advanced and emerging economies as similar systems in different positions of a linear development path. Developing economies are structurally different as they are populated by a large informal sector that accounts for up to half of economic activity. The role of the informal sector in economic development remains controversial, let alone the implications of its existence to a successful green transition. We present a macrodynamic model to study the interplay between informality and heterogeneity in ecological thinking. The model explains the endogenous emergence of four stable equilibria. Two have minor informality but significant differences in green attitudes. We refer to them as the US *vs* Europe cases in the Global North. In the other two, informality prevails, while we observe sharp differences in general support for mitigation policies, resembling an Asia *vs* Latin America scenario. Studying the basins of attraction allows us to provide policymakers with additional insights into the political economy of climate change in the Global South.

**Keywords**: Climate change; Informality; Green attitudes, Global South; Development.

**JEL**: Q01; Q56; O11; O44.

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# 1 Introduction

Global anthropogenic carbon emissions have continued to rise, as have cumulative CO2 in the atmosphere since the mid-19th century. Despite the near-global consensus among the scientific community, public perceptions of the urgency to fight climate change differ between countries and have fluctuated over time (e.g. WRP, 2021; People's Climate Vote, 2021; Dechezlepretre et al., 2023). Heterogeneity in ecological thinking posits a problem because limiting global warming requires cohesion and coordination among the players of the socioeconomic system in both developed and developing countries. Investigating how environmental attitudes adapt and can be coordinated requires a complex systems approach (Hommes, 2021). Social scientists have only recently started exploring how boundedly rational heterogeneous agents communicate and learn, leading to emergent macro climate-related behaviours as the aggregate outcome of micro-interactions.

It must be noted, however, that most studies in the field have wrongly treated advanced and emerging economies as similar systems in distinct positions of a linear development path. This is a critical limitation as developing economies are structurally different (e.g. Lewis, 1954; see also Ros, 2016; Oliveira and Lima, 2020; Skott, 2023, pp. 155-165). They are populated by a large informal sector that accounts for up to half of economic activity (La Porta and Shleifer, 2014; Ulyssea, 2020). Such a characteristic is one of the elements that define the Global South in contraposition to nations in the Global North. The role of the informal sector in economic development remains controversial, let alone the implications of its existence to a successful green transition. Our knowledge of the implications of heterogeneity in ecological thinking in informal economies is poor. We know little about the feedback mechanisms from the economy-environment to the composition of green attitudes under high levels of informality.[1]

Our research question lies in the intersection between these two major themes. We develop a heterogeneous agents' macrodynamic model to study the interplay between informality and heterogeneity in ecological thinking. Its main novelty is providing a tractable framework for studying the interaction between two "choices": Whether to belong to the formal sector and supporting or opposing climate mitigation policies. Both change endogenously following the discrete-choice approach (Brock and Hommes, 1997; for a literature review, see Franke and Westerhoff 2017). The probability of an agent choosing (in)formality depends on scale and growth effects. In contrast, the probability of acknowledging the urgency of climate change is subject to peer effects, the size of the formal sector, and the perception of the climate threat.

Conditional to the composition of green attitudes, policymakers choose the carbon tax, which has an autonomous and sentiment-induced component. Firms react to a stable policy and adopt more or less energy-efficient production techniques. This decision affects the output growth rate in the formal sector, thus influencing the size of the informal sector. Given that only the formal sector is assumed to pollute, the latter becomes a subproduct of economic activity that feedback on environmental attitudes. Among our main findings, three deserve special attention. First, if the scale and peer effects are arbitrarily small but technology's response to the carbon tax is sufficiently large, and people care about economic performance when forming environmental preferences, then two stable equilibria coexist. One green-formal and another non-green-informal.

[1] Empirical evidence on the informality-environment nexus is scarce. Welcome exceptions include Elgin and Oztunali (2014), who document an inverse-U relationship between the informal sector size and emissions for 152 countries. One of the main difficulties is disentangling between higher pollution following less regulation, as in poor countries, and higher emissions due to more energy-intensive productive structures, as in rich nations (see also Goel and Saunoris, 2020).

Second, if the scale and peer effects are sufficiently large, the model explains the endogenous emergence of up to four stable equilibria similar to those identified in major climate surveys. Two have reduced informality but significant differences in green attitudes. We refer to them as the US *vs* Europe cases, capturing the state of affairs in the Global North. In the other two, informality prevails, but we observe sharp differences in general support for mitigation policies, resembling an Asia *vs* Latin America scenario. Studying the basins of attraction allows us to provide policymakers with additional insights into the political economy of climate change in the Global South. Finally, a sufficiently high autonomous component of carbon taxes and strong growth can make the other attractors disappear, leading to a unique green-formal solution. This last result supports the need for cooperation between developed and developing countries, suggesting that implementing an international carbon tax that does not depend on domestic politics could create a win-win situation for both regions.

This paper joins recent efforts to build environmental heterogeneous agent models rooted in discrete choice theory (e.g. Hommes and Zeppini, 2014; Zeppini and van den Bergh, 2020; Cahen-Fourot et al., 2023, among others). To the best of our knowledge, we are the first in this literature to put informality at the centre of the stage. Applications to climate-related questions include Zeppini (2015) and Mercure (2015) addressing the technology adoption problem; or Campiglio et al. (2024) showing that commitment uncertainty of policymakers to announced targets might result in the coexistence of desirable and undesirable transition paths. Studies by Davila-Fernandez and Sordi (2020) and Cafferata et al. (2021) have explore the role of environmental regulation using Porter's hypothesis as the connection between macroeconomic and environmental dimensions. A reference to the international political economy of the green transition appears in Galanis et al. (2023), who uses the discrete-choice framework to investigate countries' adherence to transnational climate agreements. While they contrast Global North and South positions, their model is not explicitly concerned with informality, which remains to be properly incorporated into the discussion.

Our study also relates to a family of Agent-Based Models (ABMs) dedicated to the political economy of climate change. They include the "battle of perspectives" family initiated by Janssen and de Vries (1998) and ABMs dedicated to climate policy support, emphasising carbon tax acceptability (e.g. Foramitti et al., 2021; for a discussion of ongoing controversies and possible research avenues, see van den Bergh and Savin, 2021). We also engage with models in a structuralist macro-development tradition (such as the green-Lewis system in Oliveira and Lima, 2020). As we will show, some of our policy recommendations align with the idea of an Environmental Big Push in the spirit of Rosenstein-Rodan (1943). We add to those efforts by providing micro-foundations to boundedly rational agents' decisions. This is also an innovation concerning Bento et al. (2018), one of the few studies that, using a more conventional optimisation toolbox, models the economic consequences of carbon taxes in the presence of an informal sector. Last but not least, the coexistence of two or more stable equilibria recalls the literature on poverty traps both neoclassical and evolutionary (e.g. Azariadis and Stachurski, 2005; Sanchez-Carrera, 2019). While self-reinforcing mechanisms are part of our narrative, our model is a novel application linking underdevelopment to climate change.

The remainder of the article is organised as follows: Section 2 provides some empirical insights into the relationship between environmental attitudes and informality. We overview the relevance of heterogeneity in ecological thinking contrasting developed and developing countries. Section 3 presents our modelling framework. We describe the main transmission channels and underlying mechanisms resulting in our 2-dimensional nonlinear map. Section 4 lists a series of Propositions demonstrating analytically the conditions for the coexistence of stable attractors. Numerical experiments confirm our analytical findings and allow us to provide a more concrete visualisation of what is going on in the model. It is shown how to achieve a green-formal equilibrium, either through the dissolution of the other attractors or relying on an environmental Big Push. The latter implies a transition between basins of attraction. Some final considerations follow.

## 2 Some empirical insights

Data from numerous global surveys now indicate that climate change is broadly accepted as an important problem (e.g. WRP, 2021; People's Climate Vote, 2021; Dechezlepretre et al., 2023). While a more "optimistic" reader could interpret the evidence as indicating convergence towards a global recognition of the necessity of immediate urgent action, we believe there are well-grounded reasons for being more sceptical. First, the "opposition" to adopting a green agenda has become more subtle. Frequently, it is not open and even pays lip service to the relevance of mitigation policies. Second, the general public might recognise there is a climate emergency. However, there is a difference between a generic recognition of the matter and being willing to effectively address the problem.

Climate scientists have been clear that the Earth is well beyond its planetary boundaries (e.g. Rockstrom et al., 2009; Steffen et al., 2020; Richardson et al., 2023). The United Nations reports there is currently no credible pathway to limit any rise in global temperature to 1.5C (UN, 2022), with reasonable projections suggesting there is only a 5% chance that temperatures will increase less than 2C (see Raftery et al., 2017). Countries' new pledges would shave 1% off emissions in 2030, below the 50% reduction necessary to keep the 1.5C target alive. A closer look at climate surveys suggests that there is no strong support for implementing a green agenda in most parts of the world. At least not in the scale and urgency climate scientists have argued. Thus, heterogeneity in ecological thinking is perhaps more relevant than ever.

Let us have a look at the last edition of the World Risk Poll (WRP, 2021), which interviewed more than 125,000 people in 120 countries. When asked: *How serious a threat do you think climate change is to the people in this country in the next 20 years?* Two-thirds of respondents perceive climate change as a threat, but just a little more than half of them acknowledge it is a *very* serious threat. Computing the difference between the share of those recognising the seriousness of the issue with the rest of the population, we can build an index $\Phi \in (-1, 1)$ that will be useful for the model to be developed later. If $\Phi = 1$, then everybody recognises the need for urgent action. Conversely, $\Phi = -1$ implies nobody cares about global warming and either openly opposes green policies or pays lip service to the problem. A $\Phi = 0$ indicates a divided society between the two groups. Fig. 1 shows the emergence of four distinct regions along the Global North-South divide.

Developed countries in the Global North form two distinct clusters. A darker green Europe marks a stronger recognition of the urgency of fighting climate change. The United States (US) in yellow depicts a certain division, to some extent reflecting the politicisation of the topic in the country (see Dechezlepretre et al., 2023). On the other hand, developing countries are also divided. Only 1/3 of the population perceives climate change as a serious threat in Asia, which has a solid energy-intensive industrial base. Thus, most countries in the region have a $\Phi < 0$ and appear in brow colours. Such figures contrast Latin America, where more than 70% understand the seriousness of the climate emergency so that $\Phi > 0$.

To the extent that developing countries have a large informal sector that frequently accounts for more than half of employment, we take the share of formal workers minus those informal to build an index $\Omega \in (-1, 1)$. It comes with the desirable property that in economies where the formal sector dominates, $\Omega > 0$, while informal countries report a

$\Omega < 0$. Taking those self-employed as a proxy of informality, Fig. 2 plots the $\Phi$-$\Omega$ space for 117 countries. The blue quadrant strongly recognises the urgency to fight climate change, and most of the economy is formal. Contrariwise, the red quadrant shows countries with high informality rates and insufficient support to implement a green agenda. In grey, we have hybrid cases, i.e., formal countries with non-green environmental attitudes or informal ones that see global warming as a serious threat.

Let us turn to the People's Climate Vote (2021) to provide some robustness to this informal discussion. Coordinated by the United Nations and the University of Oxford, they interviewed 1.2 million respondents and reported statistics for 37 countries. When asked: *Do you think climate change is a global emergency?* Again, two-thirds responded yes. However, when confronted with the follow-up question: *If yes, what should we do about it?* A small percentage chose *do everything necessarily, urgently.* Most opted for *act slowly while we learn more about what to do* and *the world is already doing enough.* If, as before, we plot green attitudes against our formality index, Fig. 3 shows a few countries in the blue and red quadrants. Most appear in the upper-left in grey, i.e. relatively high formality rates and non-green attitudes. Not a single country where more than half of workers are informal has a majority willing to do whatever it takes to respond to climate change.

Together, these figures indicate the existence of two groups in the Global North and the Global South. There is a significant contrast between the US and Europe, with minor informality but considerable differences in green attitudes. On the other hand, while Asia and Latin America have high informality rates, the former region is generally less concerned about the climate. At the same time, the latter appears in solid green. Moreover, a comparison of Figs. 2 and 3 provides some useful insights. First, a large informal sector and people willing to tackle climate change urgently do not appear to be a strong attractor. Few to no countries fall into it. Second, while blue and red cases seem the most natural cases and somehow echo the idea of a linear development path from non-green and informal to green and formal, many countries have a dominant formal sector but see climate change as a secondary problem.

The role of the informal sector in economic development remains controversial, let alone the implications of its existence to a successful green transition. We need to learn more about the mechanisms explaining the emergence and coexistence of these combinations between $\Phi$ and $\Omega$. Such an effort includes providing policymakers insights on achieving the more desirable blue equilibria, especially among developing countries. La Porta and Shleifer (2014) identify as it five major stylised facts of informality:

1. The informal sector is huge.
2. It has extremely low productivity.
3. Lowering registration and other "formalisation costs" do not bring many informal firms into the formal sector nor spur output growth.
4. Formal and informal sectors are largely disconnected, with the latter facing strong financial constraints.
5. Informality becomes less significant as the economy grows.

Taking them as a starting point, the model developed in the next section is a first attempt to map heterogeneity in ecological thinking under high informality levels. Similarly to the poverty traps (Azariadis and Stachurski, 2005) and structuralist macro-development literature (Oliveira and Lima, 2020), we rely on a combination of inertia with self-reinforcement mechanisms. In our case, they come from within and between the (in)formality and environmental attitudes dimensions. Moreover, a distinctive feature is treating each problem dimension as a "choice" made by boundedly rational agents.

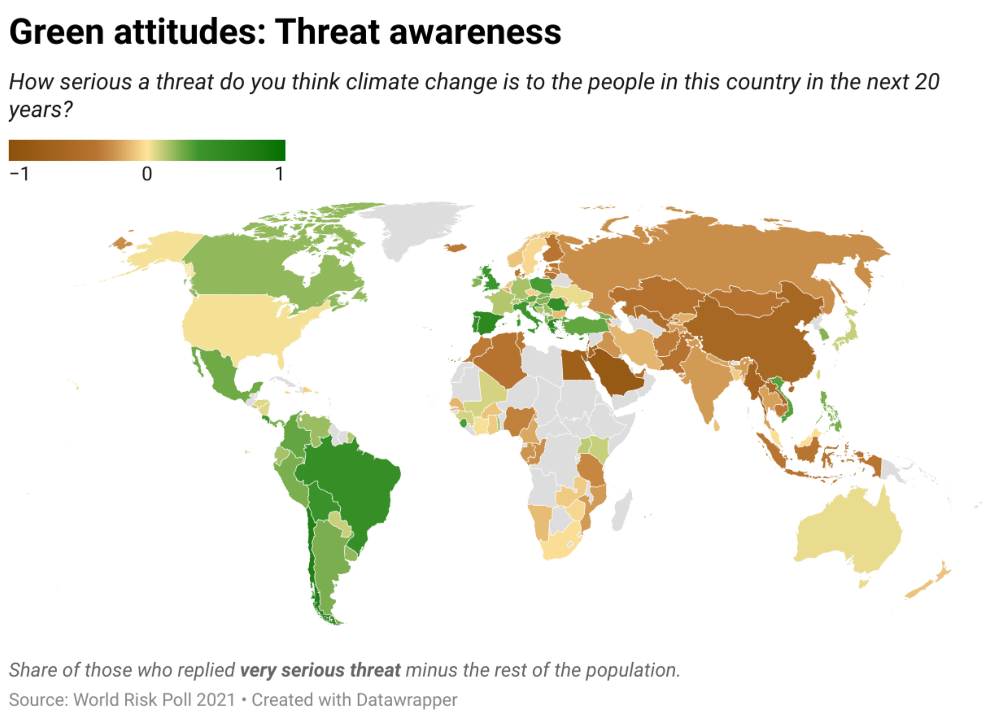



Figure 1: Attitudes towards climate change in the Global North and South. Green colours represent the prevalence of climate change awareness, $\Phi > 0$, while in brown, we have countries where more neutral or indifferent attitudes prevail, $\Phi < 0$.

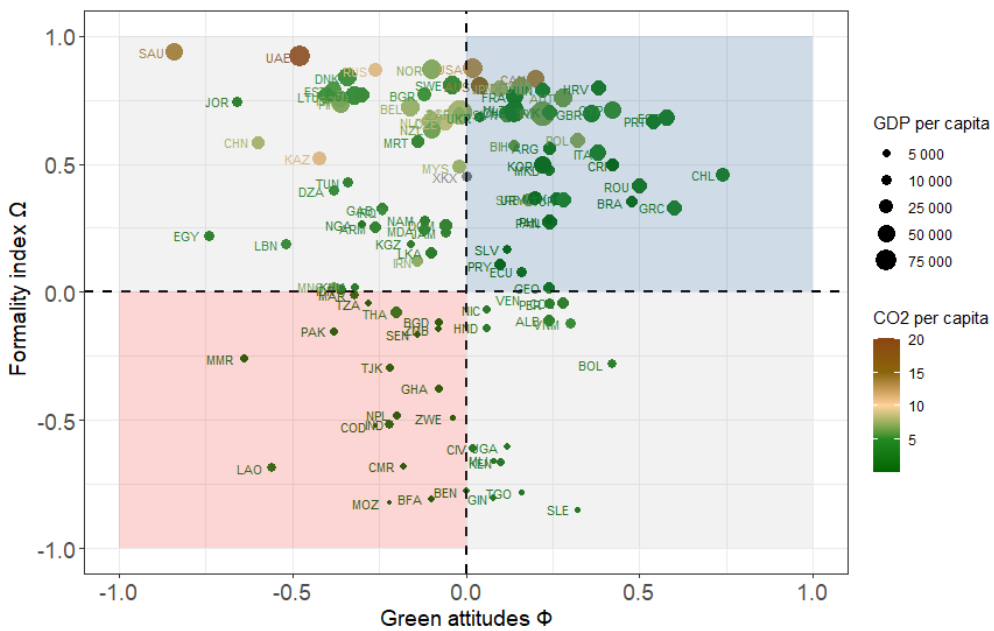



Figure 2: Green attitudes $\Phi \in [-1, 1]$ and the formality index $\Omega \in [-1, 1]$. The former reflects threat awareness. The latter corresponds to the share of those who are not minus those who are self-employed. GDP per capita is reported in 2017 PPP international dollars.

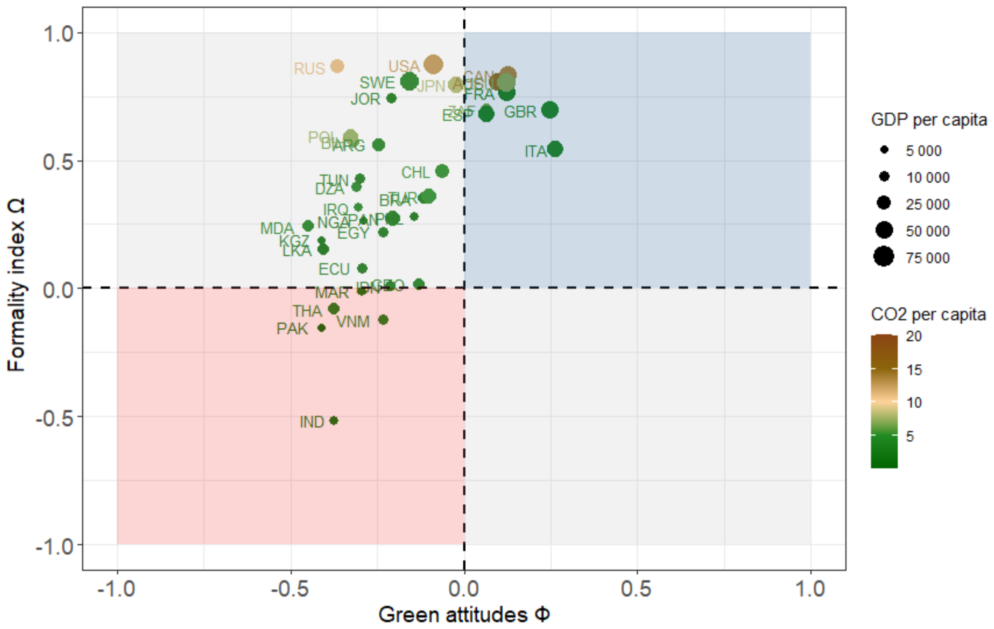



Figure 3: Green attitudes $\Phi \in [-1, 1]$ and the formality index $\Omega \in [-1, 1]$. The former reflects recognition of the urgency of action on climate change. The latter corresponds to the share of those who are not minus those who are self-employed. GDP per capita is reported in 2017 PPP international dollars.

# 3 The model

Our primary purpose is to provide a tractable framework for studying the interaction between two choices. The first relates to belonging or not to the formal sector. One could debate if this can be considered a choice, as informality is frequently the only available alternative to firms and workers in that situation. Still, even in that case, agents will necessarily belong to one of the states with a given probability. Large informality levels are a defining characteristic of most developing economies. An extensive literature has investigated over the past decades the determinants of informality, differentiating between its extensive and intensive margins (for a review, see La Porta and Shleifer, 2014; Ulyssea, 2020). The latter refers to formal *vs* informal employment relations and will be the main focus of this study. The probability of being formal depends on a vector of explanatory variables; evidence is not always clear-cut. Here, we will concentrate on the mechanism that, in our view, has more empirical support: Informality becomes less significant as the economy grows.

The second choice is how seriously people support strong environmental action. In general, global warming is broadly accepted as an important problem (e.g. WRP, 2021; People's Climate Vote, 2021; Dechezlepretre et al., 2023). However, to effectively address the climate emergency and to influence policy, arguably, we need more than just a generic recognition of the matter. Figs. 1-3 suggest that heterogeneity in ecological thinking emerges once we ask how severe or urgent agents perceive the problem. The literature on environmental attitudes provides valuable insights into the probabilities of supporting or opposing a green agenda (e.g. Drews and van den Bergh, 2016; Whitmarsh and Capstick, 2018). Our model addresses two main channels. On the one hand, high levels of informality reduce the probability of recognising the urgency of the climate threat because people care more about immediate, relatable needs rather than something that seems to happen in the distant future. On the other hand, they also respond to carbon emissions or the consequences of environmental degradation that follow from it. Fig. 4 provides a summarising diagram of the model.

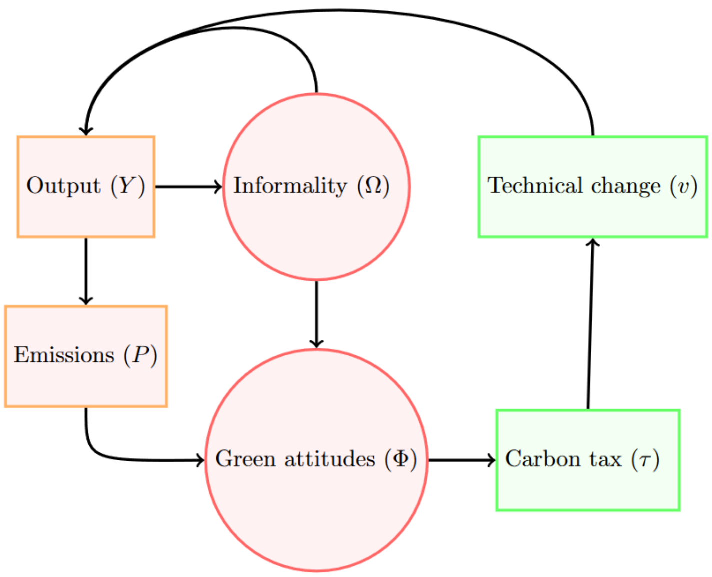



Figure 4: A summarising diagram of our behavioural macro model.

Consider an economy consisting of a formal and an informal sector, represented by the respective subscripts $i = \{F, I\}$. The former uses energy and labour inputs, while the latter uses only labour. Relative prices are assumed to be constant and, for simplicity, are normalised to one. The population is supposed to be constant and equal to the labour force. Workers choose their sector, but the decision can be reversed. This means that we will not discuss the conditions behind firms' decision to formalise, i.e. the extensive margin of informality, dealing only with the intensive margin. Finally, agents differ in their support of adopting a carbon tax. Green sentiments and the sectoral composition of the economy respond endogenously to the macroeconomic conditions, asynchronously updating their state following a discrete-choice approach (Brock and Hommes, 1997; Hommes et al., 2005).

## 3.1 Formal sector

Output ($Y$) in the formal sector is produced using a Leontief production function that combines energy ($E$) and labour ($L$) weighted by the respective technical coefficients:

$$Y_{F_t} = \min \{E_t v_t, L_{F_t} q_{F_t}\} \tag{1}$$

where $v$ is the output-energy ratio and $q$ stands for labour productivity. This technology is in line with evidence suggesting there is limited substitutability between human and non-human production factors (e.g. Gechert et al., 2022). More importantly, it also indicates energy is an underlying precondition for any human activity so that $E$ and $L$ are complements rather than substitutes.

From the Leontief efficiency condition, it follows:

$$\begin{aligned} Y_{F_t} &= E_t v_t \\ q_{F_t} &= \frac{Y_{F_t}}{L_{F_t}} \end{aligned} \tag{2}$$

In growth rates, we have that:

$$
\begin{aligned}
\frac{Y_{F_t} - Y_{F_{t-1}}}{Y_{F_{t-1}}} &= \frac{E_t - E_{t-1}}{E_{t-1}} + \frac{v_t - v_{t-1}}{v_{t-1}} + \left(\frac{E_t - E_{t-1}}{E_{t-1}}\right)\left(\frac{v_t - v_{t-1}}{v_{t-1}}\right) \\
\frac{q_{F_t} - q_{F_{t-1}}}{q_{F_{t-1}}} &= \frac{\frac{Y_{F_t} - Y_{F_{t-1}}}{Y_{F_{t-1}}} - \frac{L_{F_t} - L_{F_{t-1}}}{L_{F_{t-1}}}}{1 + \frac{L_{F_t} - L_{F_{t-1}}}{L_{F_{t-1}}}}
\end{aligned} \tag{3}
$$

The first expression in (3) indicates that expanding production in the formal sector is only possible by demanding more energy or increasing efficiency in using energy inputs. Moreover, from the second, it follows that the labour market adjusts accordingly, so variations in labour productivity depend on the difference between output growth and labour requirements.

### 3.2 Informal sector

Given its small capital requirements, we assume the informal sector uses neglectable amounts of energy. This assumption is frequently used in structuralist models dealing with an informal sector (e.g. Ros, 2016; Skott, 2023). Thus, production only depends on labour inputs:

$$Y_{I_t} = L_{I_t} q_{I_t} \tag{4}$$

If labour productivity is constant, then it follows:

$$\frac{Y_{I_t} - Y_{I_{t-1}}}{Y_{I_{t-1}}} = \frac{L_{I_t} - L_{I_{t-1}}}{L_{I_{t-1}}} \tag{5}$$

That is, a positive output growth rate in the informal sector depends on workers leaving the formal sector. This structure allows us to accommodate two stylised facts of informal economies. First, informal firms and workers have extremely low productivity compared to the formal economy. Second, the two are largely disconnected from each other.

### 3.3 Choosing (in)formality

The first choice agents face is whether to join the formal sector. This dimension of structural change is captured by the evolution over time in the composition of the employment between formal and informal sectors. Recall that the workforce was supposed to be constant. Hence, we have:

$$L = L_{F_t} + L_{I_t}$$

Defining the auxiliary variable ($l$) as the difference between the two groups:

$$l_t = L_{F_t} - L_{I_t}$$

We can build an index ($\Omega$) that has the elegant property of capturing the informality rate:

$$\Omega_t = \frac{l_t}{L} \in [-1, 1]$$

It corresponds to the share of formal minus informal labour. When all employment is formal, then $\Omega = 1$. In the other extreme, a purely informal economy is such that $\Omega = -1$. Of course, an economy can be at any point on the spectrum. Developing countries are most likely to depict $\Omega < 1$ while those in the Global North are such that $\Omega > 1$.

The share of agents in each group changes accordingly to the discrete choice approach (as in Brock and Hommes, 1997). Given the significant degree of inertia in formality decisions, we adopt the asynchronously updating version in Hommes et al. (2005). Thus, we write:

$$
\begin{aligned}
\frac{L_{F_t}}{L} &= \alpha \frac{L_{F_{t-1}}}{L} + (1-\alpha) p^F_{t-1} \\
\frac{L_{I_t}}{L} &= \alpha \frac{L_{I_{t-1}}}{L} + (1-\alpha) p^I_{t-1}
\end{aligned}
\tag{6}
$$

where $\alpha \in (0, 1)$ stands for those that repeat the choice of the previous period, and $p^i$ is the probability of belonging to either sector. In the limit, when $\alpha = 1$, there is no structural change as the composition of employment remains constant. Alternatively, $\alpha = 0$ implies synchronous updating.

Subtracting the second expression from the first in (6), we obtain:

$$
\Omega_t = \alpha \Omega_{t-1} + (1-\alpha) \left( p^F_{t-1} - p^I_{t-1} \right) \tag{7}
$$

giving us variations in the sectoral composition in terms of two probability functions.

Applying a specification for $p^F$ and $p^I$ similar to those in the discrete-choice type of models, we have:

$$
\begin{aligned}
p^F_{t-1} &= \frac{\exp\left(\rho U_{F_{t-1}}\right)}{\displaystyle\sum_{i=\{F,I\}} \exp\left(\rho U_{i_{t-1}}\right)} \\
p^I_{t-1} &= \frac{\exp\left(\rho U_{I_{t-1}}\right)}{\displaystyle\sum_{i=\{F,I\}} \exp\left(\rho U_{i_{t-1}}\right)}
\end{aligned}
\tag{8}
$$

where $U_i$ is the utility of being formal or informal, and parameter $\rho$ is commonly known as the intensity of choice. For values of $\rho$ close to zero, both probabilities converge to 0.5. At the same time, as this parameter goes to infinity, $p^i$ tends to zero or one (for a review of its properties, see Franke and Westerhoff, 2017).

Assume $U_i$ depends on the same variables but with an opposite sign, such that:

$$
U_{F_t} = -U_{I_t}
$$

Substituting the equality above into (8) and the resulting expressions on Eq. (7), we have:

$$
\Omega_t = \alpha \Omega_{t-1} + (1-\alpha) \tanh\left(\rho U_{F_{t-1}}\right) \tag{9}
$$

The existing literature has relied on cross-country and within-country (quasi) experimental data to assess informality's determinants empirically. Here, the distinction between extensive and intensive margins is critical (see Ulyssea, 2020). The extensive margin depends on firms registering and paying entry fees to achieve a formal status. On the other hand, the intensive margin refers to formal vs informal employment relations. Our study is somehow more related to the latter, though we recognise the importance of the first dimension. There is some consensus that lowering registration costs neither reduces informality nor raises economic growth (e.g. La Porta and Shleifer, 2014). The central stylised fact is that informality becomes much less significant as the economy grows. This also happens over the business cycle. Job finding rates in the formal sector are strongly pro-cyclical while remaining relatively stable in the informal sector, suggesting people prefer formality overall. Considering that the formality of a firm's suppliers and buyers is correlated with its formal status, we observe a scale effect.

To summarise the discussion above, we write:

$$\rho U_{F_{t-1}} = \underbrace{\rho_\Omega \Omega_{t-1}}_{\text{Scale effect}} + \underbrace{\rho_Y \left( \frac{Y_{F_t} - Y_{F_{t-1}}}{Y_{F_{t-1}}} \right)}_{\text{Growth effect}} \tag{10}$$

where $\rho_\Omega$ and $\rho_Y$ are sensitivity parameters to scale and growth effects. The first captures the idea that as the formal sector increases, it becomes more attractive to join it. A similar component appears in Zeppini (2015) and Zeppini and van den Bergh (2020) related to the adoption of a green technology. The existence of such a component can also be justified by the existence of increasing returns (Azariadis and Stachurski, 2005) or Marshallian externalities that arise if a certain type of workers is sufficiently numerous (Carillo and Pugno, 2004), creating a scale complementarity. Finally, the growth effect indicates that if that sector grows, there will be an increase in the probability of becoming formal.

Substituting Eq. (10) into (9), the dynamics of (in)formality are given by:

$$\Omega_t = \alpha \Omega_{t-1} + (1-\alpha) \tanh \left( \rho_\Omega \Omega_{t-1} + \rho_Y \left( \frac{Y_{F_t} - Y_{F_{t-1}}}{Y_{F_{t-1}}} \right) \right) \tag{11}$$

### 3.4 Choosing (non)green attitudes

For simplicity, we assume the population equals the labour force, thus being constant. It is divided between those supporting and opposing strong climate action, $j = \{CS, CD\}$, respectively:

$$N = N_{CS_t} + N_{CD_t}$$

Defining an auxiliary variable $(n)$ as the difference between the two groups:

$$n_t = N_{CS_t} - N_{CD_t}$$

We are able to build an index of green attitudes ranging between -1 and 1, that is:

$$\Phi_t = \frac{n_t}{N} \in [-1, 1]$$

Our index $\Phi$ is analogous to $\Omega$ and captures the composition of the population in the environmental dimension. If all agents support strong action, then $\Phi = 1$. At the other extreme, complete opposition or merely paying lip service to climate change is represented by $\Phi = -1$.

The share of agents in each group changes conditional to a set of probabilities:

$$\begin{aligned} \frac{N_{CS_t}}{N} &= \gamma \frac{N_{CS_{t-1}}}{N} + (1-\gamma) p_{t-1}^{CS} \\ \frac{N_{CD_t}}{N} &= \gamma \frac{N_{CD_{t-1}}}{N} + (1-\gamma) p_{t-1}^{CD} \end{aligned} \tag{12}$$

where $\gamma \in (0, 1)$ corresponds to the portion of agents that repeat their convictions from the previous period. Thus, $1 - \gamma$ stands for those that might switch attitudes, with $p^j$ indicating the correspondent probability functions. As a higher $\gamma$ implies more inertia in attitudes, Cafferata et al. (2021) interprets this coefficient as capturing the strength of agents' confirmation bias.

If we subtract the second expression from the first in (12), it follows that:

$$\Phi_t = \gamma\Phi_{t-1} + (1-\gamma)\left(p^F_{t-1} - p^I_{t-1}\right) \tag{13}$$

Adopting the traditional specification used in discrete choice models for $p^{CS}$ and $p^{CD}$ (see the literature review by Franke and Westerhoff, 2017), we have:

$$\begin{aligned} p^{CS}_{t-1} &= \frac{\exp\left(\beta U_{CS_{t-1}}\right)}{\sum\limits_{j=\{CS,CD\}} \exp\left(\beta U_{j_{t-1}}\right)} \\ p^{CD}_{t-1} &= \frac{\exp\left(\beta U_{CD_{t-1}}\right)}{\sum\limits_{j=\{CS,CD\}} \exp\left(\beta U_{j_{t-1}}\right)} \end{aligned} \tag{14}$$

where $U_j$ is the utility obtained from being (or not) green. The intensity of choice now is represented by $\beta$. When $\beta \to 0$, both probabilities converge to 0.5. On the other hand, $\beta \to \infty$ implies $p^j$ converges to zero or one.

It is reasonable to assume that the utility of supporting or opposing climate intervention depends on the same variables but with an opposite sign:

$$U_{CS_t} = -U_{CD_t}$$

Substituting that relationship into (14) and the resulting expressions into Eq. (13), we obtain:

$$\Phi_t = \gamma\Phi_{t-1} + (1-\gamma)\tanh\left(\beta U_{CS_{t-1}}\right) \tag{15}$$

Research on environmental psychology indicates threat appraisal is one of the strongest predictors of individual adaptation behaviour (Bamberg et al., 2017; an overview of the related literature can be found in Bechtoldt et al., 2021). The main idea is that agents are more likely to support environmental action if they understand or feel the costs imposed by climate change. Along similar lines, they also respond to the threat of losing their job in the formal sector and having to find alternatives in informality. This reasoning echoes the "basic needs hierarchy" argument. Humans only begin to pursue other goals once basic physiological needs, such as physical safety and access to food, have been taken care of (e.g. Scruggs and Benegal, 2012; Abou-Chadi and Kayser, 2017).

Taken together, we make the case that a higher emissions growth rate is associated with an increased probability of supporting stronger climate action. Analogously, higher formalisation rates increase $p^{CS}$ and reduce $p^{CD}$ because agents have an additional stimulus to think about more long-run concerns, such as global warming, after securing a good job. That is:

$$\beta U_{CS_{t-1}} = \underbrace{\beta_\Phi \Phi_{t-1}}_{\text{Peer effect}} + \underbrace{\beta_\Omega \Omega_{t-1}}_{\text{Formalisation effect}} \underbrace{+ \beta_P \left(\frac{P_t - P_{t-1}}{P_{t-1}}\right)}_{\text{Climate threat}} \tag{16}$$

where $\beta_\Phi$, $\beta_\Omega$, and $\beta_P$ are sensitivity parameters, while $P$ stands for emissions. Finally, considering the existing evidence on peer effects affecting political views (Robbett et al., 2023), we also take into account this channel.

Substituting Eq. (16) into (15), the dynamics of green attitudes are given by:

$$\Phi_t = \gamma \Phi_{t-1} + (1-\gamma) \tanh \left( \beta_\Phi \Phi_{t-1} + \beta_\Omega \Omega_{t-1} + \beta_P \left( \frac{P_t - P_{t-1}}{P_{t-1}} \right) \right) \tag{17}$$

### 3.5 Emissions, energy demand and the carbon-tax

Our narrative's last block of equations expands on the interplay between informality and environmental support by exploring the connection between emissions and technology. For this purpose, it is helpful to take as a starting point the Kaya identity:

$$P_t = N \times \frac{Y_{F_t}}{N} \times \frac{E_t}{Y_{F_t}} \times \frac{P_t}{E_t}$$

Notice that as only the formal sector uses energy inputs, emissions or pollution are a sub-product of $Y_F$. To focus on the relationship between pollution-output-technology, we allow the energy-output ratio to change over time but assume the pollution-energy ratio is constant and normalised it to one.[2]

Furthermore, given that we are assuming a constant population, the Kaya identity can be simplified to:

$$P_t = \frac{Y_{F_t}}{v_t}$$

In growth rates:

$$\frac{P_t - P_{t-1}}{P_{t-1}} = \frac{\frac{Y_{F_t} - Y_{F_{t-1}}}{Y_{F_{t-1}}} - \frac{v_t - v_{t-1}}{v_{t-1}}}{1 + \frac{v_t - v_{t-1}}{v_{t-1}}} \tag{18}$$

To keep our exercise as simple as possible, suppose the growth rate of energy demand is exogenously defined and given by:

$$\frac{E_t - E_{t-1}}{E_{t-1}} = g_E > 0$$

Recalling the first dynamic Leontief efficiency condition, substitute $g_E$ into the first expression in (3). Further substituting the result into Eq. (18) and after some manipulation we have that:

$$\frac{P_t - P_{t-1}}{P_{t-1}} = g_E \tag{19}$$

implying that emissions go hand in hand with energy demand. This result follows from the fact that improvements in energy efficiency translate into higher output in the same magnitude. While we could relax such a result and allow for the possibility of achieving negative emissions, we leave that option for future research for two main reasons. First, we would have to introduce a new parameter to address the issue, complicating the algebra without adding much to our main message. Second, evidence of absolute decoupling is still limited.

[2]Instead, we could take the energy-output ratio as constant and allow the pollution-energy ratio to vary over time. Both are suitable choices and reflect a change in technology towards a more (or less) environmentally friendly direction. Allowing $P/E$ to change would have required minor adjustments in the production function (1) and our specification of directed technical change in Eq. (20) without affecting our main narrative. We preferred using $v$ because, in our opinion, it is less intuitive to have pollution inside our Leontief technology. Environmental growth models that include $P$ in the production function normally associate it with a damaging effect or negative externality, a channel we are not exploring here.

Energy efficiency is supposed to depend on the cumulative effect of taxing emissions, i.e. a carbon tax ($\tau$). Still, adopting a carbon tax for only one year is unlikely to have meaningful technological effects. Instead, if sustained sufficiently long, the resulting increase in energy prices induces firms to adapt by increasing their search for energy-saving production techniques. This assumption is frequently called the directed technical change hypothesis and finds some empirical support. For example, studying the automotive industry, Aghion et al. (2016) estimate that a 1% rise in fuel prices results in 0.85 to 1% more clean patents. Moreover, they also document there is path dependence in the direction of technical change. Lin and Chen (2019) report similar figures. They show that a 1% increase in electricity price can increase innovation in renewable energy technologies by 0.8%–1.1%. Thus, to capture cumulative and path-dependent effect of $\tau$ on $v$, we write:

$$v_t = \prod_{\omega=1}^{t} \left(1 + \phi\tau_{\omega-1}\right) \tag{20}$$

where $\phi \in [0,1]$ represents the marginal impact of $\tau$ on $v$ on a given period.

Finally, carbon taxes depend on how urgent people believe climate change is. Considering that some of it depends on technical considerations beyond the political sphere, including agreements previously signed and the pressure of the international community, we divide the carbon tax into an autonomous component ($\tau_0$) and an attitudes-induced part:

$$\tau_{t-1} = \tau_0 + \tau_1\Phi_{t-1} \tag{21}$$

where $\tau_1 \in [0,1]$ represents the response of taxes to the population composition between those who support stronger or weaker climate action and $\tau_0 \in [0,1]$. One could argue that in less democratic societies parameter $\tau_1$ is relatively small. It is enough for our purposes to accept that at least part of the taxing emissions effort depends on public support. It is not a coincidence that numerous cases of fossil fuel subsidy reform withdrawals following social unrest (McCulloch et al., 2022).

From Eqs. (20) and (21), it follows that the variation rate of $v$ is equal to:

$$\frac{v_t - v_{t-1}}{v_{t-1}} = \phi\left(\tau_0 + \tau_1\Phi_{t-1}\right)$$

so that the composition of green attitudes influences technological choices over time. Substituting it into the first dynamic Leontief efficiency condition in (3) and recalling energy demand grows at a rate $g_E$, the output growth rate in the formal sector will be equal to:

$$\frac{Y_{F,t} - Y_{F,t-1}}{Y_{F,t-1}} = g_E + \left(1 + g_E\right)\phi\left(\tau_0 + \tau_1\Phi_{t-1}\right) \tag{22}$$

# 4 Dynamic system

Substituting Eq. (22) into Eq. (11), we obtain the dynamics of (in)formality as a function of a scale effect and output growth, the latter depending on the evolution of energy efficiency, which in turn responds to environmental support. Finally, substituting Eq. (19) into (17), variations in green attitudes respond to a peer effect and macroeconomic performance. Our 2D nonlinear map is defined and given by:

$$\begin{aligned}
\Omega_t &= \alpha\Omega_{t-1} + (1-\alpha)\tanh\left\{\rho_\Omega\Omega_{t-1} + \rho_Y\left[g_E + \left(1+g_E\right)\phi\left(\tau_0 + \tau_1\Phi_{t-1}\right)\right]\right\} \\
\Phi_t &= \gamma\Phi_{t-1} + (1-\gamma)\tanh\left(\beta_\Phi\Phi_{t-1} + \beta_\Omega\Omega_{t-1} + \beta_P g_E\right)
\end{aligned} \tag{23}$$

In steady-state, $\Omega_t = \Omega_{t-1} = \bar{\Omega}$ and $\Phi_t = \Phi_{t-1} = \bar{\Phi}$. This results in the following equilibrium conditions:

$$\begin{aligned} \bar{\Omega} &= \tanh\left\{\rho_\Omega \bar{\Omega} + \rho_Y \left[g_E + (1+g_E)\,\phi\left(\tau_0 + \tau_1 \bar{\Phi}\right)\right]\right\} \\ \bar{\Phi} &= \tanh\left(\beta_\Phi \bar{\Phi} + \beta_\Omega \bar{\Omega} + \beta_P g_E\right) \end{aligned} \tag{24}$$

The system (23) has no closed solution. Still, we would like to dig a bit deeper analytically into the existence of unique or multiple equilibrium points before moving to numerical simulations. To provide the reader with further intuition of what is going on, let us consider a hypothetical scenario with constant energy demand and carbon taxes fully attitudes-induced so that:

$$g_E = 0 \qquad \tau_0 = 0$$

It immediately follows that the equilibrium conditions (24) can be rewritten as:

$$\begin{aligned} \bar{\Omega} &= \tanh\left\{\rho_\Omega \bar{\Omega} + \rho_Y \phi \tau_1 \bar{\Phi}\right\} \\ \bar{\Phi} &= \tanh\left(\beta_\Phi \bar{\Phi} + \beta_\Omega \bar{\Omega}\right) \end{aligned} \tag{25}$$

These expressions allow us to differentiate between two particular sets of mechanisms. The first is related to scale and peer effects, as captured by parameters $\rho_\Omega$ and $\beta_\Phi$, respectively. The second set refers to cross-repercussions of one dimension over the other. On the one hand, an increase in green attitudes ($\Phi$) provides the necessary public support to increase the carbon tax by a certain factor ($\tau_1$); firms respond by increasing their search for more energy-efficient production techniques with a given success ($\phi$), leading to an increase in the output growth rate which ultimately improves the likelihood of workers choosing to become formal with a particular weight ($\rho_Y$). On the other hand, higher formalisation rates allow the general public to pay attention to problems perceived as more related in the long run, such as global warming ($\beta_\Omega$).

We proceed by stating and proving the following Propositions regarding the existence of a unique and stable equilibrium point:

**Proposition 1** *Suppose scale and group effects are arbitrarily small, such that $\rho_\Omega$ and $\beta_\Phi$ are sufficiently close to zero, while $\rho_Y \phi \tau_1$ and $\beta_\Omega$ also small enough. Then, the dynamic system only admits the solution defined and given by $(\bar{\Phi}, \bar{\Omega}) = (0, 0)$.*

**Proof.** See Appendix A.1. ■

The intuition behind this first Proposition is quite simple. Consider none of the transmission channels discussed in the model is important. In that case, the probability of choosing between formality or informality and of strongly supporting or not a green agenda will converge to 0.5. This result follows from the chosen functional forms of the probability functions in (8) and (14), which follow the discrete-choice literature. Notice that this is not a trivial nor corner solution because $\bar{\Phi}$ and $\bar{\Omega} \in [-1, 1]$. Therefore, the productive structure will be equally divided between sectors, and the population will be fragmented regarding attitudes. Things become more interesting when we allow them to become relevant. For example, increasing the relative importance of cross-repercussions between informality and environmental attitudes. We proceed by stating and proving the following Proposition regarding the existence of multiple equilibria and their local stability properties:

**Proposition 2** *If scale and group effects are arbitrarily small, such that $\rho_\Omega$ and $\beta_\Phi$ are sufficiently close to zero but $\rho_Y \phi \tau_1$ and $\beta_\Omega$ are large enough, the dynamic system admits three equilibria:*

1. *A saddle $(0, 0)$ solution.*
2. *A locally stable green-formal point $\Rightarrow (\bar{\Phi}, \bar{\Omega}) \in \mathbb{R}^{2+}$.*
3. *A locally stable non-green-informal solution $\Rightarrow (\bar{\Phi}, \bar{\Omega}) \in \mathbb{R}^{2-}$.*

**Proof.** See Appendix A.2. ■

Green-formal and non-green informal solutions are critical as initial representations of the North-South divide. Their coexistence reflects the persistence of uneven development over time. This finding depends on the complementarity between the growth effect, entering the probability of being formal, and the formalisation effect, appearing in the probability of being green. Initial conditions in the basins of attraction of the green-formal point reinforce the status of developed economies. First, high formality rates block attitudes from becoming non-green. Second, a majority supporting the green agenda keeps growth sufficiently high to block agents from migrating to the informal sector. Analogously, initial conditions in the basins of attraction of the non-green-informal solution work similarly but in the opposite direction.

The underdevelopment trap here depends on two forces that reinforce each other. On the one hand, high informality rates make agents relegate the environment as a secondary problem. As suggested throughout the paper, they might pay lip service to the relevance of it. Still, climate change mitigation does not appear to be a top priority, so there is no support for implementing a carbon tax. On the other hand, the latter's absence implies no change or even a reduction in energy input efficiency as directed technical change depends on increasing polluting costs. The absence of technological progress results in lower output growth. An economy that does not grow cannot attract workers to the formal sector, locking the trap.

Now, let us turn to the importance of the scale and peer effects muting the previous two channels. We can state and prove the following Proposition regarding multiple equilibria and their local stability.

**Proposition 3** *Suppose instead, cross-repercussions are arbitrarily small, such that $\rho_Y \phi \tau_1$ and $\beta_\Omega$ are sufficiently close to zero. If $\rho_\Omega$ and $\beta_\Phi$ are large enough, the dynamic system admits nine equilibria:*

1. *The saddle $(0, 0)$ solution.*
2. *A locally stable green-formal point $\Rightarrow (\bar{\Phi}, \bar{\Omega}) \in \mathbb{R}^{2+}$.*
3. *A locally stable green-informal solution $\Rightarrow \bar{\Phi} \in \mathbb{R}^{+} \wedge \bar{\Omega} \in \mathbb{R}^{-}$.*
4. *A locally stable non-green-formal point $\Rightarrow \bar{\Phi} \in \mathbb{R}^{-} \wedge \bar{\Omega} \in \mathbb{R}^{+}$.*
5. *A locally stable non-green-informal solution $\Rightarrow (\bar{\Phi}, \bar{\Omega}) \in \mathbb{R}^{2-}$.*
6. *Unstable environmental-polarisation with formality $\Rightarrow \bar{\Phi} \approx 0 \wedge \bar{\Omega} \in \mathbb{R}^{+}$.*
7. *Unstable environmental-polarisation with informality $\Rightarrow \bar{\Phi} \approx 0 \wedge \bar{\Omega} \in \mathbb{R}^{-}$.*

8. *Unstable green with formal-informal sectors of similar size* $\Rightarrow \bar{\Phi} \in \mathbb{R}^+ \wedge \bar{\Omega} \approx 0$.

9. *Unstable non-green with formal-informal sectors of similar size* $\Rightarrow \bar{\Phi} \in \mathbb{R}^- \wedge \bar{\Omega} \approx 0$.

**Proof.** See Appendix A.3. ■

Four stable solutions emerge, representing the coexistence of four possible sign combinations between $\Phi$ and $\Omega$. Proposition 2 indicated that cross-repercussions between the two dimensions create the blue and red equilibria in Figs. 2 and 3. Proposition 3 adds to that result by showing that scale and group effects can create blue, red, and grey states. It thus becomes quite important to study the respective basin of attraction to understand the boundary regions between each equilibrium. We will reach this point later when discussing strategies to achieve the blue region.

Before moving on to our numerical experiments, we relax the assumption that energy demand is constant and that carbon taxes are fully attitudes-induced. Instead, $g_E > 0$ while a group of technocrats or international agreements fixes and enforces $\tau_0 > 0$. We state the following Proposition regarding the existence of a unique and stable equilibrium solution.

**Proposition 4** *Suppose energy demand grows ($g_E$) at an arbitrarily large positive rate and the attitudes-free component of the carbon tax ($\tau_0$) is of considerable size. Then, the dynamic system has a unique locally stable equilibrium point $(\bar{\Phi}, \bar{\Omega}) \in \mathbb{R}^+$.*

Given our underlying assumption that there is no absolute decoupling in this economy, a large $g_E$ implies that emissions will continue rising for a sufficiently long time. As emissions accumulate in the atmosphere, the consequences of climate change become more concrete, as well as the perception of the climate threat. This increases the probability of recognising the urgency of fighting climate change, turning $\bar{\Phi} > 0$. Here comes the connection between attitudes and the productive structure. If most agents start to support the implementation of a sufficiently high carbon tax, polluting becomes more expensive. The effect is augmented by the imposed $\tau_0$ that does not depend on the general public. Because $\tau_0$ is not subject to the comes and goes from politics, it establishes a floor to the carbon tax to which firms respond by increasing their search for more energy-efficient production techniques. Our story crucially depends on technology's response to environmental policy because that is the force that keeps output growth high enough. High growth in the formal sector makes it more attractive to informal workers, reducing the probability of choosing informality, turning $\bar{\Omega} > 0$.

# 5 Numerical experiments

We calibrate the system to provide a more concrete view of the model's properties, choosing economically meaningful parameter values. Their number is relatively small, totalling 11, with five directly related to the probability functions. Table 1 reports the ranges used in our simulations. Whenever possible, we followed related studies in the field. Still, given that we are not calibrating a real economy, we used them as an approximation. Thus, the interpretation of our numerical experiments is more qualitative, with a particular interest in reproducing the main stylised facts reported in the first part of the paper. Each figure reports the specific values used to obtain it for completeness.

For example, Zeppini (2015) and Zeppini and van den Bergh (2020), using a discrete-choice framework similar to ours, have a increasing return parameter on the adoption of technology that resembles our scale effect on (in)formality. They assume it lies between 0.1

Table 1: Choice of parameters

| Parameter | Value | Source/Motivation |
|---|---|---|
| $\alpha$ | $0-1$ | Compatible with asynchronous updating |
| $\rho_\Omega$ | $0.5-1.1$ | Zeppini (2015); Zeppini and van den Bergh (2020) |
| $\rho_Y$ | $0.15-0.5$ | Fotie and Mbratana (2024) |
| $\gamma$ | $0-1$ | Compatible with asynchronous updating |
| $\beta_\Phi$ | $0.5-1.1$ | Kukacka and Sacht (2023) |
| $\beta_\Omega$ | $0.03-0.5$ | Benegal (2018) |
| $\beta_P$ | $0.01-0.2$ | Benegal (2018) |
| $\phi$ | $1-2$ | Aghion et al. (2016) |
| $g_E$ | $0-0.02$ | World Development Indicators |
| $\tau_0$ | $0-0.175$ | OECD; World Development Indicators |
| $\tau_1$ | $0.1-1$ | Compatible with a maximum $\tau = 0.15$ in our preferred scenario |

and 1. On the other hand, Kukacka and Sacht (2023) are the first to provide a reliable estimate of the intensity of choice parameter in heuristic-switching behavioural macroeconomic models. They find it to be slightly above the unity. Taking these findings together, we assume $\rho_\Omega$ and $\beta_\Phi$ vary between 0.5 and 1.1, adopting the latter in our preferred scenario. Fotie and Mbratana (2024) estimate the impact of output growth on informality, reporting a coefficient between $-0.01$ and $-0.06$; the effect can be larger depending on the level of output. Thus, we fix $\rho_Y = 0.15$, slightly above their calculations. Still, to test the model's sensitivity to this channel, we allow it up to 0.5 in one of our experiments.

To the best of our knowledge, there are no estimates on the impact of informality on climate attitudes. So, we use unemployment as a rough proxy. Benegal (2018) suggests that for each percentage point increase in state unemployment, the likelihood of an individual engaging in soft denial of climate change rises by 4.3%. He also reports a small effect of deviations from normal temperatures on climate change perceptions. We adopt similar magnitudes, choosing $\beta_\Omega = 0.03$ and $\beta_P = 0.01$ in our favoured scenario. Once more, to assess robustness, we admit values as high as $\beta_\Omega = 0.5$ and $\beta_P = 0.2$. Given that Aghion et al. (2016) document that a 1% increase in fuel prices leads to an increase between 0.85 and 1% in clean patents, we fix our induced technical change parameter $\phi = 1$ to simplify calculations. We assess the model's response to a stronger induced technical change effect in one specific case, fixing $\phi = 2$. Data on energy demand comes from the World Development Indicators, published by the World Bank.

Finally, about 16% of GHG emissions are currently priced at 30€ per tonne of $CO_2$, and only 7% have been priced above 60€ (OECD, 2023). Moreover, the World Development Indicators report that the European Union has a carbon intensity of 0.0002 tonnes of $CO_2$ per output unit, while China is 0.00075. If we multiply these figures with the maximum carbon price by emissions intensity, we get a $\leq 0.05$ rate. Thus, we allow $\tau_0$ to vary from zero to 0.075 in our preferred scenarios. Analogously, we set $\tau_1 = 0.1$ so that a society fully committed to the green agenda would double the maximum possible autonomous part of the tax. As extreme cases, we include in our experiments $\tau_0 = 0.175$ and a one-to-one response of the carbon tax to attitudes. Similar ranges can be found in Sordi and Davila-Fernandez (2023).

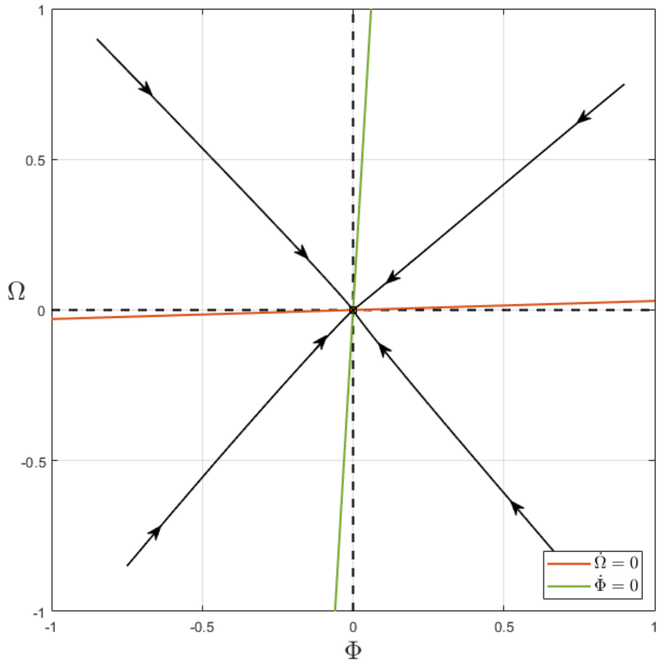



Figure 5: A unique (0,0) equilibrium point illustrating Proposition 1. Parameters $\alpha = 0.8$, $\gamma = 0.8$, $\rho_\Omega = 0.5$, $\rho_Y = 0.15$, $\beta_\Phi = 0.5$, $\beta_\Omega = 0.03$, $\beta_P = 0.01$, $\phi = 1$, $g_E = 0$, $\tau_0 = 0$, $\tau_1 = 0.1$.

### 5.1 Illustrating Propositions 1-4

We provide a concrete visualisation of the analytical results reported in the previous Section. Fig. 5 assumes scale and group effects are arbitrarily small while cross-repercussions are also minor. In green and orange, we report the equilibrium conditions $\dot{\Phi} = 0$ and $\dot{\Omega} = 0$, respectively. Proposition 1 indicates that, under these conditions, the model admits a unique locally stable equilibrium point $(0, 0)$. On the environmental attitudes axis, society is polarised, meaning that half of the population supports and the other half opposes adopting a carbon tax. As no majority is formed, carbon taxes are independent of $\Phi$. As long as $\tau_0 = 0$, there is no emissions tax. Firms have no incentive to invest in energy-saving production techniques, resulting in zero growth in the formal sector. Moreover, a stagnant formal sector keeps the probability of choosing formality *vs* informality equal so that workers are equally divided between the two sectors. The black arrows indicate different initial conditions in each quadrant converging to the steady-state solution.

We proceed by studying the emergence of multiple equilibria when cross-repercussions between the two dimensions are strong enough. This is perhaps our less relevant case because we need to assume parameter values significantly outside a range supported empirically. Nonetheless, such a step is necessary to show the validity of Proposition 2 and allows us to be as clear as possible about the mechanisms in motion. Fig. 6 indicates the coexistence of two stable attractors. On the one hand, a green-formal equilibrium (G-F) stands as an “ideal” Global North. Most of the population supports strong climate action, and workers are mainly in the formal sector. The blue area marks all initial conditions converging to it. We have a virtuous process of cumulative causation, where positive attitudes lead to implementing a carbon tax, which stimulates energy efficiency through induced technical change, creating more growth in the formal sector. The latter guarantees that citizens have satisfied their immediate needs so they can dedicate attention to the environment, reinforcing the process.

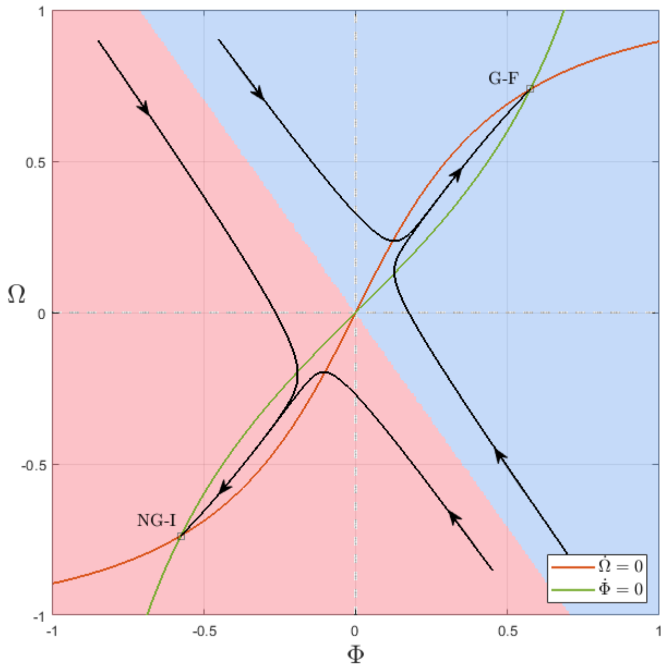



Figure 6: Basins of attraction in the $(\Phi, \Omega)$ space illustrating Proposition 2. Parameters $\alpha = 0.8$, $\gamma = 0.8$, $\rho_\Omega = 0.5$, $\rho_Y = 0.5$, $\beta_\Phi = 0.5$, $\beta_\Omega = 0.5$, $\beta_P = 0.01$, $\phi = 2$, $g_E = 0$, $\tau_0 = 0$, $\tau_1 = 1$.

On the other hand, a non-green informal (NG-I) attractor stands as the "expected" Global South case. We mark its basin of attraction in red. A vicious chain of forces makes NG-I also a stable solution. A developing economy with high levels of informality and initially low environmental support is likely to stay that way because, in the absence of carbon taxes, there is no induced technical change and efficiency in the use of inputs remains the same, resulting in no growth. Thus, there is no incentive to become formal, keeping the country in an underdevelopment trap.

A third set of experiments illustrates Proposition 3. Scale and group effects are now sufficiently strong. We deal with nine equilibrium points, indicated by the intersection between green and orange lines. Still, only four of them are locally stable. As before, we have a green-formal equilibrium on the top right, with its basins of attraction coloured in blue. On the bottom left, the red area indicates all initial conditions converging to a non-green informal solution. The two novelties are the areas in grey. They correspond to hybrid cases. For example, on the top left, we have developed countries with low levels of informality but that do not recognise the urgency of fighting climate change. Basic material needs are satisfied because they have low informality, and agents are motivated to look at the environment. However, as the peer effects are strong and initial conditions are such that people are surrounded by agents who do not have green attitudes, $\Phi$ stays negative. Moreover, because scale effects are also quite strong, it is very costly to leave the formal sector, and agents continue to prefer formality even though efficiency is stagnant.

A quick look at Figs. 2 and 3 reveal the quadrant in the bottom right is (almost) empty, suggesting the second hybrid case is less likely to happen. We will come back to this point later, but it is enough to understand its rationale for the moment, which is analogous to the previous case. As most of the population supports strong climate action, a carbon tax is adopted, and peer effects reinforce agents' attitudes. However, the induced technical change channel is relatively weak. Thus, it cannot overcome the scale effect of the informal sector, resulting in a green-informal equilibrium.

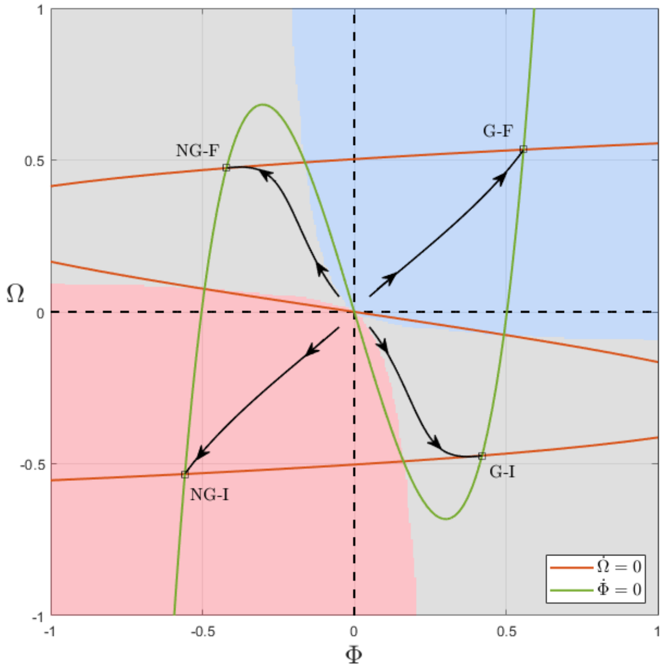



Figure 7: Basins of attraction in the $(\Phi, \Omega)$ space illustrating Proposition 3. Parameters $\alpha = 0.8$, $\gamma = 0.8$, $\rho_\Omega = 1.1$, $\rho_Y = 0.15$, $\beta_\Phi = 1.1$, $\beta_\Omega = 0.03$, $\beta_P = 0.01$, $\phi = 1$, $g_E = 0$, $\tau_0 = 0$, $\tau_1 = 0.1$.

So far, we have studied our model under the assumption of constant energy demand and zero autonomous carbon tax, i.e. $g_E = \tau_0 = 0$. In Proposition 4, we demonstrated that the model again admits a unique equilibrium solution if we relax it. Fig. 8 confirms our findings. The informal sector only uses labour to produce. Therefore, $g_E > 0$ implies that growth in the formal sector is sufficiently strong. This force creates a net incentive to become formal, leaning the probability functions against informality. Furthermore, $\tau_0 > 0$ indicates an exogenous pressure to address climate change. We can think of this parameter as a group of technocrats that does not directly respond to the general public. In the context of developed countries, we could think of working groups inside the European Commission. Looking at developing economies, one could argue that the Chinese Communist Party plays a similar role. Adopting a global emission tax system by a multilateral organisation such as the United Nations would work analogously for our purposes. The point is that these two mechanisms shift the orange line up to the left, destroying all equilibria except the G-F.[3]

## 5.2 From the model back to the stylised facts

After evaluating numerically Propositions 1-4, we turn to whether the model is compatible with what, in our view, are the main elements of Figs. 2 and 3, discussing its connection with the sytlised facts of La Porta and Shleifer (2014). First, the informal sector is huge. Accordingly, our model can generate the coexistence of equilibria with low vs high informality. Second, it has extremely low productivity. This is an underlying assumption of our framework as $q_I$ is constant. Third, lowering registration costs does not bring many informal

[3]The role of an "sentiments-autonomous" carbon tax controlling for multi-stability also appears in Sordi and Davila-Fernandez (2023). While their model also uses a discrete-choice approach to environmental attitudes, it mainly describes the dynamics of developed countries. As such, there is no reference to informality, which is the main novelty of the present paper. Still, the implications regarding policy recommendations are similar, highlighting the role of multilateral organisations such as the United Nations.

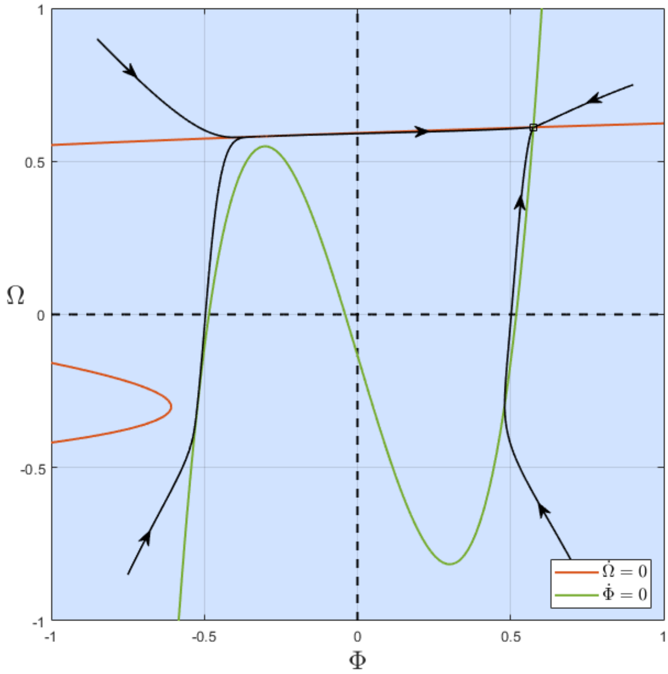



Figure 8: Basins of attraction in the $(\Phi, \Omega)$ space illustrating Proposition 4. Parameters $\alpha = 0.8$, $\gamma = 0.8$, $\rho_\Omega = 1.1$, $\rho_Y = 0.15$, $\beta_\Phi = 1.1$, $\beta_\Omega = 0.03$, $\beta_P = 0.2$, $\phi = 1$, $g_E = 0.02$, $\tau_0 = 0.175$, $\tau_1 = 0.1$.

firms into the formal sector nor spurs growth. Accordingly, we abstracted from this element and focused on the growth benefits of formality, which takes us to the last two points: Both sectors are largely disconnected, with informality disappearing as the economy grows. Furthermore, data from the two surveys revisited here suggest that a green informal (G-I) attractor is unlikely to emerge. Most countries are either green formal (G-F), non-green formal (NG-F) or non-green informal (NG-I).

Taking Fig. 7 as the starting point, $\tau_0$ is the critical policy parameter capable of destroying the G-I equilibrium. Fig. 9 shows how, as we increase the autonomous component of the carbon tax, the orange isocline moves to the left, leading to only three equilibrium points. A positive growth rate of energy demand could break up the red and grey attractors, as in Fig. 8, but this does not happen here because agents' response to pollution is realistically assumed to be low. Thus, the green isocline remains more or less the same and only the orange moves. In blue, we continue to colour the basins of attraction of G-F, which now occupies half of the $(\Phi, \Omega)$ space. Looking at the black arrows, we can appreciate a simple representation of path dependence, as economies with similar initial conditions can end up in very different states.

Studying the basins of attraction allows us to provide policymakers with insights into the social dimension of climate change in the Global South. Suppose a country like India that, in the context of our model, finds itself in the NG-I point. There are two "Big Push" options in the spirit of Rosenstein-Rodan (1943). An alternative is to jump along the vertical axis to achieve formality first. There are many natural reasons to support it. Poor countries have low emissions per capita. Historically, they have also contributed less to global warming, which nations in the Global North have fundamentally caused. Moreover, they still have to satisfy basic needs before dealing with problems that have more long-term consequences, for which their actions might even have a low scale. On the other hand, a second option is to jump along the horizontal axis. That is, using the environmental problem as a window of opportunity for development.

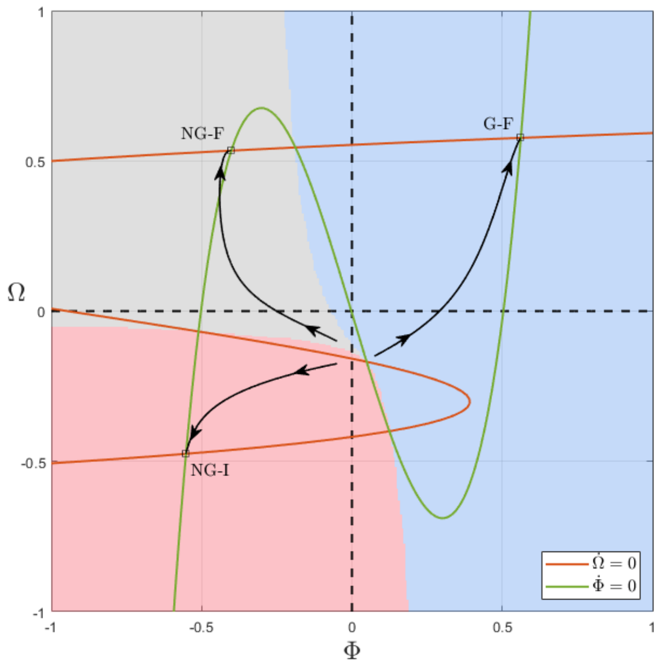



Figure 9: Basins of attraction in the $(\Phi, \Omega)$ space making a parallel with Figs. 2 and 3. Parameters $\alpha = 0.8$, $\gamma = 0.8$, $\rho_\Omega = 1.1$, $\rho_Y = 0.15$, $\beta_\Phi = 1.1$, $\beta_\Omega = 0.03$, $\beta_P = 0.2$, $\phi = 1$, $g_E = 0.02$, $\tau_0 = 0.175$, $\tau_1 = 0.1$.

Our model supports the second option and advises against the first. Ignoring climate attitudes creates the need for two shocks. One is to leave the red attractor, and the other jumps from grey to blue. Conversely, exploring the $\Phi$ dimension requires only one direct shock from the red area to the blue desirable attractor. Still, for this to happen, it is critical to tax emissions in such a way as to avoid short-term pressure from the general public. A sufficiently high carbon tax, accompanied by induced technical change, can create a win-win scenario that overcomes the red trap in a reasonable time horizon. Far from providing an excessively simplistic assessment of the development problem, we recognise the very stylised nature of our model, which calls for a note of caution. If the formal sector's scale effects or technology's response to the carbon tax is weak, something likely to happen in developing nations, underdevelopment will persist.

# 6 Final considerations

We develop a macrodynamic model to study the interplay between informality and heterogeneity in ecological thinking that reproduces some of the main stylised facts in the development literature. The paper uses discrete-choice theory to present a tractable framework for studying the political economy of climate change along the Global North-South divide. The probability of opting for (in)formality is assumed to depend on scale and growth effects. In contrast, the probability of acknowledging the urgency of climate change is subject to peer effects, the size of the formal sector, and the perception of the climate threat. Output growth and emissions ultimately are a function of the formal sector's economic performance. Moreover, based on the composition of the population in terms of their environmental attitudes, policymakers choose the carbon tax. The latter can potentially increase growth through an induced technical change channel.

Our model is compatible with the emergence of four stable equilibria. Scale and peer effects must be sufficiently large for this result to hold. Developed countries are either in the green-formal or non-green formal equilibria, while non-green informal or green informal points characterise developing nations. It is shown that if the autonomous component of the carbon tax is strong enough and firms respond to it by increasing their search for energy-saving production techniques, then the green-informal equilibria disappear. In that case, the system admits three stable attractors parallel to those identified in climate surveys. On the contrary, if scale and peer effects are arbitrarily small but technology's response to taxing emissions is still there, and people care about economic performance when forming environmental preferences, then two stable equilibria coexist. One green-formal and another non-green-informal.

Studying the basins of attraction allows us to identify different "Big Push" options for countries currently in the undesirable non-green informal equilibrium. First, a two-step process focusing on increasing formality by growing at all costs but leaving the environment as a future concern, thus requiring a second shock. Alternatively, a climate change mitigation agenda should be implemented as a window of opportunity to overcome underdevelopment. This option requires one shock emphasising attitudes to achieve the basins of attraction of the desirable equilibrium. Our analysis supports this last alternative, as ignoring climate attitudes creates the need for two instead of one big push. Numerical experiments confirm our analytical findings and allow us to provide a more concrete visualisation of the dynamic properties of the system.

The parsimonious framework developed here is flexible enough to be extended in several possible directions. The most evident is that our numerical experiments were mainly illustrative, and future research to improve them is to be encouraged. Another avenue is properly treating how the government can enter the story through alternative spending and taxing schemes. Arguably, taxation could increase the probability of choosing informality in poor countries, while the sectoral composition of government expenditures also matters. This step would allow us to get further insights on how to avoid underdevelopment traps.

# A Mathematical Appendix

Dynamic system

$$\left\{\begin{array}{l}\Omega_t=\alpha\Omega_{t-1}+(1-\alpha)\tanh\left\{\rho_\Omega\Omega_{t-1}+\rho_Y\left[g_E+(1+g_E)\,\phi\left(\tau_0+\tau_1\Phi_{t-1}\right)\right]\right\}\\ \Phi_t=\gamma\Phi_{t-1}+(1-\gamma)\tanh\left(\beta_\Phi\Phi_{t-1}+\beta_\Omega\Omega_{t-1}+\beta_P g_E\right)\end{array}\right. \tag{A.1}$$

**Lemma 5** *If* $-1 \leq A \leq 1 \Rightarrow 0 \leq A^2, 1-A^2 \leq 1$. *If* $0 \leq A, B \leq 1 \Rightarrow 0 \leq A^m B^n, 1-A^m B^n \leq 1, \forall m, n \geq 0$ *integers.*

**Proof.** Inmediate! ■

**Lemma 6** *The equilibrium points* $\left(\overline{\Omega},\overline{\Phi}\right)$ *of the dynamic system (A.1) satisfy the system*

$$\left\{\begin{array}{l}\overline{\Omega}=\tanh\left[\rho_\Omega\Omega+\phi\tau_1\rho_Y\left(g_E+1\right)\Phi+\rho_Y\left(g_E+\phi\tau_0+\phi\tau_0 g_E\right)\right]\\ \overline{\Phi}=\tanh\left(\beta_\Phi\Phi+\beta_\Omega\Omega+\beta_P g_E\right)\end{array}\right. \tag{A.2}$$

*and his Jacobian matrix* $J=J\left(\overline{\Omega},\overline{\Phi}\right)$ *satisfy the equations:*

$$\det\left(J-I\right)=\left(1-\alpha\right)\left(1-\gamma\right)C_1$$

$$\begin{aligned}C_1={}&\beta_\Phi\left(1-\overline{\Phi}^2\right)-\rho_\Omega\left(1-\overline{\Omega}^2\right)+\\&+\left[\beta_\Phi\rho_\Omega-\phi\tau_1\rho_Y\beta_\Omega\left(1+g_E\right)\right]\left(1-\overline{\Omega}^2\right)\left(1-\overline{\Phi}^2\right)+1\end{aligned} \tag{A.3}$$

$$\begin{aligned}C_2=\det\left(J+I\right)={}&\left(1+\alpha\right)\left(1+\gamma\right)+\beta_\Phi\left(1-\gamma\right)\left(1-\overline{\Phi}^2\right)+\rho_\Omega\left(1-\alpha\right)\left(1-\overline{\Omega}^2\right)+\\&+\alpha\beta_\Phi\left(1-\gamma\right)\left(1-\overline{\Phi}^2\right)+\gamma\rho_\Omega\left(1-\alpha\right)\left(1-\overline{\Omega}^2\right)+\\&+\left(1-\alpha\right)\left(1-\gamma\right)\left[\beta_\Phi\rho_\Omega-\phi\tau_1\rho_Y\beta_\Omega\left(g_E+1\right)\right]\left(1-\overline{\Omega}^2\right)\left(1-\overline{\Phi}^2\right)\end{aligned} \tag{A.4}$$

*and*

$$\begin{aligned}C_3=\det\left(J\right)={}&\alpha\gamma+\left(1-\alpha\right)\left(1-\gamma\right)\left[\beta_\Phi\rho_\Omega-\phi\tau_1\rho_Y\beta_\Omega\left(g_E+1\right)\right]\left(1-\overline{\Omega}^2\right)\left(1-\overline{\Phi}^2\right)+\\&+\alpha\beta_\Phi\left(1-\gamma\right)\left(1-\overline{\Phi}^2\right)+\gamma\rho_\Omega\left(1-\alpha\right)\left(1-\overline{\Omega}^2\right)\end{aligned} \tag{A.5}$$

*where* $I$ *represents the identiy matrix. Making the substitutions*

$$\left\{\begin{array}{c}x=\overline{\Omega},y=\overline{\Phi}\\ a_1=\alpha,a_3=\rho_\Omega,a_4=\phi\tau_1\rho_Y\left(g_E+1\right),a_5=\rho_Y\left(g_E+\phi\tau_0+\phi\tau_0 g_E\right)\\ b_1=\gamma,b_3=\beta_\Phi,b_4=\beta_\Omega,b_5=\beta_P g_E\end{array}\right. \tag{A.6}$$

*the system (A.2) and the equations for* $C_1, C_2, C_3$ *can be written as:*

$$\left\{\begin{array}{l}x=\tanh\left(a_3x+a_4y+a_5\right)\\ y=\tanh\left(b_3y+b_4x+b_5\right)\end{array}\right. \tag{A.7}$$

$$\begin{aligned}C_1&=-a_3\left(1-x^2\right)-b_3\left(1-y^2\right)+\left(a_3b_3-a_4b_4\right)\left(1-x^2\right)\left(1-y^2\right)+1\\&=\left[1-a_3\left(1-x^2\right)\right]\left[1-b_3\left(1-y^2\right)\right]-a_4b_4\left(1-x^2\right)\left(1-y^2\right)\end{aligned} \tag{A.8}$$

$$C_2 = (a_1+1)(b_1+1) + a_3(1-a_1)(b_1+1)\left(1-x^2\right) + b_3(a_1+1)(1-b_1)\left(1-y^2\right)$$
$$+ (a_3b_3 - a_4b_4)(1-a_1)(1-b_1)\left(1-x^2\right)\left(1-y^2\right) \quad \text{(A.9)}$$

$$\begin{aligned} C_3 &= 1 + a_4b_4(1-a_1)(1-b_1)\left(1-x^2\right)\left(1-y^2\right) - a_3b_1(1-a_1)\left(1-x^2\right) \\ &-a_1b_3(1-b_1)\left(1-y^2\right) - a_3b_3(1-a_1)(1-b_1)\left(1-x^2\right)\left(1-y^2\right) - a_1b_1 \\ &= 1 + a_4b_4(1-a_1)(1-b_1)\left(1-x^2\right)\left(1-y^2\right) \\ &- \left\{1-(1-a_1)\left[1-a_3\left(1-x^2\right)\right]\right\}\left\{1-(1-b_1)\left[1-b_3\left(1-y^2\right)\right]\right\} \end{aligned} \quad \text{(A.10)}$$

**Proof.** If in the system (A.1),

$$f(\Omega,\Phi) = \alpha\Omega + (1-\alpha)\tanh\left\{\rho_\Omega\Omega + \rho_Y\left[g_E + (1+g_E)\,\phi\,(\tau_0+\tau_1\Phi)\right]\right\}$$

and

$$g(\Omega,\Phi) = \gamma\Phi + (1-\gamma)\tanh\left(\beta_\Phi\Phi + \beta_\Omega\Omega + \beta_P g_E\right)$$

then

$$\begin{aligned} 0 &= f\left(\overline{\Omega},\overline{\Phi}\right) - \overline{\Omega} = (\alpha-1)\left\{\overline{\Omega} - \tanh\left[\overline{\Omega}\rho_\Omega + \left[g_E + \phi\,(g_E+1)\left(\tau_0 + \overline{\Phi}\tau_1\right)\right]\rho_Y\right]\right\} \\ 0 &= g\left(\overline{\Omega},\overline{\Phi}\right) - \overline{\Phi} = (\gamma-1)\left[\overline{\Phi} - \tanh\left(\overline{\Omega}\beta_\Omega + \overline{\Phi}\beta_\Phi + \beta_P g_E\right)\right] \end{aligned}$$

The system (A.2) occurs because $\alpha,\gamma \in ]0,1[$. On the other hand, considering (A.2) and the identities $\frac{d\tanh(x)}{dx} = \operatorname{sech}^2(x) = 1-\tanh^2(x)$, we obtain:

$$\begin{aligned} J\left(\overline{\Omega},\overline{\Phi}\right) &= \left.\begin{bmatrix} \frac{\partial f}{\partial\Omega} & \frac{\partial f}{\partial\Phi} \\ \frac{\partial g}{\partial\Omega} & \frac{\partial g}{\partial\Phi} \end{bmatrix}\right|_{\left(\overline{\Omega},\overline{\Phi}\right)} \\ &= \begin{bmatrix} \alpha + \rho_\Omega(\alpha-1)\left(\overline{\Omega}^2-1\right) & \phi\tau_1\rho_Y(\alpha-1)(g_E+1)\left(\overline{\Omega}^2-1\right) \\ \beta_\Omega(\gamma-1)\left(\overline{\Phi}^2-1\right) & \gamma + \beta_\Phi\left(\overline{\Phi}^2-1\right)(\gamma-1) \end{bmatrix} \end{aligned}$$

The equations (A.3), (A.4), (A.5), (A.7),(A.8),(A.9) and (A.10) are obtained by direct computation and later simplification. And from this it follows that the three necessary and sufficient conditions for local stability of the equilibrium point $\left(\overline{\Omega},\overline{\Phi}\right) = (x,y)$ of the system (A.1) satisfying (A.2) or (A.7), are

$$C_1 = C_1(x,y,a_3,a_4,b_3,b_4) > 0, \quad \text{(A.11)}$$

$$C_2 = C_2(x,y,a_1,a_3,a_4,b_1,b_3,b_4) > 0, \quad \text{(A.12)}$$

and

$$C_3 = C_3(x,y,a_1,a_3,a_4,b_1,b_3,b_4) > 0. \quad \text{(A.13)}$$

■

**Lemma 7** *Suppose $a_5 = b_5 = 0$. If $(x,y)$ is a solution of (A.7), that is if $(x,y)$ is a fixed point of (A.2), then $(x,y)$ is an intersection point of the curves*

$$\begin{cases} y = a(x) = \frac{1}{a_4}\left[\tanh^{-1}(x) - a_3x\right], |x|<1 \\ x = c(y) = \frac{1}{b_4}\left[\tanh^{-1}(y) - b_3y\right], |y|<1 \end{cases}, \quad \text{(A.14)}$$

*and therefore $(-x,-y)$ is also a solution of (A.7). Furthermore,*

1. *When* $0 \leq a_3, b_3 \leq 1 \Rightarrow a(x), |x| < 1$ *and* $b(x) = c^{-1}(y), -\infty < x < +\infty$, *are increasing functions with the common point* $(0,0)$ *which is an inflexion point of both curves; the graphic of* $a$ *is convex downwards when* $x < 0$ *and is convex upwards when* $x > 0$*; the convexity of the graphic of* $b$ *is the opposite.*

2. *When* $a_3 > 1 \Rightarrow a(x)$ *is increasing in* $]-1, -\alpha[\cup]\alpha, 1[$ *and decreasing in* $]-\alpha, \alpha[, \alpha = \sqrt{1 - \frac{1}{a_3}}$ *with a local maximum point in* $-\alpha$ *and a local minimum point in* $\alpha$, $(0,0)$ *is an inflection point too and his convexity is the same.*

3. *When* $b_3 > 1 \Rightarrow$ *the graph of* $x = c(y)$ *as function in* $y$ *is increasing in* $]-1, -\beta[\cup]\beta, 1[$ *and decreasing in* $]-\beta, \beta[, \beta = \sqrt{1 - \frac{1}{b_3}}$ *with a local maximum point in* $-\beta$ *and a local minimum point in* $\beta$*, it can be represented by three functions that are the inverse of the corresponding segments of* $c(y)$*:* $B_1(x), x \in ]-\infty, c(-\beta)[$*, increasing, convex upwards;* $B_2(x), x \in ]c(\beta), +\infty[$*, increasing, convex downwards, and* $B_3(x), x \in ]c(\beta), c(-\beta)[$*, decreasing, convex upwards for* $x < 0$*, convex downwards for* $x > 0$ *with* $(0,0)$ *as an inflection point.*

**Proof.** The equations (A.14) are obtained directly from (A.7) using the properties of the hyperbolic tangent function and $a_5 = b_5 = 0$. In the same way, we have that $a(-x) = -a(x)$ and $c(-y) = -c(y)$, so if $(x, y)$ is a point of intersection of the curves (A.14), $(-x, -y)$ is also and so it is also a solution of (A.7). Furthermore, we can see the behaviour of the graphics of the two equations $y = a(x)$ and $x = c(y)$ in (A.14) calculating $a'(x) = \frac{da}{dx}, a''(x) = \frac{d^2a}{dx^2}$ and $c'(y) = \frac{db}{dy}$,

$$a'(x) = \frac{1 - a_3(1 - x^2)}{a_4(1 - x^2)}, |x| < 1 \text{ and } c'(y) = \frac{1 - b_3(1 - y^2)}{b_4(1 - y^2)}, |y| < 1. \tag{A.15}$$

$$a''(x) = \frac{(1 - x^2)(2a_3x) - (1 - a_3 + a_3x^2)(-2x)}{a_4(1 - x^2)^2} = \frac{2x}{a_4(1 - x^2)^2}. \tag{A.16}$$

We have the following results:

1. When $0 \leq a_3, b_3 < 1$ we obtain using (A.15) and Lemma 5, $a'(x) > 0$ and $c'(y) > 0 \Rightarrow a(x), |x| < 1$ and $b(x) = c^{-1}(x), -\infty < x < +\infty$ are strictly increasing finctions. Furthermore, by (A.14) and (A.15), (A.16) we obtain: $\lim_{x \to -1^+} a(x) = -\infty, a(0) = 0, \lim_{x \to 1^-} a(x) = +\infty, a'(0) = \frac{1-a_3}{a_4} > 0, a''(0) = 0 \Rightarrow x = 0$ is an inflection point and the graphic is convex downwards when $x < 0$ and is convex upwards when $x > 0$; on the other hand for the function $b(x)$, we have using (A.14), (A.16), 5, the inverse derivative and the change rule: $\lim_{x \to -\infty} b(x) = -1^+, b(0) = c^{-1}(0) = 0, \lim_{x \to +\infty} b(x) = 1^-$,

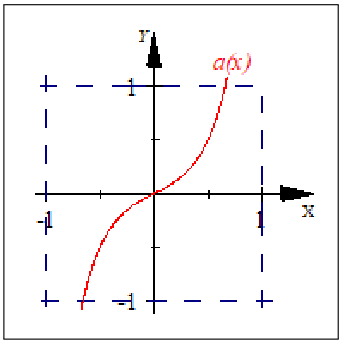
$$b'(x) = \frac{1}{c'(y)} = \frac{b_4(1 - y^2)}{1 - b_3(1 - y^2)} = \frac{b_4\left(1 - b(x)^2\right)}{1 - b_3\left(1 - b(x)^2\right)} \tag{A.17}$$

$\Rightarrow b'(x) > 0, b'(0) = \frac{b_4}{1-b_3} > 0$,

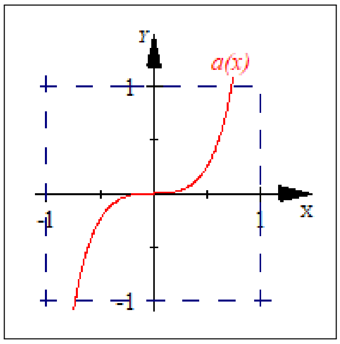
$$\begin{aligned} b''(x) &= b_4 \frac{(1 - b_3 + b_3y^2)(-2yb'(x)) - (1 - y^2)(2b_3yb'(x))}{(1 - b_3 + b_3y^2)^2} \\ &= \frac{-2b_4yb'(x)}{(1 - b_3 + b_3y^2)^2} \end{aligned} \tag{A.18}$$

$\Rightarrow x = 0$ is an inflection point and the graphic is convex upwards when $x < 0$ and is convex downwards when $x > 0$.

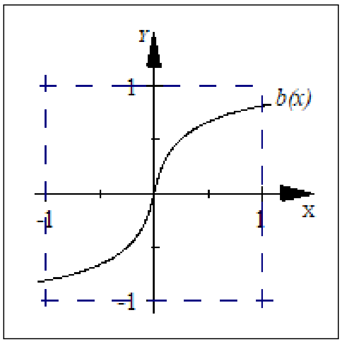
2. When $a_3 = 1, b_3 = 1, a(x)$ and $c(y)$ are non decreasing and $a(x), |x| < 1$ and $b(x) = c^{-1}(x), -\infty < x < +\infty$ maintain the previous characteristics, the only difference is, by (A.15), (A.16), (A.17) and (A.18), that $a'(0) = a''(0) = 0$ and $b'(0), b''(0)$ do not exist.



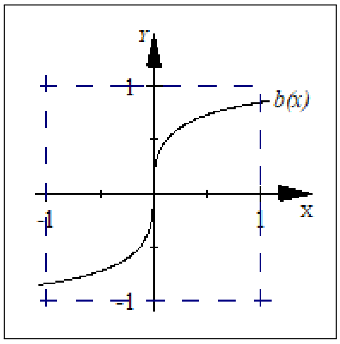

$0 \leq a_3 < 1$




$a_3 = 1$




$0 \leq b_3 < 1$




$b_3 = 1$

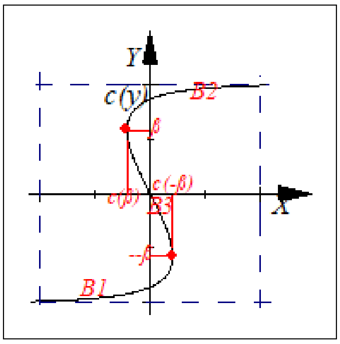
3. When $a_3 > 1 \Rightarrow$ by (A.15) and (A.16), $a'(x) = 0$ if $x = \pm\alpha, \alpha = \sqrt{1 - \frac{1}{a_3}} \Rightarrow a'(x) > 0$ if $|x| > \alpha, a'(x) < 0$ if $|x| < \alpha \Rightarrow a(x)$ is increasing in $]-1, -\alpha[\cup]\alpha, 1[$ and decreasing in $]-\alpha, \alpha[$ with a local maximum point in $-\alpha$ and a local minimum point in $\alpha. a''(0) = 0, a''(x) < 0$ if $x < 0, a''(x) > 0$ if $x > 0 \Rightarrow (0,0)$ is an inflection point and the graphic of $a$ is convex downwards when $x < 0$ and is convex upwards when $x > 0$.

4. When $b_3 > 1 \Rightarrow$ by (A.15), $c'(y) = 0$ if $y = \pm\beta, \beta = \sqrt{1 - \frac{1}{b_3}} \Rightarrow c'(y) > 0$ if $|y| > \beta, c'(y) < 0$ if $|y| < \beta \Rightarrow c(y)$ is increasing in $]-1, -\beta[\cup]\beta, 1[$ and decreasing in $]-\beta, \beta[$ with a local maximum point in $-\beta$ and a local minimum point in $\beta$. Thus, $x = c(y)$ implicitly determines three functions $y = c^{-1}(x)$ which are the inverse of the corresponding parts of $c(y)$ and that we will denoted by $B_1, B_2$ and $B_3. B_1(x), x \in ]-\infty, c(-\beta)[; B_2(x), x \in ]c(\beta), +\infty[, B_3(x), x \in ]c(\beta), c(-\beta)[, B_3(0) = 0$. We can use (A.15), the inverse derivative and the change rule to get in the same way we got (A.17) and (A.18):

$$\begin{array}{l} B_1'(x) = \frac{b_4\left(1-y^2\right)}{1-b_3(1-y^2)}, y \in ]-1, -\beta[; B_2'(x) = \frac{b_4\left(1-y^2\right)}{1-b_3(1-y^2)}, y \in ]\beta, 1[ \\ B_3'(x) = \frac{b_4\left(1-y^2\right)}{1-b_3(1-y^2)}, y \in ]-\beta, \beta[; B_i''(x) = \frac{-2b_4 y B_i'(x)}{[1-b_3(1-y^2)]^2}, i = 1, 2, 3 \end{array} \tag{A.19}$$

Follows from (A.19) that $B_1'(x) > 0, B_2'(x) > 0, B_1''(x) > 0, B_2''(x) < 0, B_3'(x) < 0, B_3'(0) = \frac{b_4}{1-b_3}, B_3''(0) = 0, B_3''(x) > 0$ for $x \in ]c(\beta), 0[$,

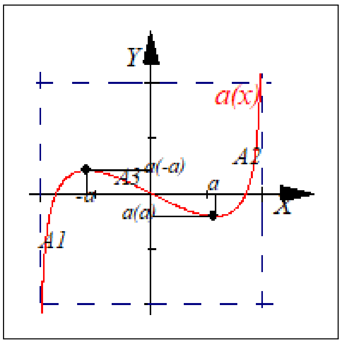
$B_3''(x) < 0$ for $x \in ]0, c(-\beta)[$. Hence $B_1$ and $B_2$ are increasing functions, $B_1$ convex upwards, $B_2$ convex downwards, $B_3$ is decreasing function with inflection point $(0,0)$, convex upwards for $x < 0$ and downwards for $x > 0$. In analogy, $a(x)$ can be decomposed into three functions $A_i(x)$ whose inverses $A_i^{-1}(y)$ are defined in intervals corresponding to $y$.




$b_3 > 1$

Inverse $B_i(x)$ of $c(y)$

$\beta = \sqrt{1 - \frac{1}{b_3}}$




$a_3 > 1$

Inverse $A_i^{-1}(y)$ of $a(x)$

$\alpha = \sqrt{1 - \frac{1}{a_3}}$

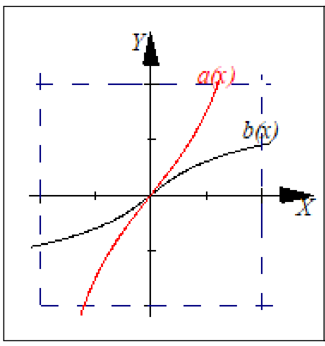
■

**Proposition 8** *Under the conditions of Lemma 7 the system (A.7) has a unique solution* $(0,0)$ *when*

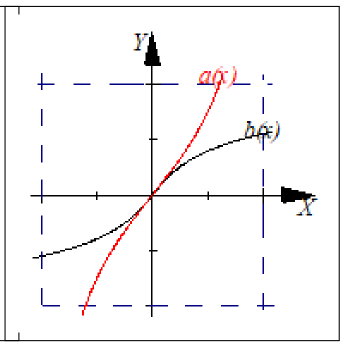
$$0 \leq a_3, b_3 < 1, (1 - a_3)(1 - b_3) \geq a_4 b_4 > 0. \tag{A.20}$$

*It's asymptotically stable if* $(1 - a_3)(1 - b_3) > a_4 b_4$ *and a Fold bifurcation if* $(1 - a_3)(1 - b_3) = a_4 b_4$.

**Proof.** Considering that the fixed points are the points of intersection of the curves $x = c(y)$ and $y = a(x)$ as given in Lemma 7, our result depend on their behaviour in situation (A.20). We know that $(0,0)$ is an intersection point of the two curves. By Lemma 7, if $a'(0) > b'(0)$ and $0 \leq a_3, b_3 < 1$, the curve $a(x)$ is above the curve $b(x)$ when $x > 0$ and below when $x < 0$ and (0,0) is a contact point of order 0; if $a'(0) = b'(0)$ the same thing happens, only $(0,0)$ becomes a contact point of order 2 between the curves. Both graphs looks like with




$a'(0) > b'(0)$




$a'(0) = b'(0)$

.

This means that if $0 \leq a_3, b_3 < 1$ and $a'(0) \geq b'(0) \Leftrightarrow \frac{1-a_3}{a_4} \geq \frac{b_4}{1-b_3} > 0 \Leftrightarrow 0 < a_4 b_4 \leq (1 - a_3)(1 - b_3) \Leftrightarrow \rho_Y \phi \tau_1 \beta_\Omega \leq (1 - \rho_\Omega)(1 - \beta_\Phi)$, then by the convexity of the graphs, $(0,0)$ is the unique fixed point of the our system.

To see the stability of this point we use the substitutions $x = y = 0$ in equations (A.8), (A.9), (A.10), (A.11), (A.12), and (A.13) of Lemma 14 to obtain

$$C_1 = (1 - a_3)(1 - b_3) - a_4 b_4 > 0, \tag{A.21}$$

$$\begin{array}{c} C_2 = (a_1 + 1)(b_1 + 1) + (a_3 b_3 - a_4 b_4)(1 - a_1)(1 - b_1) + \\ + a_3 (1 - a_1)(b_1 + 1) + b_3 (a_1 + 1)(1 - b_1) > 0 \end{array}, \tag{A.22}$$

and

$$\begin{array}{c} C_3 = a_4 b_4 (1 - a_1)(1 - b_1) + 1 \\ - [1 - (1 - a_1)(1 - a_3)][1 - (1 - b_1)(1 - b_3)] > 0 \end{array} \tag{A.23}$$

(A.21) follows directly of (A.20) if $(1 - a_3)(1 - b_3) > a_4 b_4$. If $(1 - a_3)(1 - b_3) = a_4 b_4 \Rightarrow C_1 = 0$. (A.20) is equivalent to $a_3 b_3 - a_4 b_4 + 1 \geq a_3 + b_3 \geq 0$. Since $a_1, b_1 \in ]0, 1[$, the only term of $C_2$ in (A.22) that could be negative is the second, but it becomes positive by adding the first term of the sum, and considering that $(1 + a_1)(1 + b_1) > (1 - a_1)(1 - b_1)$. Explicity,

$$\begin{array}{c} (1 + a_1)(1 + b_1) + (a_3 b_3 - a_4 b_4)(1 - a_1)(1 - b_1) > \\ (1 - a_1)(1 - b_1)[1 + a_3 b_3 - a_4 b_4] \geq 0. \end{array}$$

Hence (A.22) is satisfied. Finally, since $a_1, b_1 \in ]0, 1[$ and $a_3, b_3 \in [0, 1[$ by Lemma 5

$$0 < [1 - (1 - b_1)(1 - b_3)][1 - (1 - a_1)(1 - a_3)] < 1$$

and (A.23) is satisfied too. Thus, $(0, 0)$ is asymptotically stable when $(1 - a_3)(1 - b_3) > a_4 b_4$ and a Fold bifurcation when $(1 - a_3)(1 - b_3) = a_4 b_4$. ■

**Proposition 9** *Under the conditions of Lemma 7 the system (A.7) has three solutions:* $(0, 0)$ *an unstable point, and two asymptotically stable points* $(x_1, y_1)$, $(-x_1, -y_1)$, $x_1 > 0, y_1 > 0$, *if one of the four conditions happens*

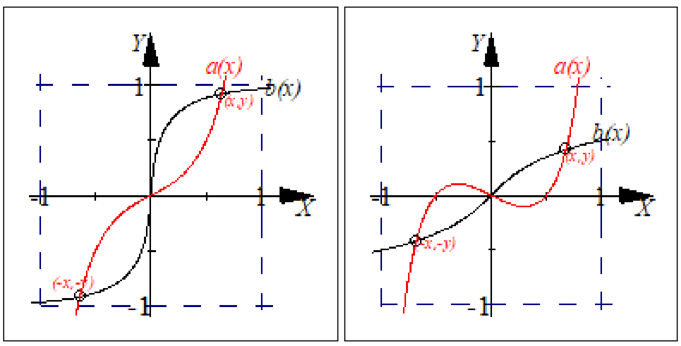
$$0 \leq a_3, b_3 \leq 1, 0 \leq (1 - a_3)(1 - b_3) < a_4 b_4, \tag{A.24}$$

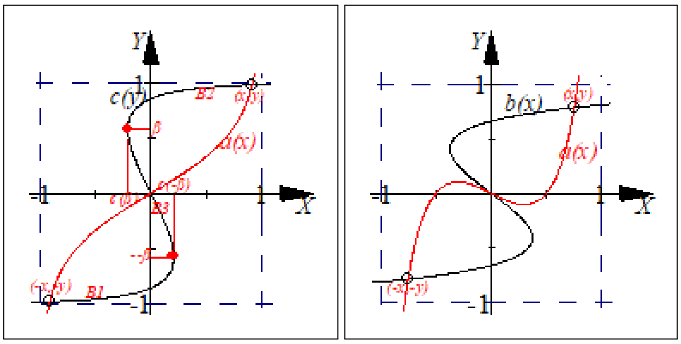
$$0 \leq a_3 \leq 1, b_3 > 1, \tag{A.25}$$

$$0 \leq b_3 \leq 1, a_3 > 1, \tag{A.26}$$

$$a_3 > 1, b_3 > 1, 0 < (a_3 - 1)(b_3 - 1) \leq a_4 b_4. \tag{A.27}$$

**Proof.** As in Proposition 8 our results depend on the behaviour of the curves $x = c(y)$ and $y = a(x)$ as given in Lemma 7 in each situation. In any case, $(0, 0)$ is an intersection point of the two curves.The following graphs illustrate, respectively, each of the four situations (A.24), (A.25), (A.26) and (A.27).





We will only demonstrate the result for (A.24) (1st graph) since for the other situations the reasoning is similar. At the end of the demonstration, we will only make some brief comments about the other cases represented by 2nd, 3rd and 4th graphs.

(A.24) is equivalent to $0 \leq a'(0) = \frac{1-a_3}{a_4} < b'(0) = \frac{b_4}{1-b_3}$. Under this condition, due to the convexity of two graphs ($a(x)$ and $b(x)$, Lemma 7), it is clear that for both $x > 0$ and $x < 0$ they separate from $(0,0)$ which is already a fixed point, and then meet again at an other fixed point $(x_1, y_1)$ in the 1st quadrant and at an third fixed point $(-x_1, -y_1)$ in the 3rd quadrant, the symmetric point of $(x_1, y_1)$.

We will now analyse the stability of each of these three equilibrium points $(0,0), (x_1, y_1)$ and $(-x_1, -y_1)$.

$(0,0)$ The three necessary and sufficient conditions are exactly as given in (A.21), (A.22) and (A.23). Follows directly of (A.24) that (A.21) is not satisfied. By (A.22),

$$C_2 = (a_1+1)(b_1+1) + a_3 b_3 (1-a_1)(1-b_1) + a_3 (1-a_1)(b_1+1) \\ + b_3 (a_1+1)(1-b_1) - a_4 b_4 (1-a_1)(1-b_1),$$

and we see that $C_2$ could be a non positive number if $a_4 b_4$ were large. In fact, we obtain from that equation and (A.24) that $C_2 > 0$ since

$$(1-a_3)(1-b_3) < a_4 b_4 < \\ \frac{1}{(1-a_1)(1-b_1)} \begin{bmatrix} b_1 + a_1 (b_1+1) + b_3 (a_1+1)(1-b_1) + \\ a_3 (1-a_1)(b_1+1) + \\ a_3 b_3 (1-a_1)(1-b_1) + 1 \end{bmatrix} = E_1 \tag{A.28}$$

where $E_1$ is a positive number greater than 1 and that if $a_4 b_4 \geq E_1$ we have that (A.22) is not satisfied and we have still the possible existence of a Flip bifurcation.

Finally, (A.23) is satisfied in the same way as was done in Proposition 8 in the case in that $(0,0)$ was the unique solution, whatever $a_4 > 0$ and $b_4 > 0$, in particular those that satisfy (A.24). Thus, we conclude that $(0,0)$ is an saddle point.

$(x_1, y_1), x_1 > 0, y_1 > 0$ The three necessary and sufficient conditions for the local stability in this case, (A.11), (A.8), (A.12), (A.9), and (A.13), (A.10) were obtained in Lemma 14 and we can repeat hear:

$$C_1 = \left[1 - a_3\left(1 - x_1^2\right)\right]\left[1 - b_3\left(1 - y_1^2\right)\right] - a_4 b_4 \left(1 - x_1^2\right)\left(1 - y_1^2\right) > 0, \tag{A.29}$$

$$C_2 = (a_1+1)(b_1+1) + \\ + a_3 (1-a_1)(b_1+1)\left(1-x_1^2\right) + b_3 (a_1+1)(1-b_1)\left(1-y_1^2\right) \\ + (a_3 b_3 - a_4 b_4)(1-a_1)(1-b_1)\left(1-x_1^2\right)\left(1-y_1^2\right) > 0 \tag{A.30}$$

and

$$C_3 = a_4 b_4 (1-a_1)(1-b_1)\left(1-x_1^2\right)\left(1-y_1^2\right)+1$$
$$-\left\{1-(1-a_1)\left[1-a_3\left(1-x_1^2\right)\right]\right\}\left\{1-(1-b_1)\left[1-b_3\left(1-y_1^2\right)\right]\right\}>0 \qquad \text{(A.31)}$$

Before analysing each of them, we must note that at the point $(x_1, y_1)$ both $a(x)$ and $b(x)$ are increasing functions and the graph of $a(x)$ is convex upwards while the graph of $b(x)$ is convex downwards, which allows us to conclude that at that point,

$$\begin{aligned} a'(x_1) > b'(x_1) &\Leftrightarrow \frac{1-a_3\left(1-x_1^2\right)}{a_4\left(1-x_1^2\right)} > \frac{b_4\left(1-y_1^2\right)}{1-b_3\left(1-y_1^2\right)} > 0 \\ &\Leftrightarrow \left[1-a_3\left(1-x_1^2\right)\right]\left[1-b_3\left(1-y_1^2\right)\right] \\ &> a_4 b_4\left(1-x_1^2\right)\left(1-y_1^2\right). \end{aligned} \qquad \text{(A.32)}$$

Now let's analyse (A.29), (A.30) and (A.31). Follows directly of (A.32) that (A.29) is satisfied. Looking at the expression in (A.30) we can say that

$$\begin{aligned} C_2 &> (a_3 b_3 - a_4 b_4)(1-a_1)(1-b_1)\left(1-x_1^2\right)\left(1-y_1^2\right)+1 \\ &= 1+(1-a_1)(1-b_1)\left[a_3 b_3\left(1-x_1^2\right)\left(1-y_1^2\right)-a_4 b_4\left(1-x_1^2\right)\left(1-y_1^2\right)\right] \\ &> (1-a_1)(1-b_1)+(1-a_1)(1-b_1)\left[\begin{array}{c} a_3 b_3\left(1-x_1^2\right)\left(1-y_1^2\right) \\ -a_4 b_4\left(1-x_1^2\right)\left(1-y_1^2\right) \end{array}\right] \\ &= (1-a_1)(1-b_1)\left[1+a_3 b_3\left(1-x_1^2\right)\left(1-y_1^2\right)-a_4 b_4\left(1-x_1^2\right)\left(1-y_1^2\right)\right] \end{aligned} \qquad \text{(A.33)}$$

However, a direct calculation shows that

$$\begin{aligned} 1+a_3 b_3\left(1-x_1^2\right)\left(1-y_1^2\right) = {} & \left[1-a_3\left(1-x_1^2\right)\right]\left[1-b_3\left(1-y_1^2\right)\right]+ \\ & a_3\left(1-x_1^2\right)+b_3\left(1-y_1^2\right) \\ & > \left[1-a_3\left(1-x_1^2\right)\right]\left[1-b_3\left(1-y_1^2\right)\right], \end{aligned} \qquad \text{(A.34)}$$

therefore, substituting (A.34) in (A.33), we have by (A.32),

$$C_2 > (1-a_1)(1-b_1)\left\{\begin{array}{c} \left[1-a_3\left(1-x_1^2\right)\right]\left[1-b_3\left(1-y_1^2\right)\right] \\ -a_4 b_4\left(1-x_1^2\right)\left(1-y_1^2\right) \end{array}\right\} > 0.$$

As $0 \le a_3, b_3 \le 1$ and by Lemma 5, the value of the expression of each factor in the last term of (A.31) is a positive number less than 1 and therefore will also be the product of the two, meaning that $C_3 > 0$. Hence, as (A.29), (A.30) and (A.31) are satisfied, the equilibrium point $(x, y) = (x_1, y_1), x_1 > 0, y_1 > 0$ is asymptotically stable.

$(-x_1, -y_1), x_1 > 0, y_1 > 0$ Both $a(x)$ and $b(x)$ are increasing functions on $(-x_1, -y_1)$ with $a\prime(-x_1) > b\prime(-x_1)$ and opposite convexity's. On the other hand, as $a', b', C_1, C_2, C_3$ given in (A.32), (A.29), (A.30) and (A.31) are even expressions in $(x, y)$, the stability of the fixed point $(-x_1, -y_1)$ follows from stability of $(x_1, y_1)$ as obtained above.

We will now briefly comment on cases (A.25), (A.26) and (A.27). The stability of $(x, y)$ and $(-x, -y)$ in all of them is exactly as in case (A.24). Therefore, we will only comment on the stability of (0,0).

(A.25) $a_3 > 1, 0 \leq b_3 \leq 1 \Rightarrow a'(0) < 0 < b'(0)$. To the fixed point $(0,0)$, $C_1 < 0$ for any $a_4 > 0, b_4 > 0$; $C_2 > 0$ if $0 < a_4 b_4 < E_1$, where $E_1$ is the positive number obtained in (A.28) and if $a_4 b_4 \geq E_1$ we have $C_2 \leq 0$ and the possible existence of a Flip bifurcation; $C_3 > 0$ if $a_4 b_4 > \frac{[1-(1-b_1)(1-b_3)][1-(1-a_1)(1-a_3)]-1}{(1-a_1)(1-b_1)} = E_2$, thus, if $E_2 \leq 0$, $C_3 > 0$ for any $a_4, b_4$ and if $E_2 > 0$ then $C_3 > 0$ for $a_4 b_4 \in ]E_2, +\infty[$. For $a_4 b_4 \leq E_2, C_3 \leq 0$ and $(0,0)$ presents a possible NS bifurcation.

(A.26) $0 \leq a_3 \leq 1, b_3 > 1 \Rightarrow a(x)$ is a non-decreasing function and according to 7 and analogously to case (A.25), we have $B_3'(0) < 0 < a'(0)$ and in this situation, to the fixed point $(0,0)$, $C_1 < 0$ for any $a_4 > 0, b_4 > 0$; $C_2 > 0$ if $a_4 b_4 \in ]0, E_1[$, where, once again, $E_1$ is the positive number obtained in (A.28) and also if $a_4 b_4 = E_1$ we have the possible existence of a Flip bifurcation; $C_3 > 0$ if $b_3 \in ]1, 1 + E_3[$ where $E_3 = \left(\frac{1-a_1}{1-b_1}\right)\left[\frac{1-a_3+a_4 b_4(1-b_1)}{1-(1-a_1)(1-a_3)}\right] > 0$ and for $b_3 = 1 + E_3$, $(0,0)$ presents a possible NS bifurcation.

(A.27) $a_3 > 1, b_3 > 1, B_3'(0) = \frac{b_4}{1-b_3} \leq a'(0) = \frac{1-a_3}{a_4} < 0 \Leftrightarrow a_4 b_4 \geq (a_3 - 1)(b_3 - 1)$. In this way, to the fixed point $(0,0)$, $C_1 \leq 0$ being that $C_1 = 0$ for $a_4 b_4 = (a_3 - 1)(b_3 - 1)$ and we have a Fold bifurcation ; $C_2 > 0$ if $a_4 b_4 \in [(a_3 - 1)(b_3 - 1), E_1[$, where $E_1$ is that positive number obtained in (A.28) and if $a_4 b_4 = E_1$ we have the possible Flip bifurcation; $C_3 > 0$ if $a_4 b_4 \in ]E_4, +\infty[$ where $E_4 = (a_3 - 1)(b_3 - 1) + \frac{a_3 - 1}{1 - b_1} + \frac{b_3 - 1}{1 - a_1}$. For $a_4 b_4 = E_4$, $(0,0)$ presents a possible NS bifurcation.

■

**Proposition 10** *Under the conditions of Lemma 7 the system (A.7) has at least three solutions:* $(0,0)$ *an unstable point, and two asymptotically stable points* $(x_1, y_1), (-x_1, -y_1), x_1 > 0, y_1 > 0$,, *since*

$$a_3 > 1, b_3 > 1, (a_3 - 1)(b_3 - 1) > a_4 b_4 > 0 \Leftrightarrow a_3 b_3 - a_4 b_4 > a_3 + b_3 - 1 > 0 \tag{A.35}$$

**Proof.** (A.35) is equivalent to $a'(0) = \frac{1-a_3}{a_4} < B_3'(0) = \frac{b_4}{1-b_3} < 0$. Under this condition, due to the convexity of two graphs: $a(x)$ and $B_3(x)$ or $a(x)$ and $B_2(x)$ or $a(x)$ and $B_1(x)$, it is clear that for both $x > 0$ and $x < 0$ they separate from $(0,0)$ which is already a fixed point, meet in the 2nd and 4th quadrants and meet again in the point $(x_1, y_1)$ of the 1st quadrant and their symmetric $(-x_1, -y_1)$ of the 3rd quadrant according to Lemma 7.

We will now analyse the stability of each of these three equilibrium points $(0,0), (x_1, y_1)$ and $(-x_1, -y_1)$.

$(0,0)$ The three necessary and sufficient conditions are exactly as given in (A.21), (A.22) and (A.23). Follows directly of (A.35) that (A.21) and (A.22) are satisfied. By (A.23)

$$\begin{aligned}
C_3 &= a_4 b_4 (1 - a_1)(1 - b_1) + 1 - [1 - (1 - a_1)(1 - a_3)][1 - (1 - b_1)(1 - b_3)] \\
&= a_4 b_4 (1 - a_1)(1 - b_1) + 1 - \\
&\left[\begin{array}{c} 1 - (1 - a_1)(1 - a_3) - (1 - b_1)(1 - b_3) + \\ (1 - a_1)(1 - a_3)(1 - b_1)(1 - b_3) \end{array}\right] \\
&= [a_4 b_4 - (1 - a_3)(1 - b_3)](1 - a_1)(1 - b_1) \\
&+ (1 - a_1)(1 - a_3) + (1 - b_1)(1 - b_3)
\end{aligned}$$

from where we get $C_3 < 0$ since each of the terms of the last sum are negative due to (A.35). Thus, we conclude that $(0,0)$ is an saddle point.

$(x_1, y_1), x_1 > 0, y_1 > 0$ The three necessary and sufficient conditions for the local stability in this case, obtained in Lemma 14 which were written in (A.29), (A.30) and (A.31), are repeated hear:

$$C_1 = \left[1 - a_3\left(1 - x_1^2\right)\right]\left[1 - b_3\left(1 - y_1^2\right)\right] - a_4 b_4\left(1 - x_1^2\right)\left(1 - y_1^2\right) > 0, \tag{A.36}$$

$$\begin{aligned} C_2 = (a_1 + 1)(b_1 + 1) + a_3(1 - a_1)(b_1 + 1)\left(1 - x_1^2\right) \\ + b_3(a_1 + 1)(1 - b_1)\left(1 - y_1^2\right) \\ + (a_3 b_3 - a_4 b_4)(1 - a_1)(1 - b_1)\left(1 - x_1^2\right)\left(1 - y_1^2\right) > 0 \end{aligned} \tag{A.37}$$

and

$$\begin{aligned} C_3 = a_4 b_4 (1 - a_1)(1 - b_1)\left(1 - x_1^2\right)\left(1 - y_1^2\right) + 1 - \\ \left\{1 - (1 - a_1)\left[1 - a_3\left(1 - x_1^2\right)\right]\right\}\left\{1 - (1 - b_1)\left[1 - b_3\left(1 - y_1^2\right)\right]\right\} > 0 \end{aligned} \tag{A.38}$$

Before analysing each of them, we must note that at the point $(x_1, y_1)$ both $a(x)$ and $B_2(x)$ are increasing functions and the graph of $a(x)$ is convex upwards while the graph of $B_2(x)$ is convex downwards, which allows us to conclude that at that point,

$$\begin{aligned} a'(x_1) > B_2'(x_1) > 0 \Leftrightarrow \frac{1 - a_3\left(1 - x_1^2\right)}{a_4\left(1 - x_1^2\right)} > \frac{b_4\left(1 - y_1^2\right)}{1 - b_3\left(1 - y_1^2\right)} > 0 \\ \Leftrightarrow \left[1 - a_3\left(1 - x_1^2\right)\right]\left[1 - b_3\left(1 - y_1^2\right)\right] > \\ a_4 b_4\left(1 - x_1^2\right)\left(1 - y_1^2\right) \\ \Leftrightarrow (a_3 b_3 - a_4 b_4)\left(1 - x_1^2\right)\left(1 - y_1^2\right) + 1 \\ > a_3\left(1 - x_1^2\right) + b_3\left(1 - y_1^2\right). \end{aligned} \tag{A.39}$$

Follows directly of (A.39) that (A.36) is satisfied. Looking at the expression in (A.37) and considering (A.35) we can conclude that $C_2 > 0$ because all its terms are positive. By (A.39), $0 < 1 - a_3\left(1 - x_1^2\right) < 1, 0 < 1 - b_3\left(1 - y_1^2\right) < 1$, thus by Lemma 5, the value of the expression of each factor in the last term of $C_3$ in (A.38) is a positive number less than 1 and therefore will also be the product of the two, meaning that $C_3 > 0$. Hence, as (A.29), (A.30) and (A.31) are satisfied, the equilibrium point $\left(\overline{\Omega}, \overline{\Phi}\right) = (x_1, y_1), x_1 > 0, y_1 > 0$ is asymptotically stable.

$(-x_1, -y_1), x_1 > 0, y_1 > 0$ Both $a(x)$ and $B_1(x)$ are increasing functions on $(-x_1, -y_1)$ with $a\prime(-x_1) > B_1'(-x_1)$ and opposite convexity's. On the other hand, as $a', B_1', C_1, C_2, C_3$ given in (A.15), (A.19), (A.36), (A.37) and (A.38) are even expressions in $(x, y)$, the stability of the fixed point $(-x_1, -y_1)$ follows from stability of $(x_1, y_1)$ as obtained above.

■

**Proposition 11** *Under the hypotheses of Proposition 10, the system (A.7) has in addition to the three solutions* $(0,0)$, $(x_1, y_1)$ *and* $(-x_1, -y_1)$, *two more unstable solutions* $(-x_2, y_2)$ *and* $(x_2, -y_2)$, $x_2 > 0, y_2 > 0$, *in the second and fourth quadrants, respectively, whenever one of the three conditions happens:*

$$[-\alpha \leq c(\beta)] \wedge [a(-\alpha) \leq \beta] \tag{A.40}$$

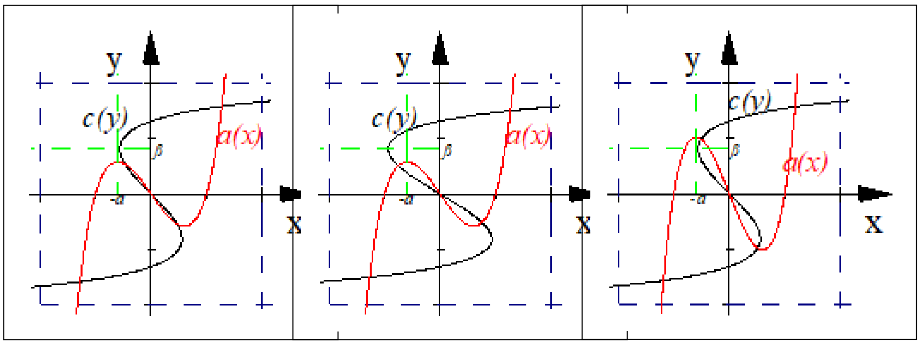
*or*

$$[-\alpha > c(\beta)] \wedge [a(x) < B_2(x)\, \forall x < -\alpha] \tag{A.41}$$

*or*

$$[a(-\alpha) > \beta] \wedge \left[c(y) > A_1^{-1}(y)\, \forall y > \beta\right]. \tag{A.42}$$

**Proof.** As we saw in Proposition 10, due to (A.35) the graphs of $a(x)$ and $c(y)$ intersect in the 2nd quadrant at least once. This intersection happens only once at the point $(-x_2, y_2), x_2 > 0, y_2 > 0$ if one of the conditions (A.40), (A.41) or (A.42) is satisfied as illustrated in the following graphs.



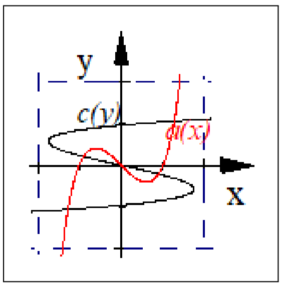

| | | |
|---|---|---|
| $[-\alpha \leq c(\beta)]$ $\wedge$ $[a(-\alpha) \leq \beta]$ | $[-\alpha > c(\beta)] \wedge$ $[a(x) < B_2(x)]$ $\forall x < -\alpha$ | $[a(-\alpha) > \beta] \wedge$ $[c(y) > A_1^{-1}(y)]$ $\forall y > \beta$ |

By Lemma 7, the point $(x_2, -y_2)$ of the 4th quadrant is also a solution of the system (A.7) in this case.The meanings of $\alpha$, $\beta$, $A_1$ and $B_2$ are described in that Lemma. Accordingly, for example, the condition (A.40) tells us that the point $-\alpha$ of the local maximum of $a(x)$ is less than or equal to the local minimum $c(\beta)$ of $c(y)$ and that simultaneously the local maximum $a(-\alpha)$ of $a(x)$ is less than or equal to the $\beta$ point, the minimum point of $c(y)$; it is clear in this situation that the graphs intersect only once in the 2nd quadrant. This same thing happens in the cases (A.41) and (A.42).

**Stability of** $(-x_2, y_2), x_2 > 0, y_2 > 0$ In order to analyse the stability of the point $(-x_2, y_2)$ (and its symmetric $(x2, -y2)$) we observe that in it, $a$ and $c$ meet in different ways. Unlike the point $(x_1, y_1)$ where $a$ and $B_2$ are both increasing, at $(-x_2, y_2)$ they either have opposite growth; or $a'(-x_2) = 0$ $(x_2 = \alpha)$ and $B_2$ increasing, $B_3$ decreasing or even $B_3'(-x_2)\nexists$ $(c(\beta) = -x_2)$; or $B_3'(-x_2)\nexists$ and $a$ increasing or decreasing. That is, 7 possibilities. Let's look at some of them.

**$a$ increasing and $B_3$ decreasing**

$$B_3'(-x_2) < 0 < a'(-x_2) \Leftrightarrow \frac{b_4(1-y_2^2)}{1-b_3(1-y_2^2)} < 0 < \frac{1-a_3(1-x_2^2)}{a_4(1-x_2^2)}$$




$C_1$ By (A.8) for $x = -x_2$ and $y = y_2$,

$$C_1 = \left[1 - a_3\left(1 - x_2^2\right)\right]\left[1 - b_3\left(1 - y_2^2\right)\right] - a_4 b_4\left(1 - x_2^2\right)\left(1 - y_2^2\right)$$

Due to the growth condition of $a$ and $B_3$ above, the product in the first term in the expression of $C_1$ is negative, so $C_1 < 0$ and (A.11) is not satisfied.

$C_2$ By (A.9) for $x = -x_2$ and $y = y_2$,

$$C_2 = (a_1+1)(b_1+1) + a_3(1-a_1)(b_1+1)\left(1-x_2^2\right) + \\ +b_3(a_1+1)(1-b_1)\left(1-y_2^2\right) \\ +(a_3b_3 - a_4b_4)(1-a_1)(1-b_1)\left(1-x_2^2\right)\left(1-y_2^2\right)$$

By (A.35) the last term of $C_2$ is positive and so $C_2 > 0$ and (A.12) is satisfied.

$C_3$ By (A.10) for $x = -x_2$ and $y = y_2$,

$$C_3 = \left[1 - a_1b_1 - a_1b_3(1-b_1)\left(1-y_2^2\right)\right] - a_3b_1(1-a_1)\left(1-x_2^2\right) \\ -(a_3b_3 - a_4b_4)(1-a_1)(1-b_1)\left(1-x_2^2\right)\left(1-y_2^2\right) \tag{A.43}$$

or

$$C_3 = a_4b_4(1-a_1)(1-b_1)\left(1-x_2^2\right)\left(1-y_2^2\right) \\ -(1-a_1)(1-b_1)\left[1-b_3\left(1-y_2^2\right)\right]\left[1-a_3\left(1-x_2^2\right)\right] \\ +\left\{(1-b_1)\left[1-b_3\left(1-y_2^2\right)\right] + (1-a_1)\left[1-a_3\left(1-x_2^2\right)\right]\right\} \tag{A.44}$$

By (A.35), the last two terms of (A.43) are negative. If we choose $b_3 = \frac{1}{a_1(1-b_1)}$, the first term becomes

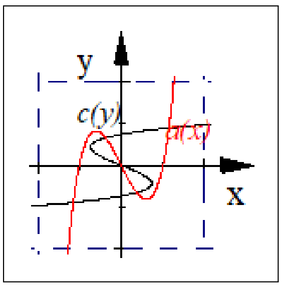
$$1 - a_1b_1 - a_1b_3(1-b_1)\left(1-y_2^2\right) = y_2^2 - a_1b_1$$

and we can choose $a_3$ small enough $a_3 \to 1$ and $a_1, b_1$ large enough $a_1 \to 1, b_1 \to 1$ so that we have $y_2 \to 0$ and this term will also be negative and thus $C_3 < 0$. On the other hand, due to the growth condition of $a$ and $B_3$, the first two terms of (A.44) are positive and the last term could be negative but if we choose $b_1$ large enough and $a_1$ small enough, this term will also become positive which implies that in this case $C_3 > 0$. This means that in reality $C_3 \gtreqless 0$.

It follows that the point $(-x_2, y_2)$ is a saddle point with the possibility of an NS bifurcation.

**$a$ decreasing and $B_2$ increasing**

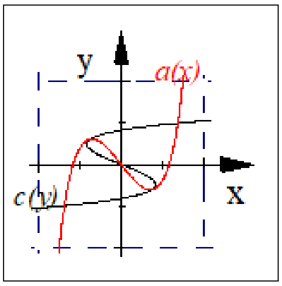
$$B_2'(-x_2) > 0 > a'(-x_2) \Leftrightarrow \frac{b_4\left(1-y_2^2\right)}{1-b_3\left(1-y_2^2\right)} > 0 > \frac{1-a_3\left(1-x_2^2\right)}{a_4\left(1-x_2^2\right)}$$




In a similar way we obtain that $C_1 < 0, C_2 > 0, C_3 \gtreqless 0$ and therefore the point $(-x_2, y_2)$ is also a saddle point with the possibility of an NS bifurcation.

**$a\prime(-x_2) = 0$ $(x_2 = \alpha)$ and $B_2$ increasing**

$$B_2'(-x_2) > 0 \text{ e } a'(-x_2) = 0 \Leftrightarrow \frac{b_4\left(1-y_2^2\right)}{1-b_3\left(1-y_2^2\right)} > 0 \text{ and } 1 - a_3\left(1-x_2^2\right) = 0$$




$C_1$ By (A.8) for $x = -x_2$ and $y = y_2$, and $1 - a_3\left(1 - x_2^2\right) = 0$, $C_1 = -a_4 b_4 \left(1 - x_2^2\right)\left(1 - y_2^2\right) < 0$ and (A.11) is not satisfied.

$C_2$ Following (A.35) and (A.9) for $x = -x_2$ and $y = y_2$ as before, $C_2 > 0$ and (A.12) is satisfied.

$C_3$ By (A.10) for $x = -x_2$ and $y = y_2$, $1 - b_3\left(1 - y_2^2\right) > 0$ and $1 - a_3\left(1 - x_2^2\right) = 0$,

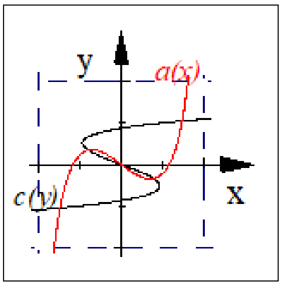
$$\begin{aligned} C_3 &= a_4 b_4 \left(1 - a_1\right)\left(1 - b_1\right)\left(1 - x_2^2\right)\left(1 - y_2^2\right) \\ &\quad - \left(1 - a_1\right)\left(1 - b_1\right)\left[1 - b_3\left(1 - y_2^2\right)\right]\left[1 - a_3\left(1 - x_2^2\right)\right] \\ &\quad + \left\{\left(1 - b_1\right)\left[1 - b_3\left(1 - y_2^2\right)\right] + \left(1 - a_1\right)\left[1 - a_3\left(1 - x_2^2\right)\right]\right\} \\ &= a_4 b_4 \left(1 - a_1\right)\left(1 - b_1\right)\left(1 - x_2^2\right)\left(1 - y_2^2\right) + \left(1 - b_1\right)\left[1 - b_3\left(1 - y_2^2\right)\right] \\ &= \left(1 - b_1\right)\left\{a_4 b_4 \left(1 - a_1\right)\left(1 - x_2^2\right)\left(1 - y_2^2\right) + \left[1 - b_3\left(1 - y_2^2\right)\right]\right\} > 0 \end{aligned} \tag{A.45}$$

and (A.13) is satisfied.

It follows that the point $(-x_2, y_2)$ is a saddle point.

**$a\prime(-x_2) = 0$ $(x_2 = \alpha)$ and $B_3$ decreasing**

$$B_3'\left(-x_2\right) < 0 \text{ and } a'\left(-x_2\right) = 0 \Leftrightarrow \frac{b_4\left(1 - y_2^2\right)}{1 - b_3\left(1 - y_2^2\right)} < 0 \text{ and } 1 - a_3\left(1 - x_2^2\right) = 0$$




$C_1$ As in the previous case $C_1 = -a_4 b_4 \left(1 - x_2^2\right)\left(1 - y_2^2\right) < 0$ and (A.11) is not satisfied.

$C_2$ Also as in the previous case, $C_2 > 0$ and (A.12) is satisfied.

$C_3$ By (A.10) for $x = -x_2$ and $y = y_2$, $1 - a_3\left(1 - x_2^2\right) = 0$ and (A.45),

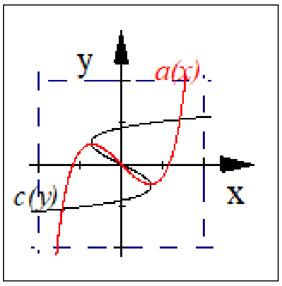
$$C_3 = \left(1 - b_1\right)\left\{a_4 b_4 \left(1 - a_1\right)\left(1 - x_2^2\right)\left(1 - y_2^2\right) + \left[1 - b_3\left(1 - y_2^2\right)\right]\right\}.$$

We note that if $a_1$ is large, $C_3 \to \left(1 - b_1\right)\left[1 - b_3\left(1 - y_2^2\right)\right] < 0$ because $B_3'\left(-x_2\right) < 0$. On the other hand, if $a_1$ is small and $a_3, b_3$ are chosen so that $y_2 \to 1$, $C_3 \to 1 - b_1 > 0$. Then $C_3 \overset{\geq}{\underset{<}{=}} 0$.

It follows that the point $(-x_2, y_2)$ is a saddle point with the possibility of an NS bifurcation.

$a'(-x_2) = 0$ **$(x_2 = \alpha)$ and** $B_3'(-x_2)\nexists$ **$(c(\beta) = -x_2)$**

$$B_3'(-x_2) \nexists \text{ and } a'(-x_2) = 0 \Leftrightarrow 1 - b_3\left(1 - y_2^2\right) = 0 \text{ and } 1 - a_3\left(1 - x_2^2\right) = 0$$

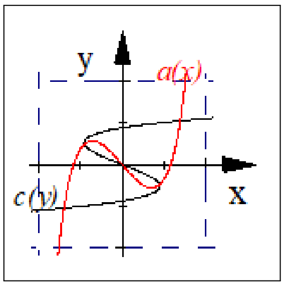


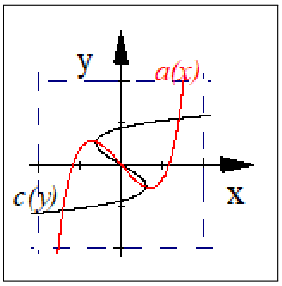

In a very similar way to the two previous cases, it is concluded that (A.11) is not satisfied and that (A.12) and (A.13) are satisfied.

It follows that the point $(-x_2, y_2)$ is a saddle point.

$B_3'(-x_2)\nexists$ **and** $a$ **increasing or decreasing**

$$B_3'(-x_2) \nexists \text{ and } \left(a'(-x_2) > 0 \text{ or } a'(-x_2) < 0\right)$$




or

In a similar way we obtain that $(-x_2, y_2)$ is a saddle point with the possibility of NS bifurcation in one of the cases.

**Stability of** $(x_2, -y_2), x_2 > 0, y_2 > 0$ Due to the symmetry and parity of the expressions involved in $a', B_1', B_3', C_1, C_2, C_3$ we conclude that as $(-x_2, y_2)$, $(x_2, -y_2)$ is also a saddle point of (A.7).

■

**Proposition 12** *Under the hypotheses of Proposition 10, the system (A.7) has in addition to the three solutions $(0,0)$, $(x_1, y_1)$ and $(-x_1, -y_1)$, four more unstable solutions: $(-x_2, y_2)$ and $(-x_3, y_3)$ in the 2nd quadrant and their symetrics $(x_2, -y_2)$ and $(x_3, -y_3)$ in the 4th quadrant, $x_i, y_i > 0, i = 1, 2, 3, x_2 < x_3, y_2 > y_3$, whenever one of the two conditions happens:*

$$[-\alpha > c(\beta)] \wedge [a(x) \leq B_2(x) \forall x < -\alpha, a(-\alpha - \delta) = B_2(-\alpha - \delta)] \tag{A.46}$$

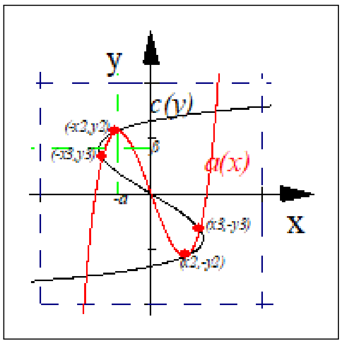
*or*

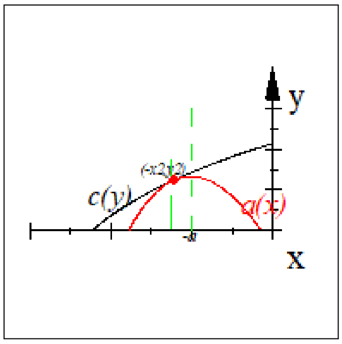
$$[a(-\alpha) > \beta] \wedge \left[c(y) \geq A_1^{-1}(y) \forall y > \beta, c(\beta + \varepsilon) = A_1^{-1}(\beta + \varepsilon)\right] \tag{A.47}$$

*for certain small positive values $\delta, \varepsilon$.*

**Proof.** As we saw in Proposition 10, due to (A.35) the graphs of $a(x)$ and $c(y)$ intersect in the 2nd quadrant at least once. If one of the conditions (A.46) or (A.47) is satisfied, this intersection happens exactly twice, one of them being an intersection proper and the other being a point of tangency. That points of interception and of tangency are $(-x_2, y_2)$ or $(-x_3, y_3), 0 < x_2 < x_3, y_2 > y_3 > 0$. Let's look at the two situations in which they happen.

**(A.46)** $[-\alpha > c(\beta)] \wedge [a(x) \leq B_2(x) \forall x < -\alpha, a(-\alpha-\delta) = B_2(-\alpha-\delta)]$
Due to the characteristics of growth and convexity of the curves (Lemma 7), this condition means that the point $(-x_2, y_2)$ is a point of tangency of the curves, also known as a point of contact of order 1 and happens, with $a$ and $c^{-1} = B_2$ increasing, close (to the left) to the point $(-\alpha, a(-\alpha))$ the point of maximum of $a$, that is, there is a small $\delta > 0$ such that $x_2 = \alpha + \delta$. And at the point $(-x_3, y_3)$ with $x_3 > x_2$ the curves only intersect with $a$ increasing and $c^{-1} = B_3$ decreasing. This is illustrated in the following graph:






$\Rightarrow$ .

**Stability of** $(-x_2, y_2)$ Due to the growth of $a$ and $B_2$ at the point of tangency $(-x_2, y_2)$, we have that

$$a'(-x_2) = \frac{1 - a_3(1-x_2^2)}{a_4(1-x_2^2)} = B_2'(-x_2) = \frac{b_4(1-y_2^2)}{1 - b_3(1-y_2^2)} > 0$$
$$\Leftrightarrow \left[1 - a_3\left(1-x_2^2\right)\right]\left[1 - b_3\left(1-y_2^2\right)\right] = a_4 b_4\left(1-x_2^2\right)\left(1-y_2^2\right) \quad \text{(A.48)}$$

Let us analyse each of the stability conditions (A.11), (A.8), (A.12), (A.9), and (A.13), (A.10) which were obtained in Lemma 14 for $x = -x_2$ and $y = y_2$.

$C_1$ By (A.48) we conclude that (A.11) is not satisfied and that $C_1 = 0$.

$C_2$ As in previous cases, (A.12) is satisfied by (A.35).

$C_3$ By (A.48), $0 < 1 - a_3(1-x_2^2) < 1$ and $0 < 1 - b_3(1-y_2^2) < 1$, thus by Lemma 5, the value of the expression of each factor in the last term of $C_3$ in (A.10) is a positive number less than 1 and therefore will also be the product of the two, meaning that $C_3 > 0$ and (A.13) is satisfied.

It therefore follows that at $(-x_2, y_2)$ we have a Fold bifurcation.

**Stability of** $(-x_3, y_3)$ Due to the growth of $a$ and $B_3$ at the point of intersection $(-x_3, y_3)$, we have that

$$B_3'(-x_3) < 0 < a'(-x_3) \Leftrightarrow \frac{b_4(1-y_3^2)}{1 - b_3(1-y_3^2)} < 0 < \frac{1 - a_3(1-x_3^2)}{a_4(1-x_3^2)} \quad \text{(A.49)}$$

Let us analyse each of the stability conditions (A.11), (A.8), (A.12), (A.9), and (A.13), (A.10) which were obtained in Lemma 14 for $x = -x_3$ and $y = y_3$.

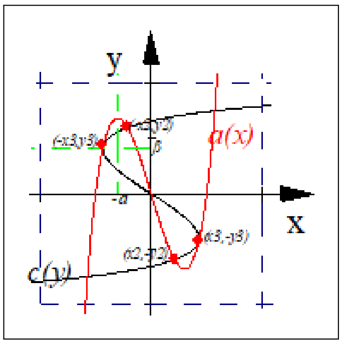
$C_1$ Due to (A.49), the product in the first term of the expression of $C_1$ in (A.8) is negative, so $C_1 < 0$ and (A.11) is not satisfied.

$C_2$ As in previous cases, (A.12) is satisfied by (A.35).

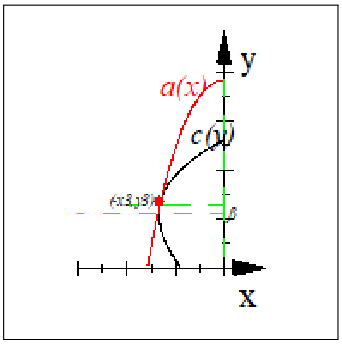
$C_3$ The case of the point $(-x_2, y_2)$ in Proposition 11 is analogous. Thus, $C_3 \gtreqless 0$ in our point $(-x_3, y_3)$.

It follows that the point $(-x_3, y_3)$ is a saddle point with the possibility of an NS bifurcation.

**(A.47)** $[a(-\alpha) > \beta] \wedge \left[c(y) \geq A_1^{-1}(y)\, \forall y > \beta, c(\beta+\varepsilon) = A_1^{-1}(\beta+\varepsilon)\right]$
Now, due to the characteristics of growth and convexity of the curves (Lemma 7), this condition means that the point $(-x_3, y_3)$ is a point of tangency of the curves and happens, with $a$ and $c^{-1} = B_2$ (or $c(y)$ and $a^{-1}(y) = A_1^{-1}(y)$) increasing , close (above) to the point $(\beta, c(\beta))$ the point of minimum of $c$, that is, there is a small $\varepsilon > 0$ such that $y_3 = \beta + \varepsilon$. And at the point $(-x_2, y_2)$ with $x_3 > x_2, y_3 < y_2$, the curves only intersect with $a$ decreasing and $c^{-1} = B_2$ increasing. This is illustrated in the following graph:




$\Rightarrow$




.

**Stability of** $(-x_2, y_2)$ **and** $(-x_3, y_3)$ Due to the growth of $a$ and $B_2$ at the point of intersection $(-x_2, y_2)$ and at the point of tangency $(-x_3, y_3)$, we have that

$$a'(-x_2) < 0 < B_2'(-x_2) \Rightarrow \frac{1 - a_3(1 - x_2^2)}{a_4(1 - x_2^2)} < 0 < \frac{b_4(1 - y_2^2)}{1 - b_3(1 - y_2^2)} \tag{A.50}$$

and that

$$a'(-x_3) = \frac{1 - a_3(1 - x_3^2)}{a_4(1 - x_3^2)} = B_2'(-x_3) = \frac{b_4(1 - y_3^2)}{1 - b_3(1 - y_3^2)} > 0 \tag{A.51}$$

The analysis of the stability conditions of $(-x_2, y_2)$ and $(-x_3, y_3)$ is identical to that of the points $(-x_3, y_3)$ and $(-x_2, y_2)$ , respectively, in the previous case. Therefore, considering (A.50), (A.51) and (A.35) we obtain $C_1 < 0, C_2 > 0$ and $C_3 \gtreqless 0$ at the point $(-x_2, y_2)$ and $C_1 = 0, C_2 > 0$ and $C_3 > 0$ at the point $(-x_3, y_3)$
It follows that the point $(-x_2, y_2)$ is a saddle point with the possibility of an NS bifurcation and that at $(-x_3, y_3)$ we have a Fold bifurcation.

**The points** $(x_2, -y_2)$ **and** $(x_3, -y_3)$ In both cases, by Lemma 7, the points $(x_2, -y_2)$ and $(x_3, -y_3)$, one of them being a point of tangency and the other a point of intersection, are also solutions of system (A.7) and due to the symmetry and parity of the expressions involved in $a', B_1', B_3', C_1, C_2, C_3$ we conclude that at the point of tangency we have a Fold bifurcation and the point of intersection is a saddle point.

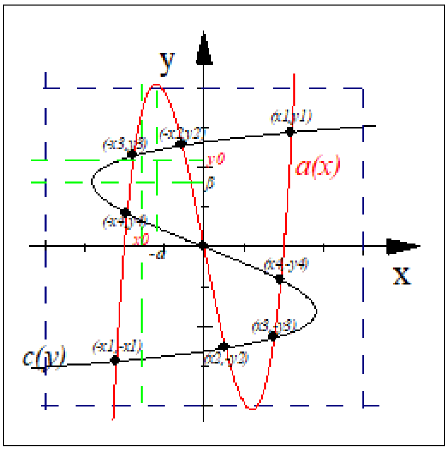
■

**Proposition 13** *Under the hypotheses of Proposition 10, the system (A.7) has in addition to the three solutions* $(0,0)$, $(x_1, y_1)$ *and* $(-x_1, -y_1)$, *six more solutions:* $(-x_2, y_2), (-x_3, y_3)$ *and* $(-x_4, y_4)$ *in the 2nd quadrant and their symmetric* $(x_2, -y_2), (x_3, -y_3)$ *and* $(x_4, -y_4)$ *in the 4th quadrant,* $x_i, y_i > 0, i = 1, 2, 3, 4, x_2 < x_3 < x_4, y_2 > y_3 > y_4$, *if none of the conditions (A.40), (A.41), (A.42), (A.46) and (A.47) of Propositions 11 and 12 occur, or equivalently, whenever the following condition happens:*

$$[a(x_0) > B_2(x_0) \text{ for some } x_0 < -\alpha] \wedge \left[c(y_0) < A_1^{-1}(y_0) \text{ for some } y_0 > \beta\right]. \tag{A.52}$$

*The points* $(-x_3, y_3), (x_3, -y_3)$ *are asymptotically stable and the other four are unstable points.*

**Proof.** As we saw in Proposition 10, due to (A.35) the graphs of $a(x)$ and $c(y)$ intersect in the 2nd quadrant at least once. It is clear that if none of the conditions (A.40), (A.41), (A.42), (A.46), (A.47) happens, the curves $a(x)$ and $c(y)$ meet at least three times in the 2nd quadrant, but due to their growth and convexity characteristics (Lemma 7) they met exactly three times at points $(-x_2, y_2), (-x_3, y_3)$ and $(-x_4, y_4), 0 < x_2 < x_3 < x_4, y_2 > y_3 > y_4 > 0$. Equivalently, if $a(x_0) > B_2(x_0)$ for some $x_0 < -\alpha$ then there is the solution $(-x_2, y_2)$ with $x_0 < -x_2, y_2 > B_2(x_0)$ and if at the same time $c(y_0) < A_1^{-1}(y_0)$ for some $y_0 > \beta$, then there is the solution $(-x_4, y_4)$ with $y_4 < y_0, A^{-1}(y_0) < -x_4$, thus due to the characteristics of growth and convexity of the curves (lemma 2) $x_4 > x_2, y_4 < y_2$ and there is also the solution $(-x_3, y_3), x_4 > x_3 > x_2, y_4 < y_3 < y_2$, with all three solutions being in the 2nd quadrant. And again by Lemma 7 the symmetric points $(x_2, -y_2), (x_3, -y_3)$ and $(x_4, -y_4)$ are also solutions of the system (A.7). The following graph illustrates the reasoning.




Let's look at the stability of each of the six points.

**Stability of** $(-x_2, y_2)$ At this point $a$ is decreasing and $B_2$ is increasing and therefore we have

$$B_2'(-x_2) > 0 > a'(-x_2) \Leftrightarrow \frac{b_4(1-y_2^2)}{1-b_3(1-y_2^2)} > 0 > \frac{1-a_3(1-x_2^2)}{a_4(1-x_2^2)}$$

This situation is exactly analogous to that of $(-x_2, y_2)$ in the case of Proposition 11 in which $a$ is decreasing and $B_2$ is increasing, as well as to that of $(-x_2, y_2)$ in the case of Proposition 12 in which the point of tangency is close to the minimum of $c$. So, analysing the the conditions (A.11), (A.12), and (A.13) for $x = -x_2$ and $y = y_2$ we see that $C_1 < 0, C_2 > 0, C_3 \gtreqless 0$ and we conclude that the point $(-x_2, y_2)$ is a saddle point with the possibility of an NS bifurcation.

**Stability of** $(-x_3, y_3)$ At this point $a$ and $B_2$ are increasing with $a' > B_2'$ and therefore we have

$$a'(-x_3) = \frac{1 - a_3(1 - x_3^2)}{a_4(1 - x_3^2)} > B_2'(-x_3) = \frac{b_4(1 - y_3^2)}{1 - b_3(1 - y_3^2)} > 0,$$

so this situation is analogous to that of $(x_1, y_1)$ in Proposition 10. We therefore have that the conditions (A.11), (A.12), and (A.13) for $x = -x_3$ and $y = y_3$ are satisfied and we conclude that the point $(-x_3, y_3)$ is asymptotically stable.

**Stability of** $(-x_4, y_4)$ At this point $a$ is increasing and $B_3$ is decreasing and therefore we have

$$B_3'(-x_4) < 0 < a'(-x_4) \Leftrightarrow \frac{b_4(1 - y_4^2)}{1 - b_3(1 - y_4^2)} < 0 < \frac{1 - a_3(1 - x_4^2)}{a_4(1 - x_4^2)}$$

This situation is exactly analogous to that of $(-x_2, y_2)$ in the case of Proposition 11 in which $a$ is increasing and $B_3$ is decreasing, as well as to that $(-x_3, y_3)$ in the case of Proposition 12 in which the point of tangency is close to the maximum of $a$. So, analysing the the conditions (A.11), (A.12), and (A.13) for $x = -x_4$ and $y = y_4$ we see that $C_1 < 0, C_2 > 0, C_3 \gtreqless 0$ and we conclude that the point $(-x_4, y_4)$ is a saddle point with the possibility of an NS bifurcation.

**Stability of** $(x_2, -y_2), (x_3, -y_3)$ **and** $(x_4, -y_4)$ As in previous cases, the stability of each point $(x, -y)$ in the 4th quadrant is exactly the same as its symmetric counterpart $(-x, y)$ in the 2nd quadrant. Therefore, $(x_2, -y_2)$ and $(x_4, -y_4)$ are saddle points with the possibility of NS bifurcation and $(x_3, -y_3)$ is an asymptotically stable point.

■

**Lemma 14** *The equilibrium points* $\left(\overline{\Omega}, \overline{\Phi}\right)$ *of the dynamic system (23) satisfy the system*

$$\begin{cases} \overline{\Omega} = \tanh\left[\rho_\Omega \Omega + \phi\tau_1\rho_Y (g_E + 1)\Phi + \rho_Y (g_E + \phi\tau_0 + \phi\tau_0 g_E)\right] \\ \overline{\Phi} = \tanh\left(\beta_\Phi \Phi + \beta_\Omega \Omega + \beta_P g_E\right) \end{cases} \tag{A.53}$$

*and his Jacobian matrix* $J = J\left(\overline{\Omega}, \overline{\Phi}\right)$ *satisfy the equations:*

$$\det(J - I) = (1 - \alpha)(1 - \gamma) \left\{ \begin{array}{c} \beta_\Phi\left(1 - \overline{\Phi}^2\right) - \rho_\Omega\left(1 - \overline{\Omega}^2\right) + \\ + \left[\beta_\Phi\rho_\Omega - \phi\tau_1\rho_Y\beta_\Omega(1 + g_E)\right]\left(1 - \overline{\Omega}^2\right)\left(1 - \overline{\Phi}^2\right) + 1 \end{array} \right\}, \tag{A.54}$$

$$\begin{aligned} \det(J + I) = {} & (1 + \alpha)(1 + \gamma) + \beta_\Phi(1 - \gamma)\left(1 - \overline{\Phi}^2\right) + \rho_\Omega(1 - \alpha)\left(1 - \overline{\Omega}^2\right) + \\ & + \alpha\beta_\Phi(1 - \gamma)\left(1 - \overline{\Phi}^2\right) + \gamma\rho_\Omega(1 - \alpha)\left(1 - \overline{\Omega}^2\right) + \\ & + (1 - \alpha)(1 - \gamma)\left[\beta_\Phi\rho_\Omega - \phi\tau_1\rho_Y\beta_\Omega(g_E + 1)\right]\left(1 - \overline{\Omega}^2\right)\left(1 - \overline{\Phi}^2\right) \end{aligned} \tag{A.55}$$

*and*

$$\begin{aligned} \det(J) = {} & \alpha\gamma + (1 - \alpha)(1 - \gamma)\left[\beta_\Phi\rho_\Omega - \phi\tau_1\rho_Y\beta_\Omega(g_E + 1)\right]\left(1 - \overline{\Omega}^2\right)\left(1 - \overline{\Phi}^2\right) + \\ & + \alpha\beta_\Phi(1 - \gamma)\left(1 - \overline{\Phi}^2\right) + \gamma\rho_\Omega(1 - \alpha)\left(1 - \overline{\Omega}^2\right) \end{aligned} \tag{A.56}$$

*where* $I$ *represents the identity matrix.*

**Proof.** If in the system (23), $f(\Omega,\Phi) = \alpha\Omega + (1-\alpha)\tanh\{\rho_\Omega\Omega + \rho_Y[g_E + (1+g_E)\phi(\tau_0+\tau_1\Phi)]\}$ and $g(\Omega,\Phi) = \gamma\Phi + (1-\gamma)\tanh(\beta_\Phi\Phi + \beta_\Omega\Omega + \beta_P g_E)$ then

$$0 = f\left(\overline{\Omega},\overline{\Phi}\right) - \overline{\Omega} = (\alpha-1)\left\{\overline{\Omega} - \tanh\left[\overline{\Omega}\rho_\Omega + \left[g_E + \phi\left(g_E+1\right)\left(\tau_0 + \overline{\Phi}\tau_1\right)\right]\rho_Y\right]\right\}$$
$$0 = g\left(\overline{\Omega},\overline{\Phi}\right) - \overline{\Phi} = (\gamma-1)\left[\overline{\Phi} - \tanh\left(\overline{\Omega}\beta_\Omega + \overline{\Phi}\beta_\Phi + \beta_P g_E\right)\right]$$

The system (A.53) occurs because $\alpha,\gamma \in ]0,1[$. On the other hand, considering (A.53) and the identities $\frac{d\tanh(x)}{dx} = \operatorname{sech}^2(x) = 1-\tanh^2(x)$, we obtain:

$$\begin{aligned} J\left(\overline{\Omega},\overline{\Phi}\right) &= \left.\begin{bmatrix} \frac{\partial f}{\partial\Omega} & \frac{\partial f}{\partial\Phi} \\ \frac{\partial g}{\partial\Omega} & \frac{\partial g}{\partial\Phi} \end{bmatrix}\right|_{\left(\overline{\Omega},\overline{\Phi}\right)} \\ &= \begin{bmatrix} \alpha + \rho_\Omega(\alpha-1)\left(\overline{\Omega}^2-1\right) & \phi\tau_1\rho_Y(\alpha-1)(g_E+1)\left(\overline{\Omega}^2-1\right) \\ \beta_\Omega(\gamma-1)\left(\overline{\Phi}^2-1\right) & \gamma+\beta_\Phi\left(\overline{\Phi}^2-1\right)(\gamma-1) \end{bmatrix} \end{aligned}$$

The equations (A.54), (A.55) and (A.56) are obtained by direct computation and later simplification. ■

## A.1 Proof of Proposition 1

Suppose scale and group effects are small, such that $0 < \rho_\Omega, \beta_\Phi < 1$, while $\rho_Y\phi\tau_1$ and $\beta_\Omega$ also small enough such that $\rho_Y\phi\tau_1\beta_\Omega < (1-\rho_\Omega)(1-\beta_\Phi)$. We consider, in the place of (A.53), the system

$$\left\{\begin{array}{l} x = \tanh(a_3x + a_4y) \\ y = \tanh(b_3y + b_4x) \end{array}\right. \tag{A.57}$$

where

$$x = \overline{\Omega} \quad y = \overline{\Phi}$$
$$a_4 = \phi\tau_1\rho_Y(g_E+1) \quad a_5 = \rho_Y(g_E + \phi\tau_0 + \phi\tau_0 g_E)$$
$$b_4 = \beta_\Omega \quad b_5 = \beta_P g_E$$

remembering that as $g_E = \tau_0 = 0$ then $a_4 = \phi\tau_1\rho_Y, a_5 = b_5 = 0$. If $(x,y)$ is a solution of (A.57), that is if $(x,y)$ is an fixed point of (23), then $(x,y)$ is a intersection point of the curves

$$\left\{\begin{array}{l} y = \frac{1}{a_4}\left[\tanh^{-1}(x) - a_3x\right] = a(x), |x|<1 \\ x = \frac{1}{b_4}\left[\tanh^{-1}(y) - b_3y\right] = c(y), |y|<1 \end{array}\right. .$$

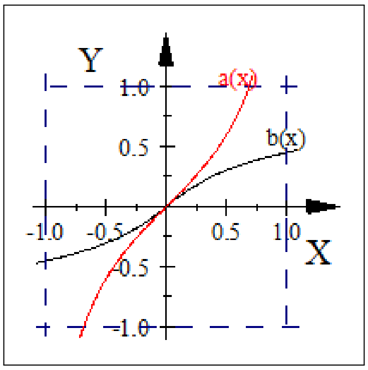
To see the behaviour of the graphics of the two equations $y = a(x)$ and $x = c(y)$, we calculate $a'(x) = \frac{da}{dx}$ and $c'(y) = \frac{db}{dy}$

$$a'(x) = \frac{a_3\left(x^2 - \frac{a_3-1}{a_3}\right)}{a_4(1-x^2)}, |x|<1 \text{ and } c'(y) = \frac{b_3\left(y^2 - \frac{b_3-1}{b_3}\right)}{b_4(1-y^2)}, |y|<1.$$

Given that $0 < a_3, b_3 < 1 \Rightarrow a'(x) > 0$ and $c'(y) > 0 \Rightarrow a(x), |x| < 1$ and $b(x) = c^{-1}(x), -\infty < x < +\infty$ are strictly increasing functions. Furthermore we obtain: $\lim_{x\to-1^+} a(x) = -\infty, a(0) = 0, \lim_{x\to 1^-} a(x) = +\infty, a'(0) = \frac{1-a_3}{a_4}, a''(x) = \frac{(1-x^2)(2a_3x)-(1-a_3+a_3x^2)(-2x)}{a_4(1-x^2)^2} = \frac{2x}{a_4(1-x^2)^2} \Rightarrow x = 0$ is an inflection point of $a(x)$ and his graphic is convex downwards when $x < 0$ and is convex upwards when $x > 0$; on the other hand for the function $b(x)$, we have: $\lim_{x\to-\infty} b(x) = -1^+, b(0) = c^{-1}(0) = 0, \lim_{x\to+\infty} b(x) = 1^-, b'(x) = \frac{1}{c'(y)} = \frac{b_4(1-y^2)}{b_3\left(y^2-\frac{b_3-1}{b_3}\right)} =$

$\frac{b_4\left(1-b(x)^2\right)}{b_3\left(b(x)^2-\frac{b_3-1}{b_3}\right)} > 0, b'(0) = \frac{b_4}{1-b_3}, b''(x) = b_4 \frac{\left(1-b_3+b_3y^2\right)\left(-2yb'(x)\right)-\left(1-y^2\right)\left(2b_3yb'(x)\right)}{\left(1-b_3+b_3y^2\right)^2} = \frac{-2b_4yb'(x)}{\left(1-b_3+b_3y^2\right)^2} \Rightarrow$ $x = 0$ is an inflection point and the graphic is convex upwards when $x < 0$ and is convex downwards when $x > 0$.

Hence $(0, 0)$ is an intersection point of the two curves and if $a'(0) > b'(0)$ the curve $a(x)$ is above the curve $b(x)$ when $x > 0$ and below it when $x < 0$. Both graphs looks like with




This means that if $a'(0) > b'(0) \Leftrightarrow \frac{1-a_3}{a_4} > \frac{b_4}{1-b_3} \Leftrightarrow a_4b_4 < (1-a_3)(1-b_3) \Leftrightarrow \rho_Y\phi\tau_1\beta_\Omega < (1-\rho_\Omega)(1-\beta_\Phi)$, then $(0, 0)$ is the unique fixed point of the our system.

To see the stability we use equations (A.54), (A.55) and (A.56) of Lemma 14 to obtain the three necessary and sufficient conditions for the local stability of the equilibrium point $\left(\overline{\Omega}, \overline{\Phi}\right) = (x, y) = (0, 0)$ of the system (23). They are:

$$a_3b_3 - b_3 - a_3 - a_4b_4 + 1 > 0,$$

$$\begin{array}{c} b_1 + a_1(b_1+1) + (a_3b_3 - a_4b_4)(1-a_1)(1-b_1) + a_3(1-a_1)(b_1+1) + \\ +b_3(a_1+1)(1-b_1) + 1 > 0 \end{array},$$

and

$$-a_3b_1(1-a_1) - (a_3b_3 - a_4b_4)(1-a_1)(1-b_1) - a_1b_1 - a_1b_3(1-b_1) + 1 > 0.$$

We will analyse each one:

**The first:** $a_3b_3 - b_3 - a_3 - a_4b_4 + 1 = 1 - a_3 - b_3(1-a_3) - a_4b_4 = (1-a_3)(1-b_3) - a_4b_4 > 0$ for the parameter condition of the proposition.

**The second:** Since $a_1, b_1 \in ]0, 1[$, the only term of that sum that could be negative is $(a_3b_3 - a_4b_4)(1-a_1)(1-b_1)$, but it becomes positive by adding 1, the last term of the sum, and considering that $a_4b_4(1-a_1)(1-b_1) < 1$. Explicity, $(a_3b_3 - a_4b_4)(1-a_1)(1-b_1) + 1 = a_3b_3(1-a_1)(1-b_1) + [1 - a_4b_4(1-a_1)(1-b_1)] > 0$. Hence the second condition is satisfied.

**The third:** If we join all negative terms of the sum we obtain

$$\begin{aligned} & -a_3b_1(1-a_1) - a_3b_3(1-a_1)(1-b_1) - a_1b_1 - a_1b_3(1-b_1) \\ & = (b_1 + b_3 - b_1b_3)(-a_1 - a_3 + a_1a_3) \\ & = -(b_1 + b_3 - b_1b_3)(a_1 + a_3 - a_1a_3) \\ & = -[1 - (1-b_1)(1-b_3)][1 - (1-a_1)(1-a_3)]. \end{aligned}$$

Since $a_1, b_1, a_3, b_3 \in ]0, 1[$, $0 < [1 - (1-b_1)(1-b_3)][1 - (1-a_1)(1-a_3)] < 1$, the total sum is $a_4b_4(1-a_1)(1-b_1) + \{1 - [1 - (1-b_1)(1-b_3)][1 - (1-a_1)(1-a_3)]\} > 0$ and this condition also is satisfied.

## A.2 Proof of Proposition 2

We consider, in the place of system (A.53), the system

$$\left\{ \begin{array}{l} x = \tanh\left(a_4 y + a_5\right) \\ y = \tanh\left(b_4 x + b_5\right) \end{array} \right. \tag{A.58}$$

where were did the substitutions $x = \overline{\Omega}, y = \overline{\Phi}, a_4 = \phi\tau_1\rho_Y\left(g_E + 1\right), a_5 = \rho_Y\left(g_E + \phi\tau_0 + \phi\tau_0 g_E\right), b_4 = \beta_\Omega$ and $b_5 = \beta_P g_E$. If $(x, y)$ is a solution of (A.58), that is if $(x, y)$ is an equilibrium point of (23), then

$$y = \frac{1}{a_4}\left[\tanh^{-1}(x) - a_5\right] \text{ and } y = \tanh\left(b_4 x + b_5\right)$$

So, if we consider the function

$$p(x) = \frac{1}{a_4}\left[\tanh^{-1}(x) - a_5\right] - \tanh\left(b_4 x + b_5\right), x \in ]-1, 1[$$

then $x$ is one of the root of $p$. To see the behaviour of $p$ we obtain $\lim_{x\to -1^+} p(x) = -\infty, \lim_{x\to 1^-} p(x) = +\infty$, $p(0) = -\frac{a_5}{a_4} - \tanh(b_5) < 0$,

$$p'(x) = \frac{1}{a_4\left(1 - x^2\right)}\left[1 - a_4 b_4\left(1 - x^2\right)\operatorname{sech}^2\left(b_4 x + b_5\right)\right] \tag{A.59}$$

$$p'(0) = \frac{1}{a_4} - b_4 \operatorname{sech}^2(b_5).$$

The sign of $p'$ depend of the sign of $1 - a_4 b_4\left(1 - x^2\right)\operatorname{sech}^2\left(b_4 x + b_5\right)$. Let's do

$$m(x) = \left(1 - x^2\right)\operatorname{sech}^2\left(b_4 x + b_5\right), x \in ]-1, 1[. \tag{A.60}$$

$0 < m(x) \le 1, \lim_{x\to -1^+, 1^-} m(x) = 0, m(0) = \operatorname{sech}^2(b_5) \in ]0, 1]$ and

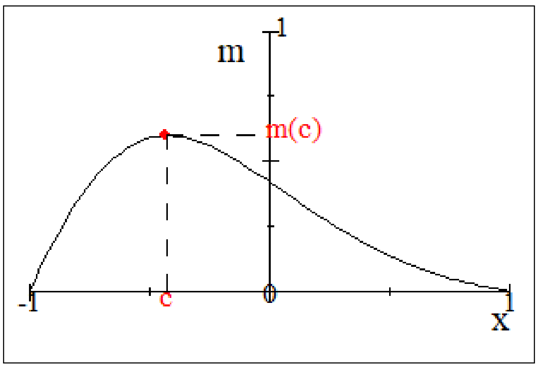
$$m'(x) = -2\operatorname{sech}^2\left(b_4 x + b_5\right)\left[b_4\left(1 - x^2\right)\tanh\left(b_4 x + b_5\right) + x\right] \tag{A.61}$$

$$m'(0) < 0, \lim_{x\to -1^{+-}} m'(x) > 0, \lim_{x\to 1^{--}} m'(x) < 0 \tag{A.62}$$

Given that $s(x) = b_4\left(1 - x^2\right)\tanh\left(b_4 x + b_5\right) + x > 0$ if $x \ge 0 \Rightarrow m'(x) = 0$ for $x < 0, b_4 x + b_5 > 0$. Furthermore,

$$s'(x) = 1 + b_4^2\left(1 - x^2\right)\operatorname{sech}^2\left(b_4 x + b_5\right) - 2b_4 x\tanh\left(b_4 x + b_5\right) > 0 \text{ in } ]-1, 0] \tag{A.63}$$

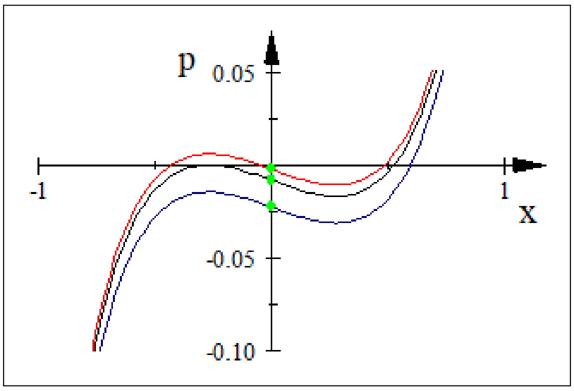
$\Rightarrow s$ is increasing in $]-1, 0], \lim_{x\to -1^+} s(x) = -1, s(0) > 0 \Rightarrow \exists! c \in ]-1, 0[$ where $s(c) = 0 \Rightarrow c$ is the unique critical point of $m(x)$ and it's an maximal point. The function $m$ looks like



It follows, from (A.59) and (A.60) that

$$p' \gtreqqless 0 \text{ if } m(x) \lesseqqgtr \frac{1}{a_4 b_4} \tag{A.64}$$

$\Rightarrow$ if $\frac{1}{a_4b_4} \geq m(c), p' \geq 0, p$ is not decreasing in $]-1,1[$ and exist an unique root $x > 0$ of it, $x = \overline{\Omega}_1$; if $\frac{1}{a_4b_4} < m(c), m(x) = \frac{1}{a_4b_4}$ for two values $x_1, x_2, x_1 < 0$ that, according to (A.64), are critical points of $p, p(x_1)$ is a minimal point and $p(x_2)$ is a maximal point. The three possibilities in that case for the function $p(x)$ are illustrated by the graphic




$p(x_1) < 0, p(x_1) = 0, p(x_1) > 0$

The three function $p$ (blue, black and red) show that $p$ has an root $x > 0, x = \overline{\Omega}_1$. The first (blue, $p(x_1) < 0$) implies that that root is unique. The second (black, $p(x_1) = 0$) implies that in addition, there is a second root $x < 0, x = x_1 = \overline{\Omega}_2$. The third (red, $p(x_1) > 0$) implies that there are two root additional, the one, $x < x_1, x = \overline{\Omega}_2$ and the other one, $x_1 < x = \overline{\Omega}_3 < 0$.

## A.3 Proof of Proposition 3

Using the substitutions in (A.57) and equations (A.54), (A.55) and (A.56) of Lemma 14 we obtain the necessary and sufficient conditions for the local stability of a given equilibrium point $(\overline{\Omega}, \overline{\Phi}) = (x, y)$ of the system (23) with $a_3 = \rho_\Omega = 0, b_3 = \beta_\Phi = 0$

$$1 - a_4b_4\left(1-x^2\right)\left(1-y^2\right) > 0 \tag{A.65}$$

$$(a_1+1)(b_1+1) - (a_4b_4)(1-a_1)(1-b_1)\left(1-x^2\right)\left(1-y^2\right) > 0 \tag{A.66}$$

$$1 - a_1b_1 + a_4b_4(1-a_1)(1-b_1)\left(1-x^2\right)\left(1-y^2\right) > 0 \tag{A.67}$$

First of all, let us note that the third condition is always satisfied because $a_1, b_1 \in ]0,1[$. Furthermore, equilibrium points $(\overline{\Omega}, \overline{\Phi}) = (x, y)$ satisfy the system (**??**) or (A.58). We will consider initially the point $P_1 = \left(\overline{\Omega}_1, \overline{\Phi}_1\right), \overline{\Omega}_1 > 0$ which to appear always. According to 8, $m\left(\overline{\Omega}_1\right) = m(x) = (1-x^2)(1-y^2) < \frac{1}{a_4b_4}$ whence it follows that (A.65) is satisfied and considering $a_1, b_1 \in ]0,1[$ (A.66) is satisfied too. Then we conclude that $P_1$ is locally asymptotically stable. To analyse $P_2 = \left(\overline{\Omega}_2, \overline{\Phi}_2\right), \overline{\Omega}_2 < 0$ we observe according to 8 that one possibility is $\overline{\Omega}_2 = x_1 < 0, p(x_1) = 0, m(x_1) = m\left(\overline{\Omega}_2\right) = m(x) = (1-x^2)(1-y^2) = \frac{1}{a_4b_4} \Rightarrow 1 - a_4b_4(1-x^2)(1-y^2) = 0$ that is, in $P_2$ it happens the fold bifurcation condition; the other one in that to appear $P_2$ is when $\overline{\Omega}_2 = x < x_1 < 0, p(x_1) > 0, m\left(\overline{\Omega}_2\right) = m(x) = (1-x^2)(1-y^2) < \frac{1}{a_4b_4}$ and once again it follows that (A.65) is satisfied and considering $a_1, b_1 \in ]0,1[$ (A.66) is satisfied too; hence, in this case $P_2$ is locally asymptotically stable. Finally, to analyse $P_3 = \left(\overline{\Omega}_3, \overline{\Phi}_3\right), \overline{\Omega}_2 < \overline{\Omega}_3 < 0$ we note that it happens when $p(x_1) > 0, \overline{\Omega}_2 < x_1 < \overline{\Omega}_3 < x_2 \Rightarrow \frac{1}{a_4b_4} = m(x_1) < m\left(\overline{\Omega}_3\right) = m(x) = (1-x^2)(1-y^2) \Rightarrow 1 - a_4b_4(1-x^2)(1-y^2) < 0$ and the condition (A.65) is violed; so $P_3$ is is a saddle point.

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